Simultaneous quadratic inequalities

by

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$$Q(x) = \sum_{i=1}^{5} \tau_i x_i^2$$

is an indefinite quadratic form with real coefficients, such that at least one of the ratios $\tau_i/\tau_j$ is irrational, then for every $\varepsilon > 0$ there exist integers $x_1, \ldots, x_5$, not all zero, such that

$$|Q(x)| < \varepsilon.$$ 

Here we shall consider the analogous problem for two diagonal quadratic forms having real coefficients. Let

$$F(x) = \sum_{i=1}^{9} \lambda_i x_i^2 \quad \text{and} \quad G(x) = \sum_{i=1}^{9} \mu_i x_i^2.$$ 

The condition that at least one of the ratios $\lambda_i/\lambda_j$ in (1) be irrational is equivalent to requiring that not all of the binary linear forms $\lambda_i u + \lambda_j v$ have coefficients which are linearly dependent over the rationals. We associate ternary linear forms

$$L_{u,v}(u, v, w) = \begin{vmatrix} u & v & w \\ \lambda_i & \lambda_j & \lambda_k \\ \mu_i & \mu_j & \mu_k \end{vmatrix}, \quad 1 \leq i < j < k \leq 9,$$

with the two forms $F$ and $G$.

Theorem. Let $F(x)$ and $G(x)$ be diagonal quadratic forms, having real algebraic coefficients, in 9 variables. Suppose that

(i) Every member of the pencil $\{LF + MG\} \setminus \{(L, M) \neq (0, 0)\}$ is an indefinite form with at least 5 non-zero coefficients; and

(ii) Not all of the ternary linear forms $L_{u,v}$ associated with $F$ and $G$ have coefficients which are linearly dependent over the rationals.
Then for any $\varepsilon > 0$ there exist integers $x_1, \ldots, x_n$, not all zero, such that

$$|F(x)| < \varepsilon \quad \text{and} \quad |G(x)| < \varepsilon.$$  

(4)

This is a partial complement to the analogous result for Diophantine equations [3]. By an appropriate application of Hua's Lemma analogous results may be obtained for $R$ additive inequalities of degree $k$. We have assumed that the coefficients of $F$ and $G$ are algebraic in order to simplify the statement of the Theorem. It is possible to obtain results for forms having real coefficients, and we shall state these results in §2.

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2. Preliminaries. We begin by normalizing the inequalities (4). Let

$$L_{123}(u, v, w) = pu + qv + rw,$$

then we may suppose that $p$, $q$, and $r$ are linearly independent over the rationals, and in particular $r \neq 0$. For any given $\varepsilon > 0$ we choose an integer $m > \varepsilon^{-1}$ and take

$$n = \frac{m}{\varepsilon} \max(|x_1| + |x_2|, |x_1| + |x_2|).$$

We define the normalized forms $A(x)$ and $B(x)$ by

$$A = nr^{-1}(\mu_1 F - \lambda_1 G) \quad \text{and} \quad B = nr^{-1}(\mu_2 F - \lambda_2 G).$$

Thus $A(x)$ and $B(x)$ are diagonal quadratic forms in $9$ variables such that every member of the pencil $\{xA + \lambda B\} ([x, \lambda] \neq (0, 0)]$ is an indefinite form with at least 5 non-zero coefficients. We write

$$A(x) = \sum_{i=1}^{9} a_i x_i^2 \quad \text{and} \quad B(x) = \sum_{i=1}^{9} b_i x_i^2,$$

so that

(5)

$$a_1 = b_2 = n, \quad a_2 = b_1 = 0, \quad a_3 = -np/r, \quad \text{and} \quad b_3 = nq/r.$$  

(6)

In order to prove the Theorem it is sufficient to prove that there exist integers $x_1, \ldots, x_9$, not all zero, such that

$$|A(x)| < 1 \quad \text{and} \quad |B(x)| < 1.$$  

(7)

DEFINITION. For any real number $a$, we say that the real linear form $pu + qv + rw$ is of order $a$ if the inequalities

$$|pu + qv + rw| < U^a, \quad 0 < \max(|u|, |v|, |w|) \leq U,$$

have an integer solution $(u, v, w)$ for all $U$ greater than some $U_0(a)$.

