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An iterated logarithm type theorem for the largest coefficient in continued fractions

by

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Let $[a_1(x), a_2(x), \ldots]$ be the regular continued fraction expansion of $x \in (0,1)$; that is, denoting by $Tx = 1/x \bmod 1$, the coefficients $a_1(x)$, $a_2(x), \ldots$ are obtained by the following algorithm:

$$a_1(x) = 1/x - Tx \quad \text{and} \quad a_{j+1}(x) = a_j(Tx) = a_1(T^j x).$$

Put

$$L_N = L_N(x) = \max\big(a_1(x), a_2(x), \ldots, a_N(x)\big).$$

Our aim in the present note is to extend our earlier investigations on L_N by proving the following result.

THEOREM. *For almost all x in $(0,1)$ (with respect to Lebesgue measure),*

$$\limsup_{N=+\infty} \frac{\log L_N - \log N}{\log\log N} = 1$$

and

$$\liminf_{N=+\infty} \frac{\log L_N - \log N}{\log\log N} = 0.$$

COROLLARY. *For almost all x in $(0,1)$, as $N \to +\infty$,*

$$\lim \log L_N / \log N = 1.$$

The first limit result on L_N was obtained in [1] in terms of the Gaussian measure $P(E)$, defined on the set $\{E\}$ of Lebesgue measurable subsets of $(0,1)$ by

(1) $$P(E) = \frac{1}{\log 2} \int_E \frac{dx}{1+x}.$$

This result states that, as $N \to +\infty$,

(2) $$\lim P(L_N < Ny/\log 2) = \exp(-1/y), \quad y > 0.$$

In a later paper [2], (2) was extended to the case when P is replaced by an arbitrary measure which is absolutely continuous with respect to P, which includes Lebesgue measure. In this same paper [2] it was shown that for any sequence A_N, $N = 1, 2, \ldots$, of positive numbers, the set, on which L_N/A_N has a positive limit, as $N \to +\infty$, has Lebesgue measure zero. This fact explains the reason of turning to $\log L_N$ in our Theorem.

Let us now turn to the proof. In the sequel, P will stand for the Gaussian measure as defined in (1) and λ will denote Lebesgue measure. We now quote those results from the literature which will be needed in the proof.

LEMMA 1 (Borel–Cantelli lemma, see [4], p. 128). *Let Q be an arbitrary measure on the interval $(0, 1)$ with $Q((0,1)) = 1$. Assume that the sets $B_j \subset (0, 1)$ are such that*

$$\sum_{j=1}^{+\infty} Q(B_j) < +\infty.$$

Then, for almost all x in $(0, 1)$, with respect to Q, only a finite number of the events $B_1, B_2, \ldots$ occur.

LEMMA 2 (see [5]). *For $n = 1, 2, \ldots$ and $t = 1, 2, \ldots$*

$$P(a_n(x) \geqslant t) = \frac{\log(1+1/t)}{\log 2}. \tag{3}$$

Furthermore, if we denote by $M_{u,v}$ the smallest σ-algebra generated by the coefficients $a_j(x)$, $u \leqslant j < v \leqslant +\infty$, then for any sets $A \in M_{1,s}$ and $B \in M_{s+m,+\infty}$,

$$|P(A \cap B) - P(A)P(B)| < dc^m P(A)P(B), \tag{4}$$

where $0 < c < 1$ and $d > 0$ is a constant.

LEMMA 3 (Bernstein; see [3], p. 67). *Let $b(n)$ be an arbitrary sequence of positive real numbers. If the series*

$$\sum_{n=1}^{+\infty} \frac{1}{b(n)} = +\infty, \tag{5}$$

then, for almost all x, infinitely many of the events

$$B_n = \{a_n(x) \geqslant b(n)\} \tag{6}$$

occur.

We now give the details of the proof of our result.

Proof of the Theorem. First of all notice that a set is of P-measure zero if, and only if, its λ-measure is zero. Hence, we can apply the Gaussian measure in our investigation.

For proving the first relation in our statement (the case of limsup), we have to show that for any $\varepsilon > 0$, and for almost all x,

$$\log L_N \geqslant \log N + (1-\varepsilon)\log\log N \quad \text{infinitely often} \tag{7}$$

and

$$\log L_N \geqslant \log N + (1+\varepsilon)\log\log N \quad \text{finitely often.} \tag{8}$$

Note, however, that for any increasing (positive valued) function $g(N)$, the inequalities $\log L_N \geqslant g(N)$ hold infinitely often if, and only if, $\log a_n(x) \geqslant g(n)$ infinitely often. Indeed, if $\log L_N \geqslant g(N)$, then, for some $n \leqslant N$, $\log a_n(x) \geqslant g(N) \geqslant g(n)$. Conversely, if $\log a_n(x) \geqslant g(n)$, then evidently $\log L_n \geqslant g(n)$. Therefore, the relations (7) and (8) are equivalent to

$$\log a_n(x) \geqslant \log n + (1-\varepsilon)\log\log n \quad \text{infinitely often} \tag{7a}$$

and

$$\log a_n(x) \geqslant \log n + (1+\varepsilon)\log\log n \quad \text{finitely often.} \tag{8a}$$

