and thus, by (13), we have that the sum of probabilities in (12) converges, hence Lemma 1 completes the proof.

The Corollary is a straight consequence of the Theorem. We stated it separately because of its interesting content.

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# On gaps between numbers with a large prime factor, II

by

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**1.** In [2] the following result was proved:

THEOREM 1. *Let $n>1$ be an integer. Let $\alpha_1, \ldots, \alpha_n$ be rational numbers such that*

(i) $\alpha_1>0, \ldots, \alpha_n>0$ *are multiplicatively independent,*

(ii) $|\log\alpha_i| \leqslant \exp\left(-\frac{1}{A}\log S_1\right)$, $1\leqslant i\leqslant n$ *and* $A>1$,

(iii) *The sizes of $\alpha_1, \ldots, \alpha_n$ do not exceed $S_1$. (The size of a rational number $a/b$, $(a, b)=1$, is defined as $|b|+|a/b|$.)*

*If $\beta_1, \ldots, \beta_{n-1}$ are rational numbers of size not exceeding $S_1$, then*

$$|\beta_1\log\alpha_1+\ldots+\beta_{n-1}\log\alpha_{n-1}-\log\alpha_n| > \exp\left(-(nA)^{cn^2}\log S_1\right)$$

*where $c>0$ is an effectively computable constant which is independent of $n$, $A$ and $S_1$.*

In this paper we shall prove the following:

THEOREM 2. *Let $n>1$ be an integer. Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{n-1}$ be rational numbers satisfying the assumptions of Theorem 1. Further assume that*

(iv) $\alpha_1=\frac{m}{m'}, \alpha_2=\frac{p_2}{p_2'}, \ldots, \alpha_n=\frac{p_n}{p_n'}$ *where $p_2, \ldots, p_n, p_2', \ldots, p_n'$ are pairwise distinct prime numbers and none of them is either a factor of $m$ or $m'$.*

*Then*

$$|\beta_1\log\alpha_1+\ldots+\beta_{n-1}\log\alpha_{n-1}-\log\alpha_n| > \exp\left(-(nA)^{c_1n}\log S_1\right)$$

*where $c_1>0$ is an effectively computable constant which is independent of $n$, $A$ and $S_1$.*

* I am very thankful to Professor H. M. Stark for sending me a preprint of his unpublished result [5]. My thanks are also due to Professor K. Ramachandra for going through the manuscript.

If $n$, $A$ are large and $a_1, \ldots, a_n$ satisfy the assumption (iv) of Theorem 2, then Theorem 2 is an improvement of Theorem 1. The improvement depends on some of the ideas of Stark [5].

If Theorem 1 is replaced by Theorem 2 in [2], the method of [2] shows that the following result can be obtained in view of Jutila's result [1] that the greatest prime factor of $(u+1)\ldots(u+k)$ exceeds $k(\log k)^2$ provided that $k^{3/2} \leqslant u \leqslant \exp\big((\log k)^{5/4}\big)$ and $k$ exceeds a certain absolute constant.

THEOREM 3. *Let $k$ be a fixed natural number and let $n_1, n_2, \ldots$ be all the natural numbers (in the increasing order) which have at least one prime factor exceeding $k$. Define*

$$f(k) = \max_{i=1,2,\ldots} (n_{i+1}-n_i).$$

*Then*

$$f(k) = O\left(\frac{k}{\log k}\left(\frac{\log\log\log k}{\log\log k}\right)\right).$$

This bound for $f(k)$ is sharper than that of [2], namely,

$$f(k) = O\left(\frac{k}{\log k}\left(\frac{\log\log\log\log k}{\log\log\log k}\right)^{1/2}\right).$$

We remark that the multiplicative independence of $a_1, \ldots, a_n$ follows from the assumption (iv) imposed on $a_1, \ldots, a_n$ in Theorem 2. Further (iv) can be somewhat relaxed (see Remark after the proof of Theorem 2). One would like to have $(nA)^{c_1}\log S_1$ in place of $(nA)^{c_1 n}\log S_1$ in Theorem 2. This would improve the bound for $f(k)$ to $k(\log k)^{-1-\delta}$, where $\delta > 0$ is a small constant.

