From (38) and Lemmas 7 and 9 we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{i=1}^{9}T(\gamma_i)K(a)K(\beta)\,da\,d\beta = \iint_{U(\tau)}\prod_{i=1}^{9}T(\gamma_i)K(a)K(\beta)\,da\,d\beta + o(P^5),$$

as $P\to\infty$ through $\mathscr{P}$. Thus, by (34), for some positive constant D

$$N(P) \geqslant DP^5 + o(P^5)$$

as $P\to\infty$ through $\mathscr{P}$ which gives $N(P) > 0$, and the proof is complete.

References

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Halving an estimate obtained from Selberg's upper bound method

by

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Introduction. In many applications of Selberg's upper bound method, an unnecessary constant factor appears in the final estimate, due to the fact that we can only sieve up to approximately $\sqrt{x}$.

At present this restriction seems unavoidable, and arises from the necessity of squaring in order to obtain a non-negative sifting function, viz.

$$s^{(+)}(n) = \Big(\sum_{d|n}\lambda_d\Big)^2.$$

As an example, let K be any positive integer whose greatest prime factor does not exceed x. Following van Lint and Richert [1], we arrive at the estimate

$$\sum_{\substack{n\leqslant x\\(n,K)=1}} 1 \leqslant \frac{\varphi(K)}{K}x\left(\frac{1}{\log z}+\frac{z^2}{x}\right)\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{-1},$$

by a careful application of Selberg's method. Choosing z optimally, Mertens' formula gives

$$\sum_{\substack{n\leqslant x\\(n,K)=1}} 1 \leqslant 2e^{\gamma}\frac{\varphi(K)}{K}x\left(1+O\left(\frac{\log\log x}{\log x}\right)\right).$$

The factor e^{γ} really is necessary, as the Prime Number Theorem shows, but apart from the error term, the estimate becomes best possible if we strike out the factor 2 on the right. The object of this note is to obtain a general result of this kind.

THEOREM. *Let $f(n)$ be defined on the positive integers and satisfy*

$$f(1) = 1, \quad 0 \leqslant f(n) \leqslant 1$$

and

$$f(nm) \leqslant f(n)f(m) \text{ provided } (n, m) = 1.$$

Then

$$T_f(x) = \sum_{n\leqslant x} f(n)$$

$$\leqslant e^{\gamma}x\left(1+O\left(\frac{\log\log x}{\log x}\right)\right)\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\dots\right).$$

Apart from the error term, this is best possible, and if we set

$$f(n) = \begin{cases} 1 & \text{if } (n, K) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain the estimate mentioned above without the factor 2. It would be interesting to have similar, best possible, results for the twin prime problem, and the problem of estimating $\pi(x+y)-\pi(x)$ and other cases, and I hope to extend the scope of my result.

I would like to take this opportunity of thanking Professor H.-E. Richert for reading an earlier version of the manuscript and writing to me about it. Professor Richert suggested that it should be possible to remove the factor $\log\log x$ from the error term: I have not succeeded in doing this yet.

Before embarking on the proof, we require the following lemma.

LEMMA. *There exists an absolute constant A such that for all $x > 1$ and positive integers h,*

$$\sum_{n\leqslant x}\left(\log x - \sum_{p\mid n}\log p\right)^h \leqslant xh!\exp(A\sqrt{h}).$$

Proof. Let m denote the square-free kernel of n, that is

$$m = \prod_{p\mid n} p.$$

Then

$$(x/m)^t = (x/n)^t\sum_{d\mid n} g(d)$$

where $g(d)$ is positive or zero and is defined by the relation

$$\prod_{p}\left(1+\frac{p^t-1}{p^{2s}-p^{s+t}}\right) = \sum_{d=1}^{\infty}\frac{g(d)}{d^s}$$

the product running over all primes p. Therefore

$$\sum_{n\leqslant x}\left(\frac{x}{m}\right)^t = \sum_{n\leqslant x}\left(\frac{x}{n}\right)^t\sum_{d\mid n} g(d)$$

$$= \sum_{d\leqslant x}\frac{g(d)}{d^t}\sum_{r\leqslant x/d}\left(\frac{x}{r}\right)^t \leqslant \frac{x}{1-t}\sum_{d\leqslant x}\frac{g(d)}{d} \quad \text{for } 0\leqslant t<1.$$

Setting $t = 1-u$ we deduce that

$$\sum_{n\leqslant x}\sum_{h=0}^{\infty}\frac{(1-u)^h}{h!}\left(\log x - \sum_{p\mid n}\log p\right)^h \leqslant \frac{x}{u}\prod_{p}\left(1+\frac{p^{1-u}}{p^2(1-p^{-u})}\right).$$

All the terms on the left are positive, therefore the inequality holds if we restrict our attention to a fixed h. Moreover,

$$1-p^{-u} \geqslant \tfrac{1}{2}\min(1, u\log p)$$

and so

$$\frac{(1-u)^h}{h!}\sum_{n\leqslant x}\left(\log x - \sum_{p\mid n}\log p\right)^h \leqslant \frac{x}{u}\prod_{p}(1+2p^{-1-u})\left(1+\frac{2}{up\log p}\right).$$

It follows that there exists an absolute constant B so that for $0<u\leqslant 1$,

$$\sum_{n\leqslant x}\left(\log x - \sum_{p\mid n}\log p\right)^h \leqslant xh!\exp\left\{\frac{B}{u} - h\log(1-u)\right\}.$$

We select $u = 1/2\sqrt{h}$ and this gives the result stated.