(8)

**Lemma 1.** Let $p, q, r$ be algebraic numbers which are linearly independent over the rationals. Then for any $\delta > 0$ there are only finitely many integer points $(u, v, w)$ with

$$|pu + qv + rw| < \max(|u|, |v|, |w|)^{1-\delta}.$$  

(9)

This is a particular case of Corollary 1 of Schmidt [7].

**Corollary.** The linear form $L_{123}$ is of order at most 2.

**Proof.** We have $L_{123} = pu + qv + rw$ where $p, q, r$ are algebraic numbers which are linearly independent over the rationals. Thus

$$M = \min\{|pu + qv + rw| > 0,$$

where the minimum is taken over those integer points $(u, v, w) \neq (0, 0, 0)$ which satisfy (9). Hence if $U^{2+\delta} > M^{-1}$ there are no solutions of (8) so $L_{123}$ is not of order $2 + \delta$. This is true for any $\delta > 0$ and so $L_{123}$ is of order at most 2.

We shall only require that $L_{123}$ is not of order $\infty$, and an analogue of the Theorem can be proved for quadratic forms $F$ and $G$ having real coefficients provided that not all of the associated ternary linear forms are of order $\infty$. It is straightforward to prove that the coefficients of the ternary linear forms of order $\infty$ form a set of Hausdorff dimension 2. Also, the proof of Theorem XIV of Cassels [2], p. 94, may readily be modified to show that there are ternary linear forms of order $\infty$ whose coefficients are linearly independent over the rationals.

For the rest of this paper we shall suppose that $L_{123}$ is not of order $\infty$. We can choose a real number $\sigma$ such that $L_{123}$ is not of order $\sigma$ and take $A = 1/3(\sigma + 2)$. We denote by $\delta$ a small positive constant chosen so that $\delta < A/4$.

We recall that the coefficients of the normalized forms $A(x)$ and $B(x)$ are $a_i$ and $b_i$, respectively. We take

$$\gamma_i = a_i a + b_i b \quad \text{for} \quad i = 1, \ldots, 9.$$  

(10)

Let $P$ be a large integer which will later be restricted to lie in a certain sequence. By $X \ll Y$ we mean $|X| < CY$ where $C$ is independent of $P$. We let $\varepsilon$ denote a small positive constant and we write $\varepsilon(x)$ for $\exp(2\pi i x)$.  

**Lemma 2.** Suppose that for every large integer $P$ there exist real numbers $\alpha, \beta$ and integers $A_i, Q_i, i = 1, 2, 3$, satisfying

$$\max(|\alpha|, |\beta|) \ll P^\varepsilon,$$

(11)

$$\gamma_i = A_i/Q_i + O(P^{d-2}), \quad i = 1, 2, 3,$$

(12)

$$0 \neq Q_i \ll P^d, \quad i = 1, 2, 3,$$

(13)
and
\[(A_1, A_2) \neq (0, 0).\]

Then \(I_{123}\) is of order \(a\).