**2. Proof of Theorem 2.** Unless otherwise specified, we shall follow the notations of [2] in this paper. The definition of $\tilde{r}_1$ in [2] (see after inequality (12)) is changed as follows:

$$\tilde{r}_1 = \left[\frac{E_1 n}{b}\right]+2,$$

where $E_1$ is a positive constant to be suitably chosen. Allow the large constant $c_1$ (occurring in the definition of $h$ in [2]) to depend on $E_1$ also. Assume that

$$\beta \leqslant \tfrac{1}{2} n^{-k}(6h_{\tilde{r}_1})^{-2h_{\tilde{r}_1}k/n^2} S_1^{-2h_{\tilde{r}_1}k/An^2}. \tag{1}$$

See that the inequalities (13) of [2] are satisfied. Proceed exactly as in [2] to conclude that

$$q(l, m_1, \ldots, m_{n-1}) = 0$$



for all integers $l$, $1 \leqslant l \leqslant h_{\tilde{r}_1}$ and for all non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1+\ldots+m_{n-1} \leqslant k_{\tilde{r}_1}$. Define

$$k_{\tilde{r}_1+m+1} = [B^{-1}k_{\tilde{r}_1+m}], \qquad 0 \leqslant m \leqslant M-1.$$

(We shall choose $M$ in such a way that $k_{\tilde{r}_1+M} > 10$.)

We shall divide the (remaining) proof of Theorem 2 in three lemmas.

LEMMA 1. *Assume that $\beta$ satisfies (1). Then for any rational number $a/p$, $0 \leqslant a/p \leqslant h$ with $0 < p \leqslant h$ and non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1+\ldots+m_{n-1} \leqslant k_{\tilde{r}_1+1}$, we have*

$$q(a/p, m_1, \ldots, m_{n-1}) = 0.$$

Proof. Put

$$f(z) = \Phi_{m_1,\ldots,m_{n-1}}(z, \ldots, z),$$

with $m_1+\ldots+m_{n-1} \leqslant k_{\tilde{r}_1+1}$ and $m_i \geqslant 0$ $(1 \leqslant i \leqslant n)$. For every $z$ with $|z| = 2h_{\tilde{r}_1}$, we have the interpolation formula:

$$\frac{1}{2\pi i}\int_{\Gamma:|\zeta|=\Delta} \frac{f(\zeta)F(z)}{(\zeta-z)F(\zeta)}\,d\zeta = f(z)+\sum_{r=1}^{h_{\tilde{r}_1}}\ \sum_{m=0}^{k_{\tilde{r}_1}-k_{\tilde{r}_1+1}} \frac{f^{(m)}(r)}{m!\,2\pi i}\int_{\Gamma_r:|\zeta-r|=1/2} \frac{(\zeta-r)^m F(z)}{(\zeta-z)F(\zeta)}\,d\zeta$$

where

$$F(\zeta) = \prod_{u=1}^{h_{\tilde{r}_1}} (z-u)^{k_{\tilde{r}_1}-k_{\tilde{r}_1+1}+1}, \qquad \Delta = 5h_{\tilde{r}_1}\exp\left(\frac{1}{A}\log S_1\right).$$

This interpolation formula gives that for every $z$, $|z| = 2h_{\tilde{r}_1}$,

$$|f(z)| \leqslant w\Big(S_1^{6nLh_{\tilde{r}_1}}(2S_1L)^{7k}\exp\Big(-\frac{h_{\tilde{r}_1}k_{\tilde{r}_1}}{An^2}\log S_1\Big)+ \\ +\beta n^k S_1^{8nLh_{\tilde{r}_1}}(2S_1L)^{8k}(6h_{\tilde{r}_1})^{2h_{\tilde{r}_1}k_{\tilde{r}_1}/n^2}\Big). \tag{2}$$

(For this, one can refer to the similar details following formula (7) of [2].)

