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A conjecture of Kátai

by

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1. An arithmetic function $f(n)$ is said to be *additive* if for every pair of positive coprime integers a and b the relation $f(ab) = f(a)+f(b)$ is satisfied. If the same relation holds for every pair of positive integers, whether coprime or not, then the function is said to be *completely additive*.

In his paper [5] Kátai lists six conjectures. The first of these, labelled H_1, asserts that if a completely additive arithmetic function vanishes on each of the integers which are of the form $p+1$, p prime, then it must be identically zero. In other words, a completely additive function is determined by its values on the shifted primes. In a subsequent paper [6], he proved the existence of an absolute constant K, so that if in addition we assume that $f(p) = 0$ for the primes p not exceeding K, then f is indeed identically zero. Unfortunately, the constant K could not be effectively determined by the method that he used, so that the conjecture H_1 could not yet be settled in this way.

We shall prove two results which will, in particular, include a stronger form of the conjecture H_1.

In what follows we assume only that f is an additive arithmetic function.

THEOREM 1. *Let $|f(p+1)| \leqslant A$ hold for each prime p. Then the series*

$$\sum_{|f(p)|\geqslant 1} \frac{1}{p}, \qquad \sum_{|f(p)|\leqslant 1} \frac{f^2(p)}{p}$$

both converge.

THEOREM 2. *Let $f(p+1) = \text{constant}$ hold for all sufficiently large primes p. Then $f(2^\nu) = \text{constant}$ for all integers $\nu \geqslant 1$, and f is zero on all other prime-powers.*

Remark. It will be clear from the method of proof of these theorems that a weaker hypothesis than that used in Theorem 2 will lead to a similar conclusion. For example, one could restrict the primes p to lie in a fixed arithmetic progression, or consider shifted primes $p+k$ for values of k other than $k = 1$.

The validity of conjecture H_1 follows from Theorem 2.

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2. The proof of both theorems depends upon the following essential lemma.

Let x be a positive real number. This may need to be sufficiently large in what follows. For each positive integer d let $N(d, x)$ denote the number of solutions to the equation

$$p+1 = d(q+1)$$

where p and q are prime numbers restricted by the conditions

$$p \leqslant x, \quad x^{1/6} < q < x^{1/5}, \quad (d, q+1) = 1.$$

LEMMA 1.

$$\sum_{\substack{x^{3/5}<d<x^{5/6}\\ N(d,x)>0}} d^{-1} \gg \log x.$$

In order to prove this lemma we shall make use of the following estimates which can be found as, or deduced from, standard results in the theory of numbers.

(1) *For each integer $D \geqslant 1$:*

$$\pi(x, D, 1) \stackrel{\text{def}}{=} \sum_{\substack{p\leqslant x\\ p\equiv 1 \pmod D}} 1 \ll 1 + \frac{x}{D};$$

(2) $$E^*(x, D) \stackrel{\text{def}}{=} \sup_{(l,D)=1} \sup_{y\leqslant x} \left| \pi(y, D, l) - \frac{\mathrm{Li}(y)}{\varphi(D)} \right| \ll 1 + \frac{x}{\varphi(D)};$$

(3) $$\sum_{D<x^{1/5}(\log x)^3} E^*(x, D) \ll x(\log x)^{-5}.$$

See, for example, Bombieri [1]. A shorter proof, on different lines is given in Gallagher [4].

(4) *Uniformly for all positive integers d not exceeding $x^{5/6}$:*

$$N(d, x) \ll \frac{x}{d(\log x)^2} \prod_{p\mid d(d-1)} \left(1 - \frac{1}{p}\right)^{-1}.$$

See, for example, Prachar [7], Kap. II, Satz 4.2, p. 45.

Proof of Lemma 1. We shall prove Lemma 1 by estimating a certain sum from above and below. We begin with the estimate from below.

