have representations which are Type A rational fractions. In particular, we show that the partial infinite product representation for $\pi/4$ with n sufficiently large is Type A and, consequently, we obtain results concerning the Brouwer conjecture that we discussed in [5, pp. 234–235].

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Remark to a theorem of P. Erdös

by

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Let $f(n)$ be a real-valued additive arithemtic function, that is,

$$f(nm) = f(n)+f(m) \quad \text{for} \quad (n, m) = 1.$$

Put

$$f^*(n) = \begin{cases} f(n) & \text{for} \quad |f(n)| \leqslant 1, \\ 0 & \text{for} \quad |f(n)| > 1. \end{cases}$$

A remarkable theorem of P. Erdös [1] states, that if

(1) $$\sum_p \frac{f^*(p)}{p} \text{ converges,}$$

(2) $$\sum_p \frac{f^*(p)^2}{p} < \infty$$

and

(3) $$\sum_{|f(p)|>1} \frac{1}{p} < \infty$$

then the distribution-function of $f(n)$ exists, that is, the limit

(4) $$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{k\leqslant N \\ f(k)\leqslant x}} 1 = G(x)$$

exists for every real x. Further he showed that if the additional condition

(5) $$\sum_{f(p)\neq 0} \frac{1}{p} = \infty$$

holds, then $G(x)$ is continuous; if

(6) $$\sum_{f(p)\neq 0} \frac{1}{p} < \infty$$

then $G(x)$ is a discrete distribution.

P. Erdös and A. Wintner [2] showed that the conditions (1), (2) and (3) are also necessary for the existence of the limit (4). Combining methods of Probability and Analytic Number Theory, A. Rényi [4] gave a new proof for the theorem of Erdös, that is, for the sufficiency of (1)-(3) in order (4) should hold. Previously Schoenberg [6] proved a weaker form of Erdös' theorem: instead of (1) he needed the stronger restriction

$$(1') \qquad \sum_p \frac{|f^*(p)|}{p} < \infty$$

under this supposition he also came to the conclusions about (5) and (6).

In the present paper I give another proof for the fact that if (1), (2), (3) and (5) hold, then the limit-function $G(x)$ of (4) is continuous. The proof is entirely different from the proof of Erdös or that of Schoenberg of his weaker statement and seems to be shorter and simpler.

1. Proof of Erdös' theorem. In the sequel $h(t)$ denotes the characteristic function of the distribution function $H(x)$, that is

$$h(t) = \int_{-\infty}^{\infty} e^{itx} dH(x).$$

LEMMA 1.1. *Denote*

$$G_N(x) = \frac{1}{N} \sum_{\substack{k \leq N \\ f(k) \leq x}} 1.$$

Then we have

$$(1.1) \qquad g_N(t) = \prod_p \left(1 + \frac{e^{itf(p)} - 1}{p} + \frac{e^{itf(p^2)} - e^{itf(p)}}{p^2} + \dots\right) + o(1)$$

as $N \to \infty$. The infinite product on the right-hand side is convergent.

Proof. See Rényi [4].

LEMMA 1.2. *We have*

$$(1.2) \qquad \lim_{T\to\infty} \frac{1}{T} \int_0^T |h(t)|^2 dt = \sum_{k=1}^{\infty} d_k^2,$$

where $d_1, d_2, \dots,$ runs over all saltusses of $H(x)$.

Proof. See for instance M. Kac [3], p. 45.

Now we are in position to finish our proof. Put

$$(1.3) \qquad g(t) = \prod_p \left(1 + \frac{e^{itf(p)} - 1}{p} + \dots\right).$$

Since $g_N(t) \to g(t)$ we have $G_N(x) \to G(x)$ at any point of continuity of $G(x)$. To show that $G(x)$ is everywhere continuous, we have only to show (Lemma 1.2) that

$$\frac{1}{T} \int_0^T |g(t)|^2 dt \to 0 \quad \text{as} \quad T \to \infty$$

or

$$(1.4) \qquad \int_0^1 |g(Ty)|^2 dy \to 0 \quad \text{as} \quad T \to \infty.$$

We have

$$|g(Ty)|^2 = \prod_p \left(1 + \frac{e^{itf(p)Ty} - 1}{p} + \dots\right)\left(1 + \frac{e^{-itf(p)Ty} - 1}{p} + \dots\right)$$
$$= \prod_p \left(1 + \frac{2(\cos f(p)Ty - 1)}{p} + O\left(\frac{1}{p^2}\right)\right),$$

that is,

$$(1.5) \qquad \int_0^1 |g(Ty)|^2 dy < K_1 \int_0^1 \exp\left(2 \sum_{\substack{p \leq N \\ f(p) \neq 0}} \frac{\cos f(p)Ty - 1}{p}\right) dy$$

where N is a fixed "large" number.