Proof. Suppose that for all sufficiently large integers \(P\) there exist solutions of (12)–(14). Now
\[
y_1 = na, \quad y_2 = nb, \quad \text{and} \quad ry_3 = n(qb - pa)
\]
so that
\[
g(A_1/Q_2) - p(A_1/Q_1) = r(Q_3/Q_2) + O(P^{d-2}).
\]
Hence
\[
|\frac{pA_3Q_2Q_3 - qA_2Q_1Q_3 + rA_1Q_1Q_3}{Q_1Q_2Q_3}| \ll Q_1Q_2Q_3P^{d-2} \ll P^{4d-2}.
\]
Also \(A_i \ll P^{d+1} \ll P^{2d}\) for \(i = 1, 2, 3\), so taking
\[
u = A_1Q_2Q_3, \quad v = -A_2Q_1Q_3 \quad \text{and} \quad w = A_3Q_1Q_2
\]
we have
\[
|pu + qv + rw| \ll P^{d-2}
\]
and
\[
0 < \max(|u|, |v|, |w|) \ll P^{d-2}.
\]
Therefore for any \(\varepsilon > 0\), \(I_{123}\) is of order \((2 - 4\alpha)/4\alpha - \varepsilon\), which gives the Lemma provided that \(\varepsilon\) is small.

Corollary. We may suppose that there exists an infinite subsequence \(\mathcal{S} = \mathcal{S}(\varepsilon)\) of the positive integers such that for all \(P \in \mathcal{S}\) (11)–(14) are not all solvable.

3. General lemmas

Lemma 3. The equations \(A = B = 0\) have a non-singular real solution with none of the variables vanishing.

This is essentially Lemma 2.4 of [3].

From such a solution we have a solution \(\chi\) of the equations
\[
a_1z_1 + \ldots + a_9z_9 = 0, \quad b_1z_1 + \ldots + b_9z_9 = 0
\]
such that \(\chi_i > 0\) for \(i = 1, \ldots, 9\). Then, choosing a suitable linear multiple of this solution, we may suppose that \(\chi_i > 1\) for \(i = 1, \ldots, 9\). We now choose a constant \(C\), independent of \(P\), so that
\[(15) \quad 1 < \chi_i < C^2 \quad \text{for} \quad i = 1, \ldots, 9.
\]
For \(i = 1, \ldots, 9\) we take
\[(16) \quad T_i = \frac{T(y_i)}{\chi_i} = \sum_{x \leq P} e(\chi_i x^2),
\]
\[
(17) \quad J_i = J(y_i) = \int_{\mathcal{P}} e(\chi_i \xi^2) d\xi,
\]
and we put
\[(18) \quad K(a) = (\sin \pi a/\pi)^2.
\]

Lemma 4.

\[
(19) \quad \int_{-\infty}^{\infty} e(\eta a)K(a)da = \max(0, 1 - |\eta|).
\]

This is a Lemma 4 of Davenport and Heilbronn [6].

Let \(\mathcal{S}\) be the box \(\{x: P \leq x \leq CP, i = 1, \ldots, 9\}\) and let \(N(P)\) be the number of integer solutions of the normalized inequalities (7) in \(\mathcal{S}\).

Lemma 5.

\[
(20) \quad N(P) \gg \frac{\int \int \int_{x_i = 1}^{9} T(y_i)K(a)K(b)da db}{CP \prod_{i=1}^{9} \max(0, 1 - |A(\xi)|) \max(0, 1 - |B(\xi)|) d\xi}.
\]

This result follows from Lemma 4 on multiplying out the products and interchanging the orders of integration and summation.

4. Reduction to a finite integral. We shall obtain a lower bound for \(N(P)\) from (20), and begin by reducing the integral to a finite region.

Lemma 6. For any real \(y, z\) and any \(\varepsilon > 0\)

\[
(22) \quad \int_{y}^{y+1} \int_{z}^{z+1} \prod_{i=1}^{9} |T(y_i)| d\beta d\alpha \ll P^{4+4}
\]

where ’ denotes the omission of any one factor from the product.

Proof. Since every member of the pencil \(\{xA + MB\} (\{x, y\} \neq (0, 0))\) contains at least 5 terms explicitly, any ratio occurs at most 4 times among the \(a_i b_i\). Therefore the 8 factors in the product can be arranged into 4 pairs \(T(y_k), T(y_l)\) such that \(a_k b_k - a_l b_l \neq 0\). Then

\[
(23) \quad \int_{y}^{y+1} \int_{z}^{z+1} \prod_{i=1}^{9} |T(y_i)| d\beta d\alpha \ll \sum_{k,l} \int_{y}^{y+1} \int_{z}^{z+1} |T(y_k)T(y_l)| d\beta d\alpha,
\]

where the sum is taken over such pairs \(k, l\). The Lemma now follows on applying the generalization of Hua’s Lemma obtained in [4].