Hence by maximum-modulus principle, (2) holds for all rational numbers $z = a/p$ with $0 \leqslant a/p \leqslant h$. Assume that $q(a/p, m_1, \ldots, m_{n-1}) \neq 0$ for some rational number $a/p$ with $0 \leqslant a/p \leqslant h$, $0 < p \leqslant h$ and for some non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1+\ldots+m_{n-1} \leqslant k_{\tilde{r}_1+1}$. Notice that

$$\left|(\log a_1)^{-m_1}\ldots(\log a_{n-1})^{-m_{n-1}}\Phi_{m_1,\ldots,m_{n-1}}\left(\frac{a}{p}, \ldots, \frac{a}{p}\right) - q\left(\frac{a}{p}, m_1, \ldots, m_{n-1}\right)\right| \\ \leqslant (L+1)^n S_1^{4nLh}(2S_1L)^{5k} S_1^{nLh}(2S_1L)^k 2Lh\beta \leqslant \beta S_1^{6nLh}(2S_1L)^{6k}.$$

Since $p \leqslant h$, $q(a/p, m_1, \ldots, m_{n-1})$ is a non-zero algebraic number of degree $\leqslant h^n$. The absolute value of each of the conjugates of $q(a/p, m_1, \ldots, m_{n-1})$ does not exceed

$$S_1^{6nLh}(2S_1L)^{6k}.$$

The denominator of $q(a/p, m_1, \ldots, m_{n-1})$ does not exceed

$$S_1^{nLh^2}(2S_1L)^{k}.$$

Hence

$$\text{(3)}\quad \left|\Phi_{m_1,\ldots,m_{n-1}}\left(\frac{a}{p}, m_1, \ldots, m_{n-1}\right)\right| \geqslant w\left(S_1^{-7nLh^{n+2}}(2S_1L)^{-7kh^n} - \beta S_1^{6nLh}(2S_1L)^{6k}\right).$$

The contradiction is obtained by showing that (2) and (3) are inconsistent. For this it is enough to have

$$\exp\left(\frac{h_{\tilde{r}_1}k_{\tilde{r}_1}}{An^2}\log S_1\right) > S_1^{15nLh^{n+2}+15nLh_{\tilde{r}_1}}(2S_1L)^{15kh^n}\left\{1+\beta n^k(6h_{\tilde{r}_1})^{2h_{\tilde{r}_1}k_{\tilde{r}_1}/n^2}\exp\left(\frac{h_{\tilde{r}_1}k_{\tilde{r}_1}}{An^2}\log S_1\right)\right\}.$$

Since $\beta$ satisfies (1), it is sufficient to show that

$$\exp\left(\frac{h_{\tilde{r}_1}k_{\tilde{r}_1}}{An^2}\log S_1\right) > S_1^{16nLh^{n+2}+15nLh_{\tilde{r}_1}}(2S_1L)^{15kh^n}.$$

This can easily be established. For proof, one can refer to similar details just after inequality (13) of [2]. This completes the proof of Lemma 1.

Remark. Let $p'(\lambda_1, \ldots, \lambda_n)$, $0 \leqslant \lambda_i \leqslant L$ $(i = 1, \ldots, n)$ be integers satisfying

$$|p'(\lambda_1, \ldots, \lambda_n)| \leqslant S_1^{4nLh}(2S_1L)^{5k}.$$

(See inequality next to (3) of [2]). Consider

$$q' = q'(z, m_1, \ldots, m_{n-1}) = \sum_{\lambda_1=0}^{L}\cdots\sum_{\lambda_n=0}^{L} p'(\lambda_1, \ldots, \lambda_n)\,\alpha_1^{\lambda_1 z}\ldots\alpha_n^{\lambda_n z}\gamma_1^{m_1}\ldots\gamma_{n-1}^{m_{n-1}}.$$

Suppose that $q'(l, m_1, \ldots, m_{n-1}) = 0$ for all integers $l$, $1 \leqslant l \leqslant h$, and for all non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1+\ldots+m_{n-1} \leqslant k$. If $\beta$ satisfies (1), then our argument shows that

$$q'(a/p, m_1, \ldots, m_{n-1}) = 0$$

for all rational numbers $a/p$, $0 \leqslant a/p \leqslant h$ with $p \leqslant h$ and all non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1+\ldots+m_{n-1} \leqslant k_{\tilde{r}_1+1}$.