From (2) and (3) we obtain

$$\sum_{x^{3/5}<d<x^{5/6}} N(d, x) \geqslant \sum_{x^{1/6}<q<x^{1/5}} \sum_{\substack{x^{4/5}<p<x\\ p\equiv -1 \pmod{(q+1)}\\ \left(\frac{p+1}{q+1},\, q+1\right)=1}}$$

$$\geqslant \sum_{x^{1/6}<q<x^{1/5}} \sum_{\substack{p<x\\ p\equiv -1 \pmod{(q+1)}}} \sum_{r\mid \left(\frac{p+1}{q+1},\, q+1\right)} \mu(r) - \pi(x^{1/5})\pi(x^{4/5})$$

$$\geqslant \sum_{x^{1/6}<q<x^{1/5}} \sum_{r\mid(q+1)} \mu(r)\,\mathrm{Li}(x)/\varphi\big(r(q+1)\big) -$$

$$- \sum_{x^{1/6}<q<x^{1/5}} \sum_{r\mid(q+1)} E^*\big(x, r(q+1)\big) + O\big(x(\log x)^{-2}\big),$$

where $\pi(y)$ denotes the number of rational primes not exceeding the real number y. The first of these last two double sums can be estimated at once by

$$\mathrm{Li}(x) \sum_{x^{1/6}<q<x^{1/5}} \frac{1}{q+1} = \log\frac{6}{5} \cdot \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right)$$

whilst by using (3) and (2) the second (and final) double sum is

$$\ll \sum_{\substack{x^{1/6}<q<x^{1/5}\\ r<(\log x)^3}} E^*\big(x, r(q+1)\big) + \sum_{\substack{rs<x^{1/5}\\ r>(\log x)^3}} \left(1 + \frac{x}{\varphi(r^2 s)}\right) + \frac{x}{(\log x)^2}$$

$$\ll x(\log x)^{-2} + x\log\log x \sum_{s<x^{1/5}} \frac{1}{s} \sum_{r>(\log x)^3} \frac{1}{r^2} \ll x(\log x)^{-3/2}$$

since for all sufficiently large integers m we have $\varphi(m) \gg m(\log\log m)^{-1}$. Hence we obtain the lower bound

$$\sum_{x^{3/5}<d<x^{5/6}} N(d, x) \gg \frac{x}{\log x}.$$

In the other direction we apply the Cauchy–Schwarz inequality in the form

$$\sum_{x^{3/5}<d<x^{5/6}} N(d, x) \leqslant \Big(\sum_{\substack{x^{3/5}<d<x^{5/6}\\ N(d,x)>0}} d^{-1}\Big)^{1/2} \Big(\sum_{2\leqslant d\leqslant x^{5/6}} dN^2(d, x)\Big)^{1/2}.$$

In view of the lower bound established above, Lemma 1 will follow if we prove that

$$\sum_{2\leqslant d\leqslant x^{5/6}} dN^2(d, x) \ll \frac{x^2}{(\log x)^3}.$$

To do this it is convenient to define the function

$$\varepsilon(d) = d^{-1/2}\prod_{p|d}(1-p^{-1})^{-1}.$$

Then by the preliminary result (4) we have

$$\sum_{2\leqslant d\leqslant x^{5/6}} dN^2(d,x) \ll \frac{x^2}{(\log x)^4}\sum_{2\leqslant d\leqslant x^{5/6}} \varepsilon(d)\varepsilon(d-1).$$

After a further application of the Cauchy–Schwarz inequality, noting that the function $\varepsilon(d)$ is multiplicative, we have

$$\sum_{2\leqslant d\leqslant x}\varepsilon(d)\varepsilon(d-1) \leqslant \Big(\sum_{d\leqslant x}\varepsilon^2(d)\Big)^{1/2}\Big(\sum_{2\leqslant d\leqslant x}\varepsilon^2(d-1)\Big)^{1/2}$$
$$\leqslant \prod_{p\leqslant x}\big(1+\varepsilon^2(p)+\varepsilon^2(p^2)+\ldots\big)$$
$$= \prod_{p\leqslant x}\big(1+p^{-1}(1-p^{-1})^{-5}\big) \ll \log x.$$

Gathering together the various inequalities we see that Lemma 1 is proved.

3. In this section we prove Theorem 1. We need one further preliminary result. An additive function $f(m)$ is said to be *finitely distributed* if there are two positive constants c_1 and c_2, and an unbounded sequence of positive integers $n_1 < n_2 < \ldots$ so that for each integer $n = n_j$ we can find a further sequence of integers $1 \leqslant a_1 < a_2 \ldots < a_k \leqslant n$ with $k \geqslant c_1 n$, and for which $|f(a_i)-f(a_j)| \leqslant c_2$. We shall make use of the following characterization of such functions, due to Erdös [3].