Corollary. For any real \(y, z\) and any \(\varepsilon > 0\),

\[
(24) \quad \int_{y}^{y+1} \int_{z}^{z+1} \prod_{i=1}^{9} |T(y_i)| d\beta d\alpha \ll P^{5+4}.
\]
From the \( \alpha \cdot \beta \) plane we now select 4 regions \( R_i \):
\[
R_1 = \{(\alpha, \beta) : \alpha > P^\delta \}; \quad R_2 = \{(\alpha, \beta) : \alpha < -P^\delta \}; \\
R_3 = \{(\alpha, \beta) : \beta > P^\delta \}; \quad R_4 = \{(\alpha, \beta) : \beta < -P^\delta \}.
\]
Here \( \delta \) is the positive number chosen in § 2. We take
\[
R = \bigcup_{i=1}^{4} R_i.
\]

**Lemma 7.** For any \( \varepsilon > 0 \)
\[
\int \int \int_{K} |T(\gamma)| K(\alpha) K(\beta) d\alpha d\beta \leq P^{3+\varepsilon-\delta}.
\]

**Proof.** It is sufficient to prove with each \( R_i \) in place of \( R \).
Using the estimate \( K(\alpha) \leq \max(\alpha^{-3}, 1) \), we have
\[
\int \int \int_{R_i} |T(\gamma)| K(\alpha) K(\beta) d\alpha d\beta = \sum_{\lambda^P>0} \sum_{\lambda^P<0} \int \int \int_{R_i} |T(\gamma)| K(\alpha) K(\beta) d\alpha d\beta \\
\leq \left( \sum_{\lambda^P>0} \sum_{\lambda^P<0} |T(\gamma)| \right) P^{2+\varepsilon} \leq P^{2+\varepsilon-\delta},
\]
and the other regions \( R_i \) are treated similarly.

With the linear form \( \gamma_i \) we associate the line
\[
\Gamma_i : \gamma_i = 0
\]
in the \( \alpha \cdot \beta \) plane. We label the \( \Gamma_i \) so that the positive angle from \( \beta = 0 \) to \( \Gamma_i \) increases monotonically with \( i \). Note that we may have \( \Gamma_i = \Gamma_{i+1} \). If \( \Gamma_i \neq \Gamma_{i+1} \), let \( B_i \) be the line bisecting the angle formed by the lines \( \Gamma_i \) and \( \Gamma_{i+1} \). If \( j \) is the largest integer less than \( i \) such that \( \Gamma_j \neq \Gamma_i \), we let \( S_{j+1} = \ldots = S_i' \) be the sector bounded by \( B_j \) and \( B_i \). Thus \( \Gamma_{j+1} = \ldots = \Gamma_i \) lie in the interior of \( S_i' \).

We choose a positive constant \( \varepsilon \) such that if \( \max(|\alpha|, |\beta|) > 0 \), \( |\gamma_i| < 1 \) and \( a_i b_j - a_j b_i \neq 0 \) then \( |\gamma_i| > 1 \). Therefore, for any large integer \( P \), if \( \max(|\alpha|, |\beta|) > P^{-\varepsilon} \), \( |\gamma| < P^{-1} \) and \( a_i b_j - a_j b_i \neq 0 \) then \( |\gamma| > P^{-1} \).

Let \( r \) be a small positive constant, and take \( S_i \) to be the intersection of \( S_i' \) with the region
\[
cP^{r-\varepsilon-\delta} < \max(|\alpha|, |\beta|) < cP^{-\varepsilon}.
\]

**Lemma 8.** If \( \gamma_i = O(P^{-1}) \) and \( \gamma_i \neq 0 \) then
\[
T(\gamma) \ll |\gamma_i|^{-1+\varepsilon}.
\]
This is Lemma 7 of Davenport and Heilbronn [6].

