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## Forms representable by integral binary quadratic forms

by

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*In memory of the late Professor L. J. Mordell*

**1. Introduction.** By a *form* we shall mean an integral binary quadratic form in the indeterminates  $X$  and  $Y$ . This paper concerns the representability of a given form by forms of discriminant  $D$ , where  $D$  is the discriminant of a quadratic field (in fact, of the field  $Q(\sqrt{D})$ , where  $Q$  denotes the field of rationals).

If  $f(X, Y)$  is a given form, and  $g(X, Y)$  is a form of discriminant  $D$ , we say that  $f(X, Y)$  is *representable* by  $g(X, Y)$  if there exist rational integers  $p, q, r, s$  with  $ps - qr \neq 0$  such that

$$(1.1) \quad f(X, Y) = g(pX + qY, rX + sY).$$

If such integers exist, we call  $(p, q, r, s)$  a *representation* of  $f$  by  $g$ .

From (1.1), a necessary condition for the representability of  $f$  by some  $g$  of discriminant  $D$  is

$$\begin{aligned} \text{discrim}(f(X, Y)) &= \text{discrim}(g(pX + qY, rX + sY)) \\ &= (ps - qr)^2 \text{discrim}(g(X, Y)) = Dk^2. \end{aligned}$$

From now on, we fix  $D$  and assume that  $f(X, Y) = aX^2 + bXY + cY^2$  is a given form of discriminant  $Dk^2$ , where  $k$  is a positive integer. (Note that  $a$  and  $c$  must be non-zero.)

For those discriminants  $D$  given by

$$(1.2) \quad -D = 3, 4, 7, 8, 11, 19, 43, 67, 163$$

one of us [5], extending results of Mordell [2] (see also [3]) and Pall [4] (see also [6]), has determined necessary and sufficient conditions for a positive-definite form of discriminant  $Dk^2$  to be representable by a positive-definite form of discriminant  $D$ , as well as the number of such representations. We extend these results to all field discriminants  $D$ , replacing the use of unique factorization in the ring of integers of  $Q(\sqrt{D})$  by a rela-

\* Research supported under National Research Council of Canada grant A-7233.

tionship between certain ideals of this ring and representations of  $f(X, Y)$  by forms of discriminant  $D$ . The way in which these ideals arise is given in Lemma 2.2, and their use in counting representations is shown in Lemma 3.1. The main results of the paper are contained in § 4.

We conclude this introduction by giving some notation to be used throughout. Suppose  $I$  is an ideal of the ring of integers of  $Q(\sqrt{D})$ , with norm given by  $N(I) = |a|$  (recall  $f(X, Y) = aX^2 + bXY + cY^2$  is given). An ideal basis  $\{\alpha_1, \alpha_2\}$  of  $I$  is said to be  $a$ -ordered if

$$\alpha_1 \alpha_2' - \alpha_1' \alpha_2 = a\sqrt{D},$$

where the dash (') denotes quadratic conjugation. If  $\{\alpha_1, \alpha_2\}$  is such a basis for  $I$  we write  $I = [\alpha_1, \alpha_2]$ . For any ideal basis  $\{\alpha_1, \alpha_2\}$ , either  $I = [\alpha_1, \alpha_2]$  or  $I = [\alpha_2, \alpha_1]$  (not both). Moreover, applying strictly unimodular transformations to a given  $a$ -ordered basis yields all  $a$ -ordered bases of  $I$ . If  $I = [\alpha_1, \alpha_2]$ , we associate with this basis the form

$$(1.3) \quad \frac{1}{a} \{(\alpha_1 \alpha_1')X^2 + (\alpha_1 \alpha_2' + \alpha_1' \alpha_2)XY + (\alpha_2 \alpha_2')Y^2\},$$

which is integral because  $II' = (a)$ , where  $I'$  denotes the conjugate ideal of  $I$ , and which is of discriminant

$$\frac{(\alpha_1 \alpha_2' + \alpha_1' \alpha_2)^2 - 4\alpha_1 \alpha_1' \alpha_2 \alpha_2'}{a^2} = D.$$