LEMMA 2. *An additive function $f(n)$ is finitely distributed if and only if it can be expressed in the form $c\log n + g(n)$, where c is a constant, and the additive function $g(n)$ satisfies the conditions*

$$\sum_{|g(p)|>1}\frac{1}{p} < \infty, \qquad \sum_{|g(p)|\leqslant 1}\frac{g^2(p)}{p} < \infty.$$

Proof. The proof given in Erdös' original paper [3] is elementary but complicated. Another proof, based on quite different ideas, can be found in Ryavec [8].

Proof of Theorem 1. Let us define

$$D(y) = \sum_{\substack{x^{3/5}<d\leqslant y\\ N(d,x)>0}} 1, \qquad \beta = \sup_{x^{3/5}<y\leqslant x^{5/6}} D(y)/y.$$

Then appealing to Lemma 1 and integrating by parts:

$$\log x \ll \sum_{\substack{x^{3/5}<d\leqslant x^{5/6}\\ N(d,x)>0}} d^{-1} = \int_{x^{3/5}}^{x^{5/6}} y^{-1}dD(y)$$
$$= X^{-5/6}D(x^{5/6}) + \int_{x^{3/5}}^{x^{5/6}} D(y)y^{-2}dy \leqslant \beta + \beta\int_{x^{3/5}}^{x^{5/6}} y^{-1}dy$$
$$= \beta\big(1+(\tfrac{5}{6}-\tfrac{3}{5})\log x\big).$$

From this it follows that we can find an absolute constant $c_1 > 0$ such that, in each interval $x^{3/5} < y \leqslant x^{5/6}$, there exists a number y_0 for which we have $D(y_0) > c_1 y_0$.

We define

$$\mathscr{D} = \{d\colon x^{3/5} < d \leqslant y_0,\ N(d,x) > 0\}.$$

Then each integer d in $\mathscr{D}$ can be represented in the form $d = (p+1)/(q+1)$ where $(d, q+1) = 1$, and the numbers p and q are prime and become large with x. From this and the assumptions of Theorem 1 we obtain

$$|f(d)| = |f(p+1)-f(q+1)| \leqslant 2A.$$

This implies that f satisfies the assumptions of Lemma 2, and therefore it can be represented in the form $f(n) = c\log n + g(n)$ where $g(n)$ satisfies the conditions given in Lemma 2.

To complete the proof of Theorem 1 we shall prove that the constant c has the value zero.

Let ε be a real number, $0 < \varepsilon < 1/10$. Let P be chosen so large that

$$\sum_{\substack{p>P\\ |g(p)|>1}} p^{-1} < \varepsilon.$$

The number of integers m not exceeding y_0, and divisibe by a prime $p > P$ for which $|g(p)| > 1$, is then not more than

$$\sum_{\substack{p>P\\ |g(p)|>1}}\left[\frac{y_0}{p}\right] < \varepsilon y_0.$$

The number of integers m not exceeding y_0, and divisible by a prime-power p^ν with $\nu \geqslant \nu_0 = 10-(\log\varepsilon)/\log 2$, is not more than

$$\sum_p\sum_{\nu\geqslant\nu_0}\left[\frac{y_0}{p^\nu}\right] \leqslant y_0\sum_p\frac{1}{p^{\nu_0}-p^{\nu_0-1}} < 2^{-\nu_0+10}y_0 = \varepsilon y_0.$$

The number of integers m not exceeding y_0, and divisible by p^2 for some prime $p > 1+\varepsilon^{-1}$, is at most

$$\sum_{p>\varepsilon^{-1}+1}\left[\frac{y_0}{p^2}\right] < y_0 \sum_{n>\varepsilon^{-1}+1}\frac{1}{n(n-1)} < \varepsilon y_0.$$

Let $\varepsilon = \min(\frac{1}{4}c_1, \frac{1}{20})$, and define

$$\mathscr{B} = \{d:\ x^{3/5} < d \leqslant y_0,\ N(d,x) > 0;\ p^2|d \Rightarrow p \leqslant \varepsilon^{-1}+1,$$
$$p^\nu \| d \Rightarrow \nu \leqslant \nu_0,\ p|d,\ |g(p)| > 1 \Rightarrow p \leqslant P\}.$$