**Lemma 9.** For each \( S_j \),
\[
\left| \int \int \int_{S_j} |T(\gamma)| K(\alpha) K(\beta) d\alpha d\beta \right| = O(P^5).
\]

**Proof.** We take new coordinates in the region \( S_j \). These are \( r \), the distance along \( \Gamma_j \) to the origin, and \( s \), perpendicular to \( r \). The region \( S_j \) lies in a region bounded by two lines, say \( -mr \leq s \leq mr \). Also, we can choose positive constants \( c_0, c_1 \) and \( c_2 \), independent of \( P \), so that
\[
r > c_0 P^{r-\varepsilon-\delta} \text{ in } S_j,
\]
and if \( a_i b_j - a_j b_i \neq 0 \), \( c_1 r \leq |\gamma| \leq c_2 r \).

In \( S_j \), each \( \gamma_i \) is \( O(\varepsilon^{-1}) \) and if \( a_i b_j - a_j b_i \neq 0 \) we have \( \gamma_i \neq 0 \) in \( S_j \) and so we can use Lemma 8 to estimate \( T(\gamma) \). Since any ratio occurs at most 4 times among the \( a_i b_j \) we can use Lemma 8 on at least 5 factors in the product, and use the trivial estimate \( O(P) \) on the remaining terms. Hence
\[
\int \int \int_{S_j} |T(\gamma)| K(\alpha) K(\beta) d\alpha d\beta \leq \int \int \int_{S_j} P^4(r^{-1+\varepsilon}) d\alpha d\beta \\
\leq c_0 P^{r-\varepsilon-\delta} \int \int \int_{S_j} P^4(r^{-1+\varepsilon}) d\alpha d\beta \\
\leq P^{r+4+4+5} = o(P^5),
\]
provided that \( r \) and \( \varepsilon \) are sufficiently small.

We now take \( \Sigma_i \) to be the intersection of \( S_i' \) with the region
\[
\max(|\alpha|, |\beta|) > cP^{r-\varepsilon-\delta}.
\]

**Lemma 10.** For each \( S_j \),
\[
\left| \int \int \int_{S_j} |J(\gamma)| K(\alpha) K(\beta) d\alpha d\beta \right| = O(P^5).
\]

**Proof.** If \( \gamma_i \neq 0 \) we have, as in Lemma 11 of Davenport and Heilbronn [6], \( J(\gamma_i) = O(|\gamma|^{-1}) \). The result now follows in the same way as Lemma 9.

5. The main term

**Lemma 11.** For some positive constant \( D \), independent of \( P \),
\[
\left| \int_{P}^{C_0 P} \ldots \int_{P}^{C_0 P} \max(0, 1 - |A(\xi)|) \max(0, 1 - |B(\xi)|) d\xi > D P^5,
\]
for all sufficiently large \( P \).
Proof. We put \( \xi_i^2 = P^3 \eta_i \), then \( d \xi_i = P (2 | \eta_i |^{1/2})^{-1} d \eta_i \). We take
\[
\mathcal{E} = \{ \eta : 1 < \eta < C^3, \; i = 1, \ldots, 9 \}
\]
and
\[
\mathcal{G} = \{ \eta : \max(\vert A_1 (\eta) \vert, \vert B_1 (\eta) \vert) < (2 P^3)^{-1} \},
\]
where \( A_1 (\eta) = a_1 \eta_1 + \cdots + a_9 \eta_9 \) and \( B_1 (\eta) = b_1 \eta_1 + \cdots + b_9 \eta_9 \). Then the left hand side of (31) is at least
\[
2^{-11} P^9 \int_{\mathcal{E} \cap \mathcal{G}} \ldots \int_{\mathcal{G}} | \eta_1 \ldots \eta_9 |^{-1/2} d \eta.
\]