We say that the  $a$ -ordered basis  $\{\alpha_1, \alpha_2\}$  of  $I$  leads to the form (1.3), and write

$$\{\alpha_1, \alpha_2\} \rightarrow \frac{1}{a} \{(\alpha_1 \alpha_1')X^2 + (\alpha_1 \alpha_2' + \alpha_1' \alpha_2)XY + (\alpha_2 \alpha_2')Y^2\}.$$

In general, given an ideal  $I$  of norm  $|a|$  and a form  $LX^2 + mXY + nY^2$  of discriminant  $D$ , we say that  $I$  leads to  $LX^2 + mXY + nY^2$  (written  $I \rightarrow LX^2 + mXY + nY^2$ ) if  $I$  possesses an  $a$ -ordered basis  $\{\alpha_1, \alpha_2\}$  such that  $\{\alpha_1, \alpha_2\} \rightarrow LX^2 + mXY + nY^2$ . It is easy to show that bases of strictly equivalent ideals of norm  $|a|$  lead to (properly) equivalent forms of discriminant  $D$ .

A representation  $(p, q, r, s)$  of  $f(X, Y)$  by a form  $g(X, Y)$  of discriminant  $D$  will be called *proper* if  $ps - qr = k$ , and *improper* if  $ps - qr = -k$ .

If  $(p, q, r, s)$  is a given proper representation of  $f$  by the form  $LX^2 + mXY + nY^2$ , we define a class of proper representations corresponding to  $(p, q, r, s)$  by

$$\mathcal{R}(p, q, r, s) = \left\{ (p', q', r', s') : \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix} = A \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \right.$$

where  $A$  is a proper automorph of  $LX^2 + mXY + nY^2$ .

Finally, let  $d = \text{GCD}(a, b, c)$ . It is also useful to define classes of representations of  $d$  by  $LX^2 + mXY + nY^2$ . If  $(u, v)$  is such a representation, with

$$d = lu^2 + mvv + nv^2,$$

we define its class by

$$\mathcal{C}(u, v) = \left\{ (u', v') : lu' + \frac{m - \sqrt{D}}{2} v' = \theta \left( lu + \frac{m - \sqrt{D}}{2} v \right), \right.$$

where  $\theta$  is a unit of the ring of integers of  $Q(\sqrt{D})$ .

**2. Preliminary lemmas.** Let  $f(X, Y) = aX^2 + bXY + cY^2$  be a form of discriminant  $Dk^2$ ,  $k > 0$ ,  $d = \text{GCD}(a, b, c)$ , and  $h = \frac{1}{2}(b - k\sqrt{D})$ , so that  $hh' = ac$ .

LEMMA 2.1. (i).  $h/d$  and  $h'/d$  are integers of  $Q(\sqrt{D})$ .

(ii). If  $K = (a, h)$ , then  $N(K) = N(K') = |a|d$ .

Proof. (i).  $h/d$  and  $h'/d$  lie in  $Q(\sqrt{D})$  and satisfy the monic integral quadratic equation

$$t^2 - \frac{b}{d}t + \frac{ac}{d^2} = 0.$$

(ii). Using (i), we have

$$KK' = (a^2, ah, ah', hh') = (ad) \left( \frac{a}{d}, \frac{h}{d}, \frac{h'}{d}, \frac{c}{d} \right) = (ad),$$

$$\text{as } \text{GCD} \left( \frac{a}{d}, \frac{b}{d}, \frac{c}{d} \right) = 1$$

implies the existence of rational integers  $r, s$ , and  $t$  satisfying

$$1 = r \frac{a}{d} + s \left( \frac{h}{d} + \frac{h'}{d} \right) + t \frac{c}{d}.$$