First suppose that for $f(p) \neq 0$, $f(q) \neq 0$ we have $f(p) \neq f(q)$. Then we have for any given N

$$(1.6) \qquad \int_0^1 \left(\sum_{\substack{p \leq N \\ f(p) \neq 0}} \frac{\cos f(p)Ty}{p}\right)^2 dy = O(1) \quad \text{as} \quad T \to \infty.$$

Therefore by the well-known Chebyshev inequality (see for instance A. Rényi [5], p. 373), we have(1)

$$(1.7) \qquad P\left(\left|\sum_{\substack{p \leq N \\ f(p) \neq 0}} \frac{\cos f(p)Ty}{p}\right| > \frac{1}{2} \sum_{\substack{p \leq N \\ f(p) \neq 0}} \frac{1}{p}\right) < \frac{K_2}{\left(\sum_{\substack{p \leq N \\ f(p) \neq 0}} 1/p\right)^2},$$

which can be made arbitrarily small by taking N large enough, because of (5); therefore we have because of (1.5) and (1.7)

$$(1.8) \qquad \int_0^1 |g(Ty)|^2 dy \leq K_1\left(\exp\left(-\sum_{\substack{p \leq N \\ f(p) \neq 0}} \frac{1}{p}\right) + K_2\left(\sum_{\substack{p \leq N \\ f(p) \neq 0}} \frac{1}{p}\right)^{-2}\right)$$

(1) $P(\dots)$ means the Lebesgue-measure of the set in y in the parenthesis.

which can be made arbitrarily small by choosing N large enough. This proves Erdös' theorem for $f(p) \neq f(q)$ $(f(p) \neq 0,\ f(q) \neq 0)$. If for some sequence $f(p_1) = f(p_2) = \ldots$, then, considering the expression

$$\sum_{f(p)=g_l} (\cos g_l Ty - 1) \sum_{f(p)=g_l} \frac{1}{p}$$

instead of

$$\sum \frac{\cos f(p) Ty - 1}{p}$$

one can repeat the argument above and our statement follows again.

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Some remarks on the decomposition of a rational prime in a Galois extension

by

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1. Introduction. Not much is known about the law of decomposition of rational primes in a Galois extension if the extension is not abelian. It is known that only for abelian extensions we can give a simple law of decomposition depending on the residue of the given prime with respect to a certain modulus. The object of the present paper is to get some information about the relationship between the number of prime divisors of a given rational prime and a rational prime which is ramified in a Galois extension. This information also helps us to get some idea about the class numbers of certain algebraic number fields. For example, the well-known result that the class number of the field $Q(\sqrt[r]{a})$ (r odd prime and a is divisible by a prime of the form $rt+1$) is divisible by r could be deduced from our result.

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2. Notations and preliminaries. Throughout this paper, Q denotes the rational number field, k denotes a finite Galois extension of Q with Galois group G and $\mathcal{O}_k$ denotes the ring of integers of k. The prime ideals of $\mathcal{O}_k$ are called *k-primes*. p and q denote distinct rational primes and $\mathfrak{P}$ and $\mathfrak{Q}$ denote the k-primes lying above p and q respectively. g_l denotes the number of distinct k-primes $\mathfrak{L}$ lying above the rational prime l. e_l and f_l denote the ramification index and residue class degree respectively of $\mathfrak{L}$. $G_{\mathfrak{L}}$ and $T_{\mathfrak{L}}$ denote the decomposition group and inertia group of $\mathfrak{L}$. They are subgroups of G of order $e_l f_l$ and e_l respectively. $T_{\mathfrak{L}}$ is a subgroup of $G_{\mathfrak{L}}$ and its elements induce the trivial automorphism on the residue class field of $\mathfrak{L}$. g_l will be the number of cosets of $G_{\mathfrak{L}}$ in G. Let $G = \bigcup_{j=1}^{g_l} \tau_j G_{\mathfrak{L}}$ be a coset decomposition of $G_{\mathfrak{L}}$ in G. Then the k-primes $\tau_j \mathfrak{L}$ are precisely the distinct k-primes lying above l.