The surfaces \( A_1 (\eta) = 0 \) and \( B_1 (\eta) = 0 \) are 8-dimensional linear subspaces meeting in a 7-dimensional linear subspace which contains the point \( \chi \) chosen by Lemma 3. Further, \( \chi \) is interior to \( \mathcal{E} \). The set \( \mathcal{E} \cap \mathcal{G} \) will therefore contain a box around \( \chi \) of volume \( D_9 P^{-4} \) for some positive constant \( D_9 \) independent of \( P \). Then
\[
\int_{\mathcal{E} \cap \mathcal{G}} \ldots \int_{\mathcal{G}} | \eta_1 \ldots \eta_9 |^{-1/2} d \eta > C^{-3} D_9 P^{-4}
\]
and the result follows with \( D = 2^{-11} C^{-3} D_9 \).

Lemma 12. If \( | \gamma_i | = O (P^{-32}) \) then
\[
| T (\gamma_i) - J (\gamma_i) | = O (1).
\]

This is Lemma 5 of Davenport and Heilbronn [6].
We take \( U (v) = \{ (a, \beta) : \max (|a|, |\beta|) \leq c P^{-32-2^{-1}} \} \).

Lemma 13.

\[
\int_0 \int_{\mathcal{E} (c)} \int_0 \int_{\mathcal{E} (c)} \int_0 \int_{\mathcal{G}} \int_0 \int_{\mathcal{G}} \int_0 \int_{\mathcal{G}} \ldots \int_{\mathcal{G}} T (\gamma_i) K (a) K (\beta) d \alpha d \beta
\]
\[
= \int_0 \int_{\mathcal{E} (c)} \int_0 \int_{\mathcal{G}} \ldots \int_{\mathcal{G}} (P^3)^{1/2} + o (P^3).
\]

Proof. In \( U (v) \) we have each \( | \gamma_i | = O (P^{-32}) \), so, by Lemma 12,
\[
\int_0 \int_{\mathcal{E} (c)} \int_0 \int_{\mathcal{G}} \ldots \int_{\mathcal{G}} | T (\gamma_i) - J (\gamma_i) | = O (P^3).
\]
Thus the difference between the two integrals in (33) is
\[
\ll P^3 (P^{-32-2^{-1}})^2 = P^{-r_2}.
\]

Collecting together the results of Lemmas 13, 10, 5 and 11, we see that for some positive constant \( D \), independent of \( P \),
\[
\int_0 \int_{\mathcal{E} (c)} \int_0 \int_{\mathcal{G}} \ldots \int_{\mathcal{G}} T (\gamma_i) K (a) K (\beta) d \alpha d \beta \gg D P^3 + o (P^3).
\]

6. The residual integral. We now have to estimate the integral of the exponential sums \( T (\gamma_i) \) over the region
\[
R_0 = \{ (a, \beta) : c P^{-1} < \max (|a|, |\beta|) \leq P^3 \}.
\]

Lemma 14. Suppose that \( | T (\gamma_i) | = P^{-32} \), where \( \theta < \frac{1}{4} - 2 \delta \); then \( \gamma_i \) has a rational approximation \( A_i / Q_i \) such that
\[
| \gamma_i - A_i / Q_i | < P^{-3-2}.
\]

Proof. By Dirichlet's theorem on Diophantine approximation, there exists a rational approximation \( A / Q \) to \( \gamma_i \) such that
\[
1 < Q < P^{1+\delta} \quad \text{and} \quad | \gamma_i - A / Q | < Q^{-1} P^{-3-\delta}.
\]
If \( Q > P^{1+\delta} \) then by Weyl's inequality (Lemma 1 of [4]),
\[
| T (\gamma_i) | < P^{4+\delta},
\]
which gives a contradiction. Thus \( Q \leq P^{1+\delta} \) and so, from the Corollary to Lemma 9 of Birch and Davenport [1],
\[
| T (\gamma_i) | < Q^{-1/2} \min (P, P^{-1} | \beta_i |^{-1}),
\]
where \( \beta_i = \gamma_i - A / Q \). Thus
\[
P^{1-\delta} \ll Q^{-1/2} P
\]
and
\[
P^{1-\delta} \ll Q^{-1/2} P^{1-\delta} P^{-1} | \beta_i |^{-1}
\]
which gives (36).