LEMMA 2.2. Suppose  $(p, q, r, s)$  is a proper representation of  $f$  by the form  $LX^2 + mXY + nY^2$  of discriminant  $D$ , that is,

$$(2.1) \quad aX^2 + bXY + cY^2 = l(pX + qY)^2 + m(pX + qY)(rX + sY) + n(rX + sY)^2$$

with  $ps - qr = k$ . Then

$$\alpha_1 = lp + \frac{r}{2}(m + \sqrt{D}) \quad \text{and} \quad \alpha_2 = nr + \frac{p}{2}(m - \sqrt{D})$$

are integers of  $Q(\sqrt{D})$  such that

(i) the ideal  $I = (\alpha_1, \alpha_2)$  has  $N(I) = |a|$ , and  $I = [\alpha_1, \alpha_2]$ ,

(ii)  $a = p\alpha_1 + r\alpha_2$ ,  $h = q\alpha_1 + s\alpha_2$  (so  $I \mid (a, h)$ ),

(iii)  $\{\alpha_1, \alpha_2\} \rightarrow LX^2 + mXY + nY^2$ .

Proof. It is trivial to verify that  $\alpha_1$  and  $\alpha_2$  are indeed integers of  $Q(\sqrt{D})$ .

(i) As  $D$  is a field discriminant, an integral basis for  $Q(\sqrt{D})$  is  $\{1, \omega\}$ , where  $\omega = (\varepsilon + \sqrt{D})/2$ , with

$$\varepsilon = \begin{cases} 0, & \text{if } D \equiv 0 \pmod{4}, \\ 1, & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

Moreover,  $m = 2m_1 + \varepsilon$  for some integer  $m_1$ . It is easily shown that for rational integers  $x_1, x_2, y_1, y_2$ ,

$$\begin{aligned} (x_1 + y_1\omega)\alpha_1 + (x_2 + y_2\omega)\alpha_2 \\ = (x_1 + (m_1 + \varepsilon)y_1 + ny_2)\alpha_1 + (x_2 - ly_1 - m_1y_2)\alpha_2. \end{aligned}$$

Furthermore, appealing to (2.1), we have

$$\begin{aligned} \alpha_1\alpha_2 - \alpha_1'\alpha_2' &= 2\left(\frac{p}{2}\left(lp + \frac{rm}{2}\right) + \frac{r}{2}\left(nr + \frac{pm}{2}\right)\right)\sqrt{D} \\ &= (lp^2 + mpr + nr^2)\sqrt{D} = a\sqrt{D}. \end{aligned}$$

Thus  $I$  has  $\{\alpha_1, \alpha_2\}$  as a basis,  $N(I) = |a|$ , and  $I = [\alpha_1, \alpha_2]$ . (ii) and (iii) follow by direct computation.

LEMMA 2.3. Let  $N$  denote the number of ideals  $I$  such that  $I|(a, h)$  and  $N(I) = |a|$ , and for a given form  $lX^2 + mXY + nY^2$  of discriminant  $D$ , let  $N_{(l,m,n)}$  denote the number of such  $I \rightarrow (l, m, n)$ . Then

$$N = \sum N_{(l,m,n)},$$

where the sum is taken over a representative set of inequivalent forms  $lX^2 + mXY + nY^2$  of discriminant  $D$ .

Proof. Let  $\{l_iX^2 + m_iXY + n_iY^2\}_{i=1, \dots, e}$  be a representative set of inequivalent forms of discriminant  $D$ . Let

$$\mathcal{I} = \{I: N(I) = |a|, I|(a, h)\}$$

and for each  $i$  ( $1 \leq i \leq e$ ) let

$$\mathcal{I}_i = \{I: N(I) = |a|, I|(a, h), I \rightarrow l_iX^2 + m_iXY + n_iY^2\}.$$