Proof of the theorem. We may write

$$T_f(x)\log x = \sum_{n\leqslant x} f(n)\sum_{p^a\parallel n}\Lambda(p^a) + \sum_{n\leqslant x} f(n)\left(\log x - \sum_{p\mid n}\log p\right)$$

where as usual, $p^a\parallel n$ denotes that $p^a\mid n$, $p^{a+1}\nmid n$, that is, $(p, n/p^a) = 1$. We estimate the second sum on the right by Holder's inequality, using the lemma. Thus for $h\in Z^+$,

$$\sum_{n\leqslant x} f(n)\left(\log x - \sum_{p\mid n}\log p\right) \leqslant (T_f(x))^{1-1/h}(xh!\exp(A\sqrt{h}))^{1/h},$$

here we need that $0\leqslant f(n)\leqslant 1$. Hence

$$\sum_{n\leqslant x} f(n)\left(\log x - \sum_{p\mid n}\log p\right) \ll hT_f(x)\left(\frac{x}{T_f(x)}\right)^{1/h},$$

the constant implied by Vinogradov's notation $\ll$ being independent of all parameters, here and elsewhere. Next,

$$\sum_{n\leqslant x} f(n)\sum_{p^a\parallel n}\Lambda(p^a) = \sum_{d\leqslant x}\Lambda(d)\sum_{\substack{m\leqslant x/d \\ (m,d)=1}} f(md)$$

$$\leqslant \sum_{d\leqslant x}\Lambda(d)\sum_{m\leqslant x/d} f(m) \leqslant \sum_{m\leqslant x} f(m)\psi\left(\frac{x}{m}\right).$$

Therefore

$$T_f(x)\log x \leqslant x\sum_{m\leqslant x}\frac{f(m)}{m} + \sum_{m\leqslant x} f(m)\left|\psi\left(\frac{x}{m}\right) - \frac{x}{m}\right| + ChT_f(x)\left(\frac{x}{T_f(x)}\right)^{1/h}.$$

Since $f(1) = 1$, if $T_f(x) \leqslant x/\log x$ this estimate holds without the last term on the right. Therefore in any case,

$$T_f(x)\log x \leqslant x\sum_{m\leqslant x}\frac{f(m)}{m} + \sum_{m\leqslant x} f(m)\left|\psi\left(\frac{x}{m}\right) - \frac{x}{m}\right| + ChT_f(x)(\log x)^{1/h}.$$

We select

$$h = [\log\log x]+1 \geqslant 1$$

provided $x \geqslant e$, and deduce that

$$T_f(x)\big(\log x + O(\log\log x)\big) \leqslant x\sum_{m\leqslant x}\frac{f(m)}{m} + \sum_{m\leqslant x} f(m)\left|\psi\left(\frac{x}{m}\right) - \frac{x}{m}\right|.$$

We apply this estimate twice. Initially, we employ Tchebycheff's result that $\psi(x) \ll x$, and Mertens' formula. We obtain

$$T_f(x) \ll x\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\ldots\right).$$

We now use this preliminary result to estimate the sum

$$\sum_{m\leqslant x} f(m)\left|\psi\left(\frac{x}{m}\right) - \frac{x}{m}\right|$$

more effectively. We need the Prime Number Theorem, in the relatively weak form

$$\psi(x) = x + O\left(\frac{x}{\log^2 x}\right).$$

Then the sum above is

$$\begin{aligned}
&\ll \sum_{m\leqslant x} f(m)\sum_{d\leqslant x/m}\frac{1}{\log^2 2d}\\
&\ll \sum_{d\leqslant x}\frac{1}{\log^2 2d}\,T_f\left(\frac{x}{d}\right)\\
&\ll \sum_{d\leqslant \sqrt{x}}\frac{1}{\log^2 2d}\,T_f\left(\frac{x}{d}\right) + x\sum_{d>\sqrt{x}}\frac{1}{d\log^2 2d}\\
&\ll x\prod_{p\leqslant \sqrt{x}}\left(1-\frac{1}{p}\right)\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\ldots\right) + \frac{x}{\log x}\\
&\ll \frac{x}{\log x}\prod_{p\leqslant x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\ldots\right).
\end{aligned}$$

Therefore

$$T_f(x)\big(\log x + O(\log\log x)\big) \leqslant x\left(1+O\left(\frac{1}{\log x}\right)\right)\prod_{p\leqslant x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\ldots\right).$$

By Mertens' formula, in the form

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x}\left(1+O\left(\frac{1}{\log x}\right)\right)$$

we obtain our result.

Reference

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