We shall call that $q'$ is associated to $p'(\lambda_1, \ldots, \lambda_n)$.

LEMMA 2. *Assume that $\beta$ satisfies* (1). *Let $p'(\lambda_1, \ldots, \lambda_n)$ be integers, $0 \leqslant \lambda_i \leqslant L$ $(i = 1, \ldots, n)$, satisfying*

$$|p'(\lambda_1, \ldots, \lambda_n)| \leqslant S_1^{4nLh}(2S_1L)^{5k}.$$

*Let $q' = q'(z, m_1, \ldots, m_{n-1})$ be associated to $p'(\lambda_1, \ldots, \lambda_n)$. Assume that for $x_0$, $0 < x_0 \leqslant 1$, we have*

$$q'(x_0+l, m_1, \ldots, m_{n-1}) = 0$$

*for all integers $0 \leqslant l \leqslant h-1$ and all non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1+\ldots+m_{n-1} \leqslant k$ $(= k_1)$. Then*

$$q'(a/p, m_1, \ldots, m_{n-1}) = 0$$

*for all rational numbers $a/p$, $0 \leqslant a/p \leqslant h$, $0 < p \leqslant h$ and all non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1+\ldots+m_{n-1} \leqslant k_{\tilde{r}_1+2}$.*

Proof. It is sufficient to prove that

$$q'(l, m_1, \ldots, m_{n-1}) = 0$$

for all integers $l$, $1 \leqslant l \leqslant h$, and non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1+\ldots+m_{n-1} \leqslant k_2$. (Then the lemma would follow by the above remark.) Define

$$f(z) = \Phi'_{m_1,\ldots,m_{n-1}}(z, \ldots, z), \qquad m_1+\ldots+m_{n-1} \leqslant k_2$$

where

$$\Phi'(z_1, \ldots, z_{n-1}) = \sum_{\lambda_1=0}^{L}\cdots\sum_{\lambda_n=0}^{L} p'(\lambda_1, \ldots, \lambda_n)\,\alpha_1^{\gamma_1 z_1}\ldots\alpha_{n-1}^{\gamma_{n-1} z_{n-1}}.$$

For $z$ with $|z| = 2h$, we have

$$\frac{1}{2\pi i}\int_{\Gamma:|\zeta|=\Delta}\frac{f(\zeta)F(z)}{(\zeta-z)F(\zeta)}\,d\zeta = f(z)+\sum_{r=0}^{h-1}\sum_{m=0}^{k_1-k_2}\frac{f^{(m)}(r+x_0)}{m!\,2\pi i}\int_{\Gamma_r}\frac{(\zeta-r-x_0)^m F(z)}{(\zeta-z)F(\zeta)}\,d\zeta$$

where $\Gamma_r$ denotes the circle with centre $(r+x_0)$ and radius $\frac{1}{2}$,

$$F(\zeta) = \prod_{u=0}^{h-1}(\zeta-u-x_0)^{k_1-k_2+1} \quad \text{and} \quad \Delta = 5h\exp\left(\frac{1}{A}\log S_1\right).$$

For every $z$, $|z| = 2h$, the above formula gives

$$\text{(4)}\quad |f(z)| \leqslant w\left(S_1^{6nLh_1}(2S_1L)^{7k}\exp\left(-\frac{h_1k_1}{An^2}\log S_1\right)+\beta n^k S_1^{8nLh_1}(2S_1L)^{8k}(6h_1)^{2h_1k_1/n^2}\right).$$

Hence by maximum-modulus principle, (4) holds for $z = l$ with $1 \leqslant l \leqslant h$. Further for every $l$, $1 \leqslant l \leqslant h_1$,

$$|f(l)| \geqslant w\left(S_1^{-nLh_1}(2S_1L)^{-2k} - \beta S_1^{7nLh_1}(2S_1L)^{7k}\right) \tag{5}$$

provided that

$$q'(l, m_1, \ldots, m_{n-1}) \neq 0.$$

(See inequality (6) of [2] with $S = S_1$.)