Let  $I = [\alpha_1, \alpha_2] \in \mathcal{I}$  be such that  $\{\alpha_1, \alpha_2\} \rightarrow lX^2 + mXY + nY^2$ . This form is equivalent to precisely one form

$$l_iX^2 + m_iXY + n_iY^2,$$

say,

$$\begin{aligned} l_iX^2 + m_iXY + n_iY^2 \\ = l(tX + uY)^2 + m(tX + uY)(vX + wY) + n(vX + wY)^2, \end{aligned}$$

where  $t, u, v, w$  are rational integers with  $tw - uv = 1$ . Then  $\{t\alpha_1 + u\alpha_2, v\alpha_1 + w\alpha_2\}$  is an  $a$ -ordered basis of  $I$  leading to

$$l_iX^2 + m_iXY + n_iY^2,$$

so that

$$I \rightarrow l_iX^2 + m_iXY + n_iY^2,$$

that is,  $I \in \mathcal{I}_i$ , and so

$$\mathcal{I} = \bigcup_{i=1}^e \mathcal{I}_i.$$

Moreover no ideal  $I \in \mathcal{I}$  can belong to two different  $\mathcal{I}_i$  as inequivalent forms arise from ideals which are not strictly equivalent, hence certainly distinct. Thus the  $\mathcal{I}_i$  are disjoint and we have

$$N = |\mathcal{I}| = \sum_{i=1}^e |\mathcal{I}_i| = \sum_{i=1}^e N(l_i, m_i, n_i).$$

LEMMA 2.4. There is a one-to-one correspondence between ideals  $I$  such that  $I|(a, h)$  and  $N(I) = |a|$ , and ideals  $J$  such that  $N(J) = d$ .

Proof. By Lemma 2.1 any such  $I$  defines a unique  $J$  with  $N(J) = d$  by the relation  $(a, h) = IJ$ . Conversely, if  $J$  is an ideal with  $N(J) = d$ , then  $JJ' = (d)$ , and thus  $J|(a, h)$ , since  $(d)|(a, h)$  by Lemma 2.1. Setting  $(a, h) = IJ$ , we see that  $J$  defines a unique  $I$  with the desired properties.

LEMMA 2.5. There is a one-to-one correspondence between classes of representations of  $d$  by a representative set of inequivalent forms of discriminant  $D$  and ideals  $J$  with  $N(J) = d$ .

Proof. This result follows from Theorem 5, p. 143 in [1].

**3. The main lemma.** The result of this section makes more precise the connection (from Lemma 2.2) between ideals, and proper representations of  $f(X, Y) = aX^2 + bXY + cY^2$  (of discriminant  $Dk^2$ ) by a given  $lX^2 + mXY + nY^2$  of discriminant  $D$ .

LEMMA 3.1. There is a one-to-one correspondence between ideals  $I$  satisfying

$$(3.1) \quad N(I) = |a|, \quad I|(a, h), \quad I \rightarrow lX^2 + mXY + nY^2,$$

and classes of proper representations of  $aX^2 + bXY + cY^2$  by  $lX^2 + mXY + nY^2$ .

Proof. Let  $I$  be an ideal satisfying (3.1), and let  $\{\alpha_1, \alpha_2\}$  be an  $a$ -ordered basis of  $I$  leading to  $lX^2 + mXY + nY^2$ . Now  $I|(a, h)$  means  $(a, h) \subseteq [a_1, a_2]$ , and so there exist unique rational integers  $p, q, r, s$  such that

$$(3.2) \quad a = p\alpha_1 + r\alpha_2, \quad h = q\alpha_1 + s\alpha_2.$$

We note that  $ps - qr \neq 0$ , for otherwise we have  $rh = as$  contradicting that  $k \neq 0$ . Moreover as  $\{a_1, a_2\} \rightarrow lX^2 + mXY + nY^2$  we have

$$a_1 a'_1 = la, \quad a_1 a'_2 + a'_1 a_2 = ma, \quad a_2 a'_2 = na,$$

and so

$$a = \frac{1}{a} (pa_1 + ra_2)(pa'_1 + ra'_2) = lp^2 + mpr + nr^2,$$

$$b = h + h' = \frac{1}{a} ((pa_1 + ra_2)(qa'_1 + sa'_2) + (pa'_1 + ra'_2)(qa_1 + sa_2))$$

$$= 2lpq + m(ps + qr) + 2nrs,$$

$$c = \frac{hh'}{a} = \frac{1}{a} (qa_1 + sa_2)(qa'_1 + sa'_2) = lq^2 + mqs + ns^2.$$