The inequalities of (7a) and (8a) are of the same type as in (6). From (7a),

$$b(n) = \exp(\log n + (1-\varepsilon)\log\log n) = n(\log n)^{1-\varepsilon},$$

and thus (5) holds. Lemma 3 therefore yields that (7a), and thus (7), holds for almost all x. Turning to (8a), we apply Lemma 1 and the relation (3). With the notation of (6),

$$b(n) = \exp(\log n + (1+\varepsilon)\log\log n) = n(\log n)^{1-\varepsilon},$$

and thus, applying that, for $|u| \leqslant \frac{1}{2}$, $|\log(1+u)| \leqslant 2|u|$,

$$P(B_n) = (1/\log 2)\log(1+\exp(-[\log n + (1+\varepsilon)\log\log n]))$$
$$\leqslant \frac{2e}{\log 2}\cdot\frac{1}{b(n)} = \frac{2e}{(\log 2)n(\log n)^{1+\varepsilon}},$$

where $[w]$ denotes the integer part of w. From the inequality above

$$\sum_{n=1}^{+\infty} P(B_n) < +\infty,$$

and thus Lemma 1 applies, yielding (8a) for almost all x. Our statement about the limsup is therefore established.

Turning to the liminf, we first remark that the event

$$C = \{\log L_N \leqslant \log N \text{ infinitely often}\}$$

has the property that $C \in M_{k,+\infty}$ for all $k = 1, 2, \ldots$ This is evident if we show that for any k, $C = C_k$, where

$$C_k = \{\log L_{N,k} \leqslant \log N \text{ infinitely often}\}$$

with $L_{N,k} = \max\big(a_k(x), a_{k+1}(x), \ldots, a_N(x)\big)$. By definition, $C \subseteq C_k$, hence only the converse needs proof. That, however, immediately follows by noting that, for any fixed x, $L_k(x)$ is a fixed number, hence, for sufficiently large N, whatever x be, $\log L_k \leqslant \log N$, and thus for such an N, $\log L_{N,k}$ and $\log L_N$ can be smaller than $\log N$ only simultaneously. The usual way of proving zero-one laws yields by (4) that $P(C) = 1$ or 0 (approximate C by cylinder sets $D_s \in M_{1,s}$ and apply (4) with $A = D_s$ and $B = C$. Since $C = C_k \in M_{k,+\infty}$, (4) applies with any m, yielding $P(D_s \cap C) = P(D_s)P(C)$. But for $s \to +\infty$, $D_s \to C$, thus $P(C) = P^2(C)$, which can be true only for $P(C) = 1$ or 0). On the other hand, since, as $N \to +\infty$,

$$P(C) \geqslant \limsup P(\log L_N \leqslant \log N) = \limsup P(L_N \leqslant N),$$

(2) yields that $P(C) > 0$ and thus the above result implies that $P(C) = 1$. We therefore proved that, for almost all x, $\liminf(\log L_N - \log N) \leqslant 0$, as $N \to +\infty$, which, of course, implies that, for almost all x, as $N \to +\infty$

$$\liminf(\log L_N - \log N)/\log\log N \leqslant 0. \tag{9}$$

In order to complete the proof, we have to prove the inequality obtained by reversing the inequality sign in (9). That is, we have to show that for any $\varepsilon > 0$, for almost all x,

$$\log L_N \leqslant \log N - \varepsilon \log\log N \quad \text{finitely often.} \tag{10}$$

Here we apply a trick. Noting that, by Lemma 3, the inequalities

$$\log a_N(x) \geqslant \log N - \varepsilon \log\log N$$

hold for infinitely many N, for almost all x, the argument of showing the equivalence of (7) and (7a) gives that, for almost all x, infinitely often,

$$\log L_N \geqslant \log N - \varepsilon \log\log N.$$

Therefore, the P-measure of (10) is equal to

$$\begin{aligned}(11)\quad & P\big(\log L_N \leqslant \log N - \varepsilon\log\log N,\ \log L_{N+1} > \log(N+1) - \varepsilon\log\log(N+1) \\ &\qquad \text{finitely often}\big) \\ &= P\big(\log L_N \leqslant \log N - \varepsilon\log\log N,\ \log a_{N+1}(x) > \log(N+1) - \\ &\qquad - \varepsilon\log\log(N+1) \text{ finitely often}\big),\end{aligned}$$

by the function $\log N - \varepsilon\log\log N$ being increasing. But since $L_N \in M_{1,N}$ and $a_{N+1}(x) \in M_{N+1,+\infty}$, we have from (4)

$$\begin{aligned}(12)\quad & P\big(\log L_N \leqslant \log N - \varepsilon\log\log N,\ \log a_{N+1}(x) \\ &\qquad > \log(N+1) - \varepsilon\log\log(N+1)\big) \\ &\leqslant (1+cd)P(\log L_N \leqslant \log N - \varepsilon\log\log N) \times \\ &\qquad \times P\big(\log a_{N+1}(x) > \log(N+1) - \varepsilon\log\log(N+1)\big).\end{aligned}$$