(4) and (5) are inconsistent, if

$$\exp\left(\frac{h_1k_1}{An^2}\log S_1\right) > S_1^{9nLh_1}(2S_1L)^{10k}\{1+2\beta n^k(2h_1)^{2h_1k_1/n^2}S_1^{2h_1k_1/An^2}\}.$$

This inequality is secured in [2] (see the inequality that appears after (12) in [2]).

Hence Lemma 2 is proved.

LEMMA 3. *Assume that $\beta$ satisfies* (1). *Then there exists a prime $p$, $h^{1/2} < p \leqslant h$, with the following property: There exist integers $j_1, \ldots, j_n$, with $0 \leqslant j_i < p$ $(1 \leqslant i < n)$ and $j_n = 1$, such that*

$$\alpha_1^{j_1} \ldots \alpha_n^{j_n} = \eta^p,$$

*where $\eta$ is rational.*

The proof of Lemma 3 depends on the following:

LEMMA 3′. *Let $\alpha_1, \ldots, \alpha_n$ be non-zero elements of an algebraic number field $K$ and let $\alpha_1^{1/p}, \ldots, \alpha_n^{1/p}$ denote fixed $p$-th roots for some prime $p$. Further let $K' = K(\alpha_1^{1/p}, \ldots, \alpha_{n-1}^{1/p})$. Then either $K'(\alpha_n^{1/p})$ is an extension of $K'$ of degree $p$ or we have*

$$\alpha_1^{j_1} \ldots \alpha_n^{j_n} = \eta^p$$

*for some $\eta$ in $K$ and some integers $j_1, \ldots, j_n$ with $0 \leqslant j_i < p$ $(1 \leqslant i < n)$ and $j_n = 1$.*

This is Lemma 5 of [3].

*Proof of Lemma* 3. Let $p$, $h^{1/2} < p \leqslant h$, be a prime for which the lemma is not true. Then by Lemma 3′, $K'(\alpha_n^{1/p})$ is an extension of $K' = \boldsymbol{Q}(\alpha_1^{1/p}, \ldots, \alpha_{n-1}^{1/p})$ ($\boldsymbol{Q}$ denotes the field of rational numbers) of degree $p$. Let $p_1, \ldots, p_r$, $h^{1/2} < p_i \leqslant h$ $(1 \leqslant i \leqslant r)$ be all the primes for which Lemma 3 is not valid. For convenience, write $p_0(\lambda_1, \ldots, \lambda_n)$ for $p(\lambda_1, \ldots, \lambda_n)$ which are determined in [2]. Set $q_0 = q_0(z, m_1, \ldots, m_{n-1}) = q(z, m_1, \ldots, m_{n-1})$ (associated to $p(\lambda_1, \ldots, \lambda_n)$). Then by Lemma 1, we have

$$q_0\left(\frac{a}{p'}, m_1, \ldots, m_{n-1}\right) = 0$$

for all rational numbers $0 \leqslant a/p' \leqslant h$, $0 < p' \leqslant h$ and all non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1 + \ldots + m_{n-1} \leqslant k_{\tilde{r}_1+1}$. In particular $q_0(a/p_1, m_1, \ldots, m_{n-1}) = 0$ with $0 \leqslant a/p_1 \leqslant h$ and $m_1 + \ldots + m_{n-1} \leqslant k_{\tilde{r}_1+1}$, i.e.



$$\sum_{\lambda_n=0}^{L}\left(\sum_{\lambda_1=0}^{L}\ldots\sum_{\lambda_{n-1}=0}^{L} p_0(\lambda_1, \ldots, \lambda_n)\alpha_1^{\lambda_1 a/p_1}\ldots\alpha_{n-1}^{\lambda_{n-1}a/p_1}\gamma_1^{m_1}\ldots\gamma_{n-1}^{m_{n-1}}\right)\alpha_n^{\lambda_n a/p_1} = 0,$$