Therefore

$$aX^2 + bXY + cY^2 = l(pX + qY)^2 + m(pX + qY)(rX + sY) + n(rX + sY)^2.$$

Further, (3.2) implies

$$(ps - qr)a_1 = sa - rh,$$

$$(ps - qr)a_2 = -qa + ph,$$

and so

$$\begin{aligned} (ps - qr)^2 a \sqrt{D} &= (ps - qr)^2 (a_1 a'_2 - a'_1 a_2) \\ &= (sa - rh)(-qa + ph) - (sa - rh')(-qa + ph) \\ &= a(ps - qr)(h' - h) = a(ps - qr)k\sqrt{D}, \end{aligned}$$

that is,

$$ps - qr = k.$$

Thus the  $a$ -ordered basis  $\{a_1, a_2\}$  for  $I$  gives rise to a proper representation of  $f$  by  $lX^2 + mXY + nY^2$ .

Suppose that  $\{\beta_1, \beta_2\} \rightarrow lX^2 + mXY + nY^2$ . This is related to  $\{a_1, a_2\}$  by  $\beta_1 = ta_1 + va_2$ ,  $\beta_2 = ua_1 + wa_2$ , where  $t, u, v$  and  $w$  are rational integers with  $tw - uv = 1$ . As these two bases lead to the same form, we have

$$\begin{aligned} &\frac{1}{a} \{(a_1 a'_1)X^2 + (a_1 a'_2 + a'_1 a_2)XY + (a_2 a'_2)Y^2\} \\ &= \frac{1}{a} \{(\beta_1 \beta'_1)X^2 + (\beta_1 \beta'_2 + \beta'_1 \beta_2)XY + (\beta_2 \beta'_2)Y^2\} \\ &= \frac{1}{a} \{(a_1 a'_1)(tX + uY)^2 + (a_1 a'_2 + a'_1 a_2)(tX + uY)(vX + wY) + \\ &\quad + (a_2 a'_2)(vX + wY)^2\}, \end{aligned}$$

that is \*

$$\begin{pmatrix} t & u \\ v & w \end{pmatrix}$$

is a proper automorph of the form  $lX^2 + mXY + nY^2$ .

Corresponding to (3.2) we have  $a = p'\beta_1 + r'\beta_2$ ,  $h = q'\beta_1 + s'\beta_2$ , and (as indicated on previous page) this means that  $\{\beta_1, \beta_2\}$  gives rise to the proper representation  $(p', q', r', s')$  of  $f$  by  $lX^2 + mXY + nY^2$ . Moreover, we have

$$(a \ h) = (a_1 \ a_2) \begin{pmatrix} p & q \\ r & s \end{pmatrix} = (\beta_1 \ \beta_2) \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix} = (a_1 \ a_2) \begin{pmatrix} t & u \\ v & w \end{pmatrix} \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}$$

and since  $\{a_1, a_2\}$  is a basis

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} t & u \\ v & w \end{pmatrix} \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}$$

that is  $\mathcal{R}(p', q', r', s') = \mathcal{R}(p, q, r, s)$ .

Thus, each ideal  $I$  satisfying (3.1) gives rise to proper representations  $(p, q, r, s)$  of  $f$  by  $lX^2 + mXY + nY^2$ , defined by (3.2), where  $\{a_1, a_2\}$  is any  $a$ -ordered basis of  $I$  which leads to  $lX^2 + mXY + nY^2$ ; moreover, two representations arising in this way lie in the same class. The mapping  $\varphi$  given by  $\varphi(I) = \mathcal{R}(p, q, r, s)$  is thus a well-defined function from the set of ideals  $I$  satisfying (3.1) to the set of classes of proper representations of  $f$  by  $lX^2 + mXY + nY^2$ .

The function  $\varphi$  is surjective, for if  $(p, q, r, s)$  is any proper representation of  $f$  by  $lX^2 + mXY + nY^2$ , then, letting  $I$  be defined as in Lemma 2.2, we have  $\varphi(I) = \mathcal{R}(p, q, r, s)$ .