We recall that for all \( P \in \mathcal{P} (\sigma) \) there are no \( (a, \beta) \in R_0 \) which satisfy (12)–(14). Thus for all \( (a, \beta) \in R_0 \) and \( P \in \mathcal{P} \) we have
\[
\min (\{ | T (\gamma_i) |, | T (\gamma_2) |, | T (\gamma_3) | \}) \ll P^{1-\delta}.
\]
Thus from (37) and Lemma 6,
\[
\int_0 \int_{\mathcal{E} (c)} \int_0 \int_{\mathcal{G}} \ldots \int_{\mathcal{G}} | T (\gamma_i) | K (a) K (\beta) d \alpha d \beta \ll P^{20} P^{1+\delta} P^{1-\delta}
\]
for all \( P \in \mathcal{P} \). Since \( \delta < A / 4 \), the right hand side of (38) is \( o (P^4) \), provided that \( \varepsilon \) is sufficiently small.

7. Completion of the proof of the Theorem. It is sufficient to prove that the normalized inequalities (7) have a non-trivial integer solution. The number \( N (P) \) of integer solutions of (7) in \( \mathcal{B} \) satisfies
\[
N (P) \gg \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0 \int_{\mathcal{E} (c)} \int_0 \int_{\mathcal{G}} \ldots \int_{\mathcal{G}} T (\gamma_i) K (a) K (\beta) d \alpha d \beta.
\]
From (38) and Lemmas 7 and 9 we have
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T(\gamma) K(a) K(\beta) \, da \, db = \int_{0}^{1} \int_{0}^{1} T(\gamma) K(a) K(\beta) \, da \, db + o(P^a),
\]
as \(P \to \infty\) through \(a\). Thus, by (34), for some positive constant \(D\)
\[N(P) \geq DP^a + o(P^a)\]
as \(P \to \infty\) through \(a\) which gives \(N(P) > 0\), and the proof is complete.

References


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Halving an estimate obtained from Selberg's upper bound method
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Introduction. In many applications of Selberg's upper bound method, an unnecessary constant factor appears in the final estimate, due to the fact that we can only sieve up to approximately \(\sqrt{x}\).

At present this restriction seems unavoidable, and arises from the necessity of squaring in order to obtain a non-negative sifting function, viz.

\[s^{+}(n) = \left( \sum_{d \mid n} \chi_{d} \right)^{2}.\]

As an example, let \(K\) be any positive integer whose greatest prime factor does not exceed \(x\). Following van Lint and Richert [1], we arrive at the estimate

\[\sum_{(n, K) = 1} 1 \leq \frac{\phi(K)}{K} x \left( \frac{1}{\log x} + \frac{x^2}{x} \right) \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1},\]

by a careful application of Selberg's method. Choosing \(x\) optimally, Mertens' formula gives

\[\sum_{(n, K) = 1} 1 \leq 2e^{\phi(K)} \frac{\phi(K)}{K} x \left( 1 + O \left( \frac{\log \log x}{\log x} \right) \right).\]

The factor \(e^{\phi}\) really is necessary, as the Prime Number Theorem shows, but apart from the error term, the estimate becomes best possible if we strike out the factor 2 on the right. The object of this note is to obtain a general result of this kind.

Theorem. Let \(f(n)\) be defined on the positive integers and satisfy

\[f(1) = 1, \quad 0 \leq f(n) \leq 1\]

and

\[f(nm) \leq f(n)f(m)\]

provided \((n, m) = 1\).