with $0 \leqslant a/p_1 \leqslant h$ and $m_1 + \ldots + m_{n-1} \leqslant k_{\tilde{r}_1+1}$, $m_i \geqslant 0$. If $a \not\equiv 0 \pmod{p_1}$, the above sum is still zero when $\lambda_n$ is summed over any single residue class $(\bmod\, p_1)$. Therefore for $0 \leqslant a/p_1 \leqslant h$, $a \not\equiv 0\,(p_1)$ and $m_1 + \ldots + m_{n-1} \leqslant k_{\tilde{r}_1+1}$, we have

$$\sum_{\substack{\lambda_n=0\\ \lambda_n\equiv\Lambda_n(p_1)}}^{L}\left(\sum_{\lambda_1=0}^{L}\ldots\sum_{\lambda_{n-1}=0}^{L} p_0(\lambda_1, \ldots, \lambda_n)\alpha_1^{\lambda_1 a/p_1}\ldots\alpha_{n-1}^{\lambda_{n-1}a/p_1}\gamma_1^{m_1}\ldots\gamma_{n-1}^{m_{n-1}}\right)\alpha_n^{\lambda_n a/p_1} = 0$$

where $\Lambda_n$, $0 \leqslant \Lambda_n \leqslant L$, is any integer and $\lambda_n \equiv \Lambda_n(p_1)$ stands for $\lambda_n \equiv \Lambda_n \pmod{p_1}$. Define

$$p_1(\lambda_1, \ldots, \lambda_n) = \begin{cases} p_0(\lambda_1, \ldots, \lambda_n) & \text{if } \lambda_n \equiv \Lambda_n(p_1), \\ 0 & \text{otherwise} \end{cases}$$

and call $q_1$ the function associated with $p_1(\lambda_1, \ldots, \lambda_n)$. Hence

$$q_1\left(\frac{1+lp_1}{p_1}, m_1, \ldots, m_{n-1}\right) = 0,$$

with $0 \leqslant l \leqslant h-1$ and $m_1 + \ldots + m_{n-1} \leqslant k_{\tilde{r}_1+1}$ with $m_i \geqslant 0$. By Lemma 2, we obtain

$$q_1\left(\frac{a}{p'}, m_1, \ldots, m_{n-1}\right) = 0,$$

with $0 \leqslant a/p' \leqslant h$, $0 < p' \leqslant h$ and $m_1 + \ldots + m_{n-1} \leqslant k_{2(\tilde{r}_1+2)}$, $m_i \geqslant 0$. Define

$$p_2(\lambda_1, \ldots, \lambda_n) = \begin{cases} p_1(\lambda_1, \ldots, \lambda_n) & \text{if } \lambda_n \equiv \Lambda_n(p_2), \\ 0 & \text{otherwise.} \end{cases}$$

Proceed as above and conclude that

$$q_2\left(\frac{1+lp_2}{p_2}, m_1, \ldots, m_{n-1}\right) = 0,$$

with $0 \leqslant l \leqslant h-1$ and $m_1 + \ldots + m_{n-1} \leqslant k_{2(\tilde{r}_1+2)}$, $m_i \geqslant 0$. Proceed by induction and conclude that

$$q_r\left(\frac{1+lp_r}{p_r}, m_1, \ldots, m_{n-1}\right) = 0 \tag{6}$$

for all integers $l$, $0 \leqslant l \leqslant h-1$ and all non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1+\ldots+m_{n-1} \leqslant k_{r(\tilde{r}_1+2)}$. Notice that

$$p_r(\lambda_1, \ldots, \lambda_n) = \begin{cases} p_0(\lambda_1, \ldots, \lambda_n) & \text{if} \quad \lambda_n \equiv \Lambda_n(p_i) \ (1 \leqslant i \leqslant r), \\ 0 & \text{otherwise}. \end{cases}$$

Observe that

$$p_1 \ldots p_r > h^{r/2} > L \quad \text{if} \quad r = [4nE].$$

Therefore

$$p_r(\lambda_1, \ldots, \lambda_n) = 0 \text{ if } \lambda_n \neq \Lambda_n \quad \text{and} \quad p_r(\lambda_1, \ldots, \Lambda_n) = p_0(\lambda_1, \ldots, \Lambda_n).$$