Then from these last three assertions, and what have proved concerning the set $\mathscr{D}$, it follows that

$$\sum_{b\in\mathscr{B}} 1 \geqslant c_1 y_0 - 3\varepsilon y_0 \geqslant \tfrac{1}{4}c_1 y_0.$$

Furthermore, for each integer b in $\mathscr{B}$

$$|c\log b| \leqslant |f(b)| + |g(b)| \leqslant 2A + O\Big(\sum_{p|b}1\Big).$$

Therefore

$$|c|\sum_{r\leqslant\frac{1}{4}c_1y_0}\log b \leqslant |c|\sum_{b\in\mathscr{B}}\log b \leqslant 2Ay_0 + O\Big(\sum_{b\in\mathscr{B}}\nu(b)\Big)$$
$$\ll Ay_0 + \sum_{m\leqslant y_0}\nu(m) \ll y_0\log\log y_0$$

so that

$$|c| \ll \frac{\log\log y_0}{\log y_0}.$$

Since this last inequality holds for an unbounded sequence of values y_0, we have $c = 0$, and Theorem 1 is proved.

4. In this section we prove Theorem 2. The proof is carried out along the lines of the proof of Theorem 1. As before we define $D(y)$ and determine an unbounded sequence of values y_0, and a positive constant c_1, so that $D(y_0) \geqslant c_1 y_0$. This time the integers d which we counted in the set $\mathscr{D}$ are such that

$$f(d) = f(p+1) - f(q+1) = 0.$$

As before $f(n)$ has the form $c\log n + g(n)$, and we can prove that $c = 0$. We can, however, already do better than this by applying the following result.

For each real number $z > 0$, and integer n, set

$$F_n(z) = n^{-1}\sum_{\substack{m=1\\ f(m)\leqslant z}}^{n} 1.$$

LEMMA 8. *A necessary and sufficient condition that the limiting relation*

$$\limsup_{\delta\to 0+}\limsup_{x\to\infty}\big(F_n(z+\delta) - F_n(z-\delta)\big) = 0$$

should hold uniformly for all real values of z is that the series

$$\sum_{f(p)\neq 0}\frac{1}{p}$$

diverges.

Proof. This result was proved by Erdös [3], subject to the side condition $f(p) = O(1)$. A proof along different lines, and without such a side condition, can be found in Elliott and Ryavec [2].

In our present circumstances we set $z = 0$, so that for every $\delta > 0$

$$\limsup_{n\to\infty}\big(F_n(\delta) - F_n(-\delta)\big) \geqslant \limsup_{y_0\to\infty} y_0^{-1}D(y_0) \geqslant c_1 > 0.$$

It follows at once that $f(p) = 0$ for almost all primes p, in the sense of Lemma 3.

The remainder of the proof of Theorem 2 is straightforward, but complicated. In order to facilitate its presentation we collect here three further well-known results from the theory of numbers, of which we shall make use.

(5) *For each integer $D \geqslant 1$*

$$\pi(x, D, -1) = \big(1+o(1)\big)\frac{x}{\varphi(D)\log x}.$$

See, for example, Prachar [7], Kap. IV, Satz 7.5, p. 138.

(6) *Uniformly for $D \leqslant x^\alpha$, $0 < \alpha < 1$,*

$$\pi(x, D, -1) \leqslant c_3(\alpha)\frac{x}{\varphi(D)\log x}$$

where the constant $c(\alpha)$ may depend upon α.

See, for example, Prachar [7], Kap. II, Satz 4.1, p. 44.

(7) *Uniformly for $D \leqslant x^\alpha$, $0 < \alpha < 1$,*

$$\sum_{\substack{p+1=qD\\ p,q\leqslant x}} 1 \leqslant c_4(\alpha)\frac{x}{\varphi(D)(\log x)^2}.$$

See, for example, Prachar [7], Kap. II, Satz 4.6, p. 51.

