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The distribution of the fundamental units of real quadratic fields

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In this paper we shall prove the following

THEOREM. *Let $E(x)$ for any $x > 1$ be the number of fundamental units $\varepsilon \leqslant x$ of real quadratic fields $\boldsymbol{K}$, and $E_\delta(x)$ for any fixed δ in the interval $0 \leqslant \delta < \frac{1}{2}$ be the number of such units for $\boldsymbol{K}$ with class number $h_{\boldsymbol{K}}$ and discriminant $D_{\boldsymbol{K}}$ satisfying*

$$h_{\boldsymbol{K}} \leqslant D_{\boldsymbol{K}}^{\delta}. \tag{1}$$

Then

$$E(x) = 2 \sum_{l=1}^{\left[\frac{\log x}{\log 2}\right]} \mu(l) x^{1/l} + O(\log x), \tag{2}$$

where $\mu(l)$ is Möbius function,

$$E_\delta(x) < c_1 (\log x)^{2/(1-2\delta')}, \tag{3}$$

where δ' is any number in the interval $\delta < \delta' < \frac{1}{2}$ and c_1 depends only on δ'.

We see that asymptotic behavior of the functions $E(x)$ and $E_\delta(x)$ differs in an essential manner, and "almost all" quadratic fields have "very large" class numbers. Hence, particularly, "almost no" quadratic fields have one ideal class in every genus.

In the sequel we denote by $N\{\ldots\}$ the number of elements under consideration which satisfy the conditions indicated in the brackets. Thus, for instance,

$$E(x) = N\{\varepsilon\colon \varepsilon \leqslant x\}.$$

Proof of the Theorem. It is possible to get an information on the function $E(x)$ by the following arguments. We observe that every quadratic unit $\eta > 1$ and its conjugate η' satisfy the equation of the form $z^2 - mz \pm 1 = 0$ with positive integral $m = T(\eta) = \eta + \eta'$, $|\eta'| = \eta^{-1}$. We find

$$N\{\eta\colon 1 < \eta \leqslant x\} = \begin{cases} 2x + O(1), \\ E(x) + E(x^{1/2}) + E(x^{1/3}) + \ldots \end{cases}$$

That easily gives

$$E(x) = 2x+O(x^{1/2}),$$

and still sharper estimates are possible, like

$$E(x) = 2x-2x^{1/2}+O(x^{1/3}),$$

and so on. We shall use the same idea expressed in analytic form to get (2).

Let us introduce the functions of complex variable $s=\sigma+it$:

$$Z(s) = \sum_{\eta>1}\eta^{-s} \quad (\sigma>1),$$

$$\zeta_0(s) = \sum_{\eta>1}T^{-s}(\eta) \quad (\sigma>1),$$

where η runs through all real quadratic units $\eta>1$. It may be easily seen that

$$N\{\eta\colon T(\eta)=m\} = \begin{cases}1, & \text{if} \quad m=1 \text{ or } 2,\\ 2, & \text{if} \quad m\geqslant 3.\end{cases}$$

Consequently,

$$\zeta_0(s) = 1+2^{-s}+2\big(\zeta(s)-1-2^{-s}\big) = 2\zeta(s)-1-2^{-s},$$

where $\zeta(s)$ is Riemann zeta-function. Let

(4) $$g(s) = Z(s)-2\big(\zeta(s)-1\big).$$

Hence we obtain

(5) $$g(s) = 1-2^{-s}+\sum_{\eta>1}\big(\eta^{-s}-T^{-s}(\eta)\big) \quad (\sigma>1),$$

and we see that

(6) $$g(s) = O(\eta_1^{-\sigma}) \quad (\sigma\to\infty),$$

where $\eta_1 = \dfrac{1+\sqrt{5}}{2}$ is the absolutely minimal unit in the set of all units $\eta>1$.

Let us introduce now the function

$$e(s) = \sum_{\varepsilon}\varepsilon^{-s} \quad (\sigma>1),$$

where ε runs through all fundamental units of real quadratic fields. Since every unit η is a positive integral power of a fundamental unit ε, we have

$$Z(s) = \sum_{\varepsilon}\varepsilon^{-s}+\sum_{\varepsilon}\varepsilon^{-2s}+\ldots = \sum_{k=1}^{\infty}e(ks) \quad (\sigma>1).$$

Hence we have

$$e(s) = \sum_{l=1}^{\infty}\mu(l)Z(ls) \quad (\sigma>1).$$

Indeed,

$$\sum_{l=1}^{\infty}\mu(l)Z(ls) = \sum_{l,k=1}^{\infty}\mu(l)e(kls) = \sum_{m=1}^{\infty}e(ms)\sum_{l\mid m}\mu(l) = e(s).$$