Finally, we show that  $\varphi$  is injective. Suppose  $\varphi(I) = \varphi(J)$ , where  $I$  and  $J$  are ideals satisfying (3.1). Let  $I = [a_1, a_2]$  and  $J = [\beta_1, \beta_2]$  give rise by (3.2) to the proper representations  $(p, q, r, s)$ ,  $(p', q', r', s')$ , respectively. As  $\varphi(I) = \varphi(J)$ ,

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} t & u \\ v & w \end{pmatrix} \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}$$

for some automorph

$$\begin{pmatrix} t & u \\ v & w \end{pmatrix}$$

of  $lX^2 + mXY + nY^2$ . Therefore we have

$$(a \ h) = (a_1 \ a_2) \begin{pmatrix} p & q \\ r & s \end{pmatrix} = (a_1 \ a_2) \begin{pmatrix} t & u \\ v & w \end{pmatrix} \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix} = (\beta_1 \ \beta_2) \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}$$

and so

$$(\beta_1 \beta_2) = (a_1 \ a_2) \begin{pmatrix} t & u \\ v & w \end{pmatrix}.$$

That is, the two bases are related by a strictly unimodular transformation, and therefore must be bases of the same ideal.

**4. Number of representations.** We are now in a position to prove the main results of this paper.

**THEOREM 4.1.** *The number of classes of proper representations of the form  $aX^2 + bXY + cY^2$  of discriminant  $Dk^2$  by a representative set of inequivalent forms of discriminant  $D$  is equal to the number of classes of representations of  $d = \text{GCD}(a, b, c)$  by a representative set of inequivalent forms of discriminant  $D$ .*

**Proof.** By a fixed form  $lX^2 + mXY + nY^2$  of discriminant  $D$ , the number of classes of proper representations of  $aX^2 + bXY + cY^2$  is, by Lemma 3.1, equal to the number of ideals  $I$  satisfying (3.1). Letting  $lX^2 + mXY + nY^2$  run over a representative set of inequivalent forms and applying Lemma 2.3, we see that the number of classes of proper representations of  $aX^2 + bXY + cY^2$  by a representative set of inequivalent forms of discriminant  $D$  is equal to the number of ideals  $I$  such that  $I|(a, h)$  and  $N(I) = |a|$ . By Lemmas 2.4 and 2.5, this is equal to the number of classes of representations of  $d$  by a representative set of inequivalent forms of discriminant  $D$ .

**COROLLARY 4.1.** *Let  $f(X, Y) = aX^2 + bXY + cY^2$  be a form of discriminant  $Dk^2$ , and let  $d = \text{GCD}(a, b, c)$ . The form  $f(X, Y)$  is properly representable by some form of discriminant  $D$  if and only if  $d$  is representable by a form of discriminant  $D$ .*

**COROLLARY 4.2.** (Cf. [5], Theorem 2.) *Let  $f(X, Y) = aX^2 + bXY + cY^2$  be a form of discriminant  $Dk^2$ , with  $D < 0$ . The number of proper representations of  $f(X, Y)$  by a representative set of forms of discriminant  $D$  is equal to  $r_D(d)$ , the number of representations of  $d = \text{GCD}(a, b, c)$  by a representative set of inequivalent forms of discriminant  $D$ .*

**Proof.** The result follows immediately from Theorem 4.1, since each form of discriminant  $D$  has precisely  $w_D$  automorphs, where  $w_D$  denotes the number of units in the ring of integers of  $Q(\sqrt{D})$ . Thus each class of proper representations of  $f$  by a given form (and also each class of representations of  $d$  by a given form) has  $w_D$  members.

**Remark.** The preceding results hold, with slight modifications in the proofs, for improper representations of  $f(X, Y)$  by forms of discriminant  $D$ . Therefore, a doubling occurs when all (that is both proper and improper) representations of  $f(X, Y)$  are counted.

Added in proof. The authors have subsequently obtained generalizations of these results; see Duke Mathematical Journal 40 (1973), pp. 533-539.

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Received on 7. 11. 1972

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