To continue with our proof of Theorem 2 let $q_1 < q_2 < \ldots$ denote the odd primes q for which $f(q) \neq 0$. Let d be a positive integer. Let P

be a further positive integer, which we shall later choose to be in a certain sense 'large'. We shall obtain a lower bound for the number $T(x)$ of primes p, not exceeding x, for which

$$(8) \qquad p+1 = 2dk; \quad (k, 2d) = 1; \quad \forall i\ q_i \nmid k; \quad \forall q \leqslant P,\ q \text{ prime},\ q \nmid k; \quad \forall q > P,\ q^2 \nmid (p+1).$$

To do this choose a positive integer r, and define

$$Q = 2d \prod_{i=1}^{r} q_i \cdot \prod_{p \leqslant P} p.$$

Let α be a real number, $3/4 < \alpha < 1$. Then certainly

$$\begin{aligned} T(x) &\geqslant \sum_{\substack{p\leqslant x \\ p\equiv -1 \,(\mathrm{mod}\, 2d) \\ ((p+1)/2d,\, Q)=1}} 1 - \sum_{i>r} \sum_{\substack{p \leqslant x \\ p \equiv -1 \,(\mathrm{mod}\, 2dq_i)}} 1 - \sum_{q>P} \sum_{\substack{p\leqslant x \\ p\equiv -1 \,(\mathrm{mod}\, q^2)}} 1 \\ &= \sum_{\substack{p\leqslant x \\ p\equiv -1 \,(\mathrm{mod}\, 2d)}} \sum_{s \mid ((p+1)/2d,\, Q)} \mu(s) - \sum_{\substack{i>r \\ 2dq_i \leqslant x^{\alpha}}} \sum_{\substack{p\leqslant x \\ p\equiv -1 \,(\mathrm{mod}\, 2dq_i)}} 1 \\ &\quad - \sum_{\substack{i>r \\ 2dq_i > x^{\alpha}}} \sum_{\substack{p\leqslant x \\ p\equiv -1 \,(\mathrm{mod}\, 2dq_i)}} 1 - \sum_{q>P} \sum_{\substack{p\leqslant x \\ p\equiv -1 \,(\mathrm{mod}\, q^2)}} 1 \\ &= \Sigma_1 - \Sigma_2 - \Sigma_3 - \Sigma_4; \end{aligned}$$

say.

Using (7) we can derive the estimate (for fixed d):

$$\Sigma_1 = \frac{x}{\log x} \sum_{s \mid Q} \frac{\mu(s)}{\varphi(2ds)} + o\left(\frac{x}{\log x}\right) = \big(1+o(1)\big) \frac{x}{\log x} \cdot \frac{1}{2d} \prod_{\substack{q \nmid 2d \\ q \mid Q}} \left(\frac{q-2}{q-1}\right).$$

Using (8) we obtain the upper bound

$$\Sigma_2 \leqslant c_3(\alpha) \frac{x}{\log x} \sum_{i>r} \frac{1}{q_i - 1}.$$

Using (7) we can majorise our third sum by

$$\begin{aligned} \Sigma_3 &\leqslant \sum_{\substack{p\leqslant x,\, p+1 = 2dq_i m \\ 2dq_i > x^{\alpha}}} 1 \leqslant \sum_{m \leqslant x^{1-\alpha}} \sum_{\substack{p+1=(2dm)q \\ p,q \leqslant x}} 1 \\ &\leqslant \sum_{D \leqslant x^{2(1-\alpha)}} \sum_{\substack{p+1 = Dq \\ p,q\leqslant x}} 1 \ll \frac{x}{(\log x)^2} \sum_{D\leqslant x^{2(1-\alpha)}} \frac{1}{\varphi(D)} \ll (1-\alpha) \frac{x}{\log x}, \end{aligned}$$

the implied constant being absolute.

Lastly, using (8) once again

$$\begin{aligned} \Sigma_4 &\leqslant \sum_{P < q \leqslant x^{1/4}} \sum_{\substack{p \leqslant x \\ p\equiv -1 \,(\mathrm{mod}\, q^2)}} 1 + \sum_{q > x^{1/4}} \sum_{\substack{m \leqslant x+1 \\ m \equiv 0 \,(\mathrm{mod}\, q^2)}} 1 \\ &\ll \frac{x}{\log x} \sum_{q>P} \frac{1}{q^2} + \sum_{q > x^{1/4}} \frac{x}{q^2} \ll \frac{x}{P \log x} + O(x^{3/4}). \end{aligned}$$