In (6), set $l = 0$ and we obtain (writing $p$ for $p_r$)

$$(7) \qquad \sum_{\lambda_1=0}^{L} \cdots \sum_{\lambda_{n-1}=0}^{L} p_0(\lambda_1, \ldots, \Lambda_n) \alpha_1^{\lambda_1/p} \ldots \alpha_{n-1}^{\lambda_{n-1}/p} \gamma_1^{m_1} \ldots \gamma_{n-1}^{m_{n-1}} = 0.$$

This is true for all non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1+\ldots+m_{n-1} \leqslant k_{r(\tilde{r}_1+2)}$. Observe that

$$k_{r(\tilde{r}_1+2)} > nL,$$

if $E > 1$ and $c_1$ is large enough. Hence (7) is valid for all integers $m_1, \ldots, m_{n-1}$ with $0 \leqslant m_i \leqslant L$ $(1 \leqslant i \leqslant n)$. Notice that the determinant

$$(\lambda_{n-1}+\Lambda_n\beta_{n-1})^{m_{n-1}}, \quad 0 \leqslant \lambda_{n-1} \leqslant L, \quad 0 \leqslant m_{n-1} \leqslant L$$

does not vanish. Hence if $\Lambda_{n-1}$, $0 \leqslant \Lambda_{n-1} \leqslant L$, is an arbitrary integer, (7) gives

$$\sum_{\lambda_1=0}^{L} \cdots \sum_{\lambda_{n-2}=0}^{L} p_0(\lambda_1, \ldots, \Lambda_{n-1}, \Lambda_n) \alpha_1^{\lambda_1/p} \ldots \alpha_{n-2}^{\lambda_{n-2}/p} \gamma_1^{m_1} \ldots \gamma_{n-2}^{m_{n-2}} = 0,$$

for all integers $m_1, \ldots, m_{n-2}$ with $0 \leqslant m_i \leqslant L$. Proceeding similarly, we obtain that $p_0(\Lambda_1, \ldots, \Lambda_n) = 0$. Notice that $\Lambda_1, \ldots, \Lambda_n$ are arbitrary. Hence $p_0(\lambda_1, \ldots, \lambda_n) = 0$ for all $(\lambda_1, \ldots, \lambda_n)$, which is a contradiction.

Hence the number of primes $p$, $h^{1/2} < p \leqslant h$, for which Lemma 3 is not valid is at most $4nE$. But the number of primes between $h^{1/2}$ and $h$ exceeds $8nE$, if $c_1$ is large enough. Hence there must exist a prime $p$, $h^{1/2} < p \leqslant h$, satisfying

$$(8) \qquad \alpha_1^{j_1} \ldots \alpha_n^{j_n} = \eta^p$$

for some $\eta = a/b$, $(a, b) = 1$, in $Q$ and for some integers $j_1, \ldots, j_n$ with $0 \leqslant j_i < p$ $(1 \leqslant i < n)$ and $j_n = 1$. This completes the proof of Lemma 3.

Proof of Theorem 2. Assume that $\beta$ satisfies (1). By Lemma 3, (8) holds. But this is not possible, because of the restrictions (iv) (on $\alpha_1, \ldots, \alpha_n$) in Theorem 2. Hence

$$\beta > \frac{1}{2} n^{-k} (6h_{\tilde{r}_1})^{-2h_{\tilde{r}_1}k/n^2} S_1^{-2h_{\tilde{r}_1}k/An^2} > \exp\left(-(nA)^{c_2 n} \log S_1\right)$$



where $c_2 > 0$ is an effectively computable constant which is independent of $n$, $A$ and $S_1$. This completes the proof of Theorem 2.

Remark. Instead of assuming (iv), (§ 1, Theorem 2) it is sufficient to suppose that $\alpha_1, \ldots, \alpha_n$ are such that no relation of the type (8) is possible.

Added in proof. As the main purpose of this paper is to improve the upper estimate of [2] for $f(R)$, Theorem 2 is not stated in all its generality. The linear forms of the type of Theorem 2 (when $\alpha_i$ are close to 1) were considered for the first time in [4] to prove the results announced in [3].

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