Our aim is to show that the sum of the probabilities in (12) converges, hence, by applying Lemma 1, we get that the value in (11) is one, thus proving (10) for almost all x. By the method [1] of proving (2), we get that, for N large,

$$\begin{aligned}(13)\quad & P(\log L_N \leqslant \log N - \varepsilon\log\log N) = P\big(L_N \leqslant N(\log N)^{-\varepsilon}\big) \\ &\leqslant 2\left(1 - \frac{(\log N)^{\varepsilon}}{N}\right)^N = 2\exp\left(N\log\left(1 - \frac{(\log N)^{\varepsilon}}{N}\right)\right) \leqslant 2\exp\big(-(\log N)^{\varepsilon}\big),\end{aligned}$$

amounting to the $a_n(x)$ behaving as if they were independent for the value $N(\log N)^{-\varepsilon}$ as well. We do not repeat all details here, except that we point out that, when applying repeatedly (4) in that argument, we need a more careful calculation for $\sum P\big(a_{i_1}(x) \geqslant w, \ldots, a_{i_k}(x) \geqslant w\big)$, where summation is for all distinct values of the subscripts. Let us carry out this refined calculation for $k = 2$. By (4), putting $A_j = \{a_j(x) \geqslant w\}$,

$$\begin{aligned}\sum_{1\leqslant i<j\leqslant n} P(A_iA_j) &= \sum_{j=1}^{n}\sum_{m=1}^{n-j} P(A_jA_{j+m}) = \sum_{j=1}^{n} P(A_j)\sum_{m=1}^{n-j}\big(1+O(c^m)\big)P(A_{j+m}) \\ &= \sum_{1\leqslant i<j\leqslant n} P(A_i)P(A_j) + O\Big(\sum_{j=1}^{n} P(A_j)\sum_{m=1}^{n-j} c^m P(A_{j+m})\Big).\end{aligned}$$

Note that the constant involved in $O(\cdot)$ is bounded by d for all m, hence applying the operator $O(\cdot)$ to the whole sum was possible. Now for large w, $P(A_j) \sim 1/w\log 2$ for all j, hence the error term is $O(n/w^2)$, the main term, on the other hand, being as if they were exactly independent. The other change required is to apply a so called restricted sieve theorem, instead of the exact formula (7) of [1]. This helps to avoid to estimate the tail of a sieve formula ((10) of [1]). Turning to the other term of (12), we have by (3),

$$\begin{aligned}& P\big(\log a_{N+1}(x) > \log(N+1) - \varepsilon\log\log(N+1)\big) \\ &\qquad \leqslant (1/\log 2)\exp\big(-\log(N+1) + \varepsilon\log\log(N+1)\big) = \frac{(\log(N+1))^{\varepsilon}}{(N+1)\log 2},\end{aligned}$$

and thus, by (13), we have that the sum of probabilities in (12) converges, hence Lemma 1 completes the proof.

The Corollary is a straight consequence of the Theorem. We stated it separately because of its interesting content.

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On gaps between numbers with a large prime factor, II

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1. In [2] the following result was proved:

THEOREM 1. *Let $n > 1$ be an integer. Let $a_1, \ldots, a_n$ be rational numbers such that*

(i) $a_1 > 0, \ldots, a_n > 0$ *are multiplicatively independent,*

(ii) $|\log a_i| \leqslant \exp\left(-\frac{1}{A}\log S_1\right), 1 \leqslant i \leqslant n$ *and* $A > 1$,

(iii) *The sizes of $a_1, \ldots, a_n$ do not exceed S_1. (The size of a rational number a/b, $(a, b) = 1$, is defined as $|b| + |a/b|$.)*

If $\beta_1, \ldots, \beta_{n-1}$ are rational numbers of size not exceeding S_1, then

$$|\beta_1 \log a_1 + \ldots + \beta_{n-1}\log a_{n-1} - \log a_n| > \exp\left(-(nA)^{cn^2}\log S_1\right)$$

where $c > 0$ is an effectively computable constant which is independent of n, A and S_1.

In this paper we shall prove the following:

THEOREM 2. *Let $n > 1$ be an integer. Let $a_1, \ldots, a_n, \beta_1, \ldots, \beta_{n-1}$ be rational numbers satisfying the assumptions of Theorem 1. Further assume that*

(iv) $a_1 = \frac{m}{m'}, a_2 = \frac{p_2}{p_2'}, \ldots, a_n = \frac{p_n}{p_n'}$ *where* $p_2, \ldots, p_n, p_2', \ldots, p_n'$ *are pairwise distinct prime numbers and none of them is either a factor of m or m'.*

Then

$$|\beta_1 \log a_1 + \ldots + \beta_{n-1}\log a_{n-1} - \log a_n| > \exp\left(-(nA)^{c_1 n}\log S_1\right)$$

where $c_1 > 0$ is an effectively computable constant which is independent of n, A and S_1.

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