Putting these estimates together we see that

$$\begin{aligned} \lambda &= \liminf_{x\to\infty} T(x)/x(\log x)^{-1} \\ &\geqslant \frac{1}{2d} \prod_{\substack{q \nmid 2d \\ q \mid Q}} \left(1 - \frac{1}{q-1}\right) - O\left(\sum_{i>r} \frac{1}{q_i - 1}\right) - O\big((1-\alpha)\big) - O(P^{-1}). \end{aligned}$$

If we let $r \to \infty$ and then $\alpha \to 1-$, we see that we can assert that

$$\begin{aligned} \lambda &\geqslant \frac{1}{2d} \prod_{3 \leqslant q \leqslant P} \left(1 - \frac{1}{q-1}\right) \cdot \prod_{i=1}^{\infty} \left(1 - \frac{1}{q_i - 1}\right) + O(P^{-1}) \\ &\geqslant c_5 (\log P)^{-1} + O(P^{-1}) > 0 \end{aligned}$$

if we choose P suitably large, but otherwise fixed.

Since we can find infinitely many primes p which satisfy all of the hypotheses stated in (8) we can find a sufficiently large one for which

$$f(2d) + f(k) = f(p+1) = \varrho \ \text{(constant)}$$

say. Here k is square free, and has no factor of the form q_i, so that $f(k) = 0$.

Choosing $d = 2^{\nu}$, $\nu = 0, 1, \ldots$ in turn, we see that $f(2^{\nu}) = $ constant. Next, choosing $d = q^t$, any odd prime power, we see that $f(q^t) = \varrho - f(2) = 0$.

This completes the proof of Theorem 2.

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Examples of Iwasawa invariants

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0. Let $E = Q(\sqrt{-d})$, $d > 0$, be a quadratic imaginary field of discriminant $-d$ and class number $h = h(-d)$ and let l be an odd rational prime, $(l, d) = 1$. There is a unique $\boldsymbol{Z}_l$-extension of E which is absolutely abelian. Let e_n, $n \geqslant 0$, denote the exact power of l dividing the class number of the nth-layer of the $\boldsymbol{Z}_l$-extension. Under the assumption (A) $l^{n+1} \equiv 1(d)$, the author has given the following formulas for $e_n - e_{n-1}$ ([1]). Let η be a primitive l^n-th root of unity and $\mathfrak{l} = (1-\eta)$ the prime ideal of $Q(\eta)$ lying over l. Let χ be the character of E; χ is a quadratic character of conductor d. Define $a(\tau) = \sum_{i=1}^{\tau-1} \chi(i)$. Let g be a primitive root modulo l^{n+1} and for all $s \geqslant 0$, $g(s) \equiv g^s(l^{n+1})$, $0 < g(s) < l^{n+1}$. For any $s \in \boldsymbol{Z}$, $r \in \boldsymbol{N}$, define s_r by $s_r \equiv s(l^r)$, $0 \leqslant s_r < l^r$. Then

$$e_n - e_{n-1} = \operatorname{ord}_{\mathfrak{l}}(\gamma); \quad \gamma = \sum_{s=0}^{\varphi(l^n)-1} \gamma_s \eta^s;$$

(1)

$$\gamma_s = \sum_{i=0}^{l-2} \Big(a\big(g(s+il^n)\big) - a\Big(g\big(\varphi(l^n)+il^n+s_{n-1}\big)\Big)\Big).$$

Hence the difference $e_n - e_{n-1}$ depends on the $\mathfrak{l}$-order of an algebraic integer in $Q(\eta)$ whose coefficients are certain sums in χ.

For sufficiently large n, $e_n = \mu l^n + \lambda n + c$ for fixed $\mu, \lambda \geqslant 0$, $c \in \boldsymbol{Z}$ ([4], [7]). These λ, μ are the Iwasawa invariants of the given $\boldsymbol{Z}_l$-extension. Our purpose here is to describe some computations of these invariants based on (1).

The contents of this note are as follows: in § 1 we show how to alter (1) in order to dispense with the restrictive assumption (A). In § 2 we show that, in the case $(-d/l) = -1$, a knowledge of e_i for small i often suffices for the determination of μ, λ. Some auxiliary results for $l = 3$ are given in § 3. A short description of the actual computations and the tabulated results are contained in § 4.

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