Using (4), we find

(7) $$e(s) = 2\sum_{l=1}^{\infty}\mu(l)\big(\zeta(ls)-1\big)+G(s),$$

(8) $$G(s) = \sum_{l=1}^{\infty}\mu(l)g(ls) \quad (\sigma>1).$$

Now, to express $E(x)$, we apply the usual integral operator

$$I(y,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}y^s\frac{ds}{s},$$

where $c>0$, $T>0$, $y>0$. We have

$$|I(y,T)-\delta(y)| < y^cT^{-1}|\log y|$$

or the δ-function

$$\delta(y) = \begin{cases}1, & \text{if} \quad y<1,\\ 0, & \text{if} \quad y>1\end{cases}$$

(see [1], pp. 109–110). Assuming from now on that x is half an odd integer, we find

(9) $$E(x) = \frac{1}{2\pi i}\int_{2-iT}^{2+iT}e(s)x^s\frac{ds}{s}+O\left(\frac{x\log x}{T}\right).$$

We have also as an analogue of this

(10)
$$x^{1/l} = \frac{1}{2\pi i}\int_{2-iT}^{2+iT}\big(\zeta(ls)-1\big)x^s\frac{ds}{s}+O\left(\frac{x^{1/l}\log x}{T}\right)+O\left(\frac{x^{1/l}}{T\|x^{1/l}\|}\right)+O(1),$$

where $\|x^{1/l}\|$ is the distance from $x^{1/l}$ to the nearest integer, $l\geqslant 1$ is an integer.

Since

$$\|x^{1/l}\| = |x^{1/l}-m| = \frac{|x-m^l|}{x^{(l-1)/l}\left(1+\dfrac{m}{x^{1/l}}+\ldots+\left(\dfrac{m}{x^{1/l}}\right)^{l-1}\right)} > \frac{\frac{1}{2}}{2^lx^{1-1/l}},$$

we find from (10) for any integer $L \geqslant 1$

$$\sum_{l=1}^{L} \mu(l) \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \big(\zeta(ls)-1\big) x^s \frac{ds}{s} = \sum_{l=1}^{L} \mu(l) x^{1/l} + O\left(\frac{x \log x}{T}\right) + O\left(\frac{2^L x}{T}\right) + O(L).$$

Since $\zeta(ls) - 1 = O(2^{-2l})$ for $l \geqslant L$ with $\sigma = 2$, we have

$$\sum_{l=L+1}^{\infty} \mu(l) \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \big(\zeta(ls)-1\big) x^s \frac{ds}{s} = O\left(\sum_{l=L+1}^{\infty} 2^{-2l} x^2 \int_0^T \frac{dt}{\sqrt{4+t^2}}\right) = O(x^2 2^{-2L} \log T).$$

Taking $L = [\log x/\log 2]$ and $T = x^2$, we have now from (7), (9) and (10)

$$E(x) = 2\sum_{l=1}^{L} \mu(l) x^{1/l} + \frac{1}{2\pi i} \int_{2-iT}^{2+iT} G(s) x^s \frac{ds}{s} + O(\log x).$$

To prove the estimate .

$$\frac{1}{2\pi i} \int_{2-iT}^{2+iT} G(s) x^s \frac{ds}{s} = O(\log x), \tag{11}$$

we remark that (5) gives

$$\frac{1}{2\pi i} \int_{2-iT}^{2+iT} g(ls) x^s \frac{ds}{s} = N\{\eta:\ \eta < x^{1/l}\} - N\{\eta:\ T(\eta) < x^{1/l}\} + $$

$$+ O\left(\frac{x^{1/l} \log x}{lT}\right) + O\left(\frac{x^{1/l}}{T \|x^{1/l}\|_\eta}\right) + O\left(\frac{x^{1/l}}{T \|x^{1/l}\|}\right) + O(1) = O(1),$$

where $\|x^{1/l}\|_\eta$ is the distance from $x^{1/l}$ to the nearest $\eta > 1$. Using (6) and (8) and arguing as described above, we find (11). This completes the proof of the expansion (2) in the case of x being half an odd integer, hence for any $x > 1$.

To prove (3) we observe that the class-number $h = h_K$, the discriminant $D = D_K$ and the fundamental unit $\varepsilon = \varepsilon_K$ of a real quadratic field $\boldsymbol{K}$ satisfy the relation

$$2h \log \varepsilon = D^{1/2} L(1, \chi),$$

where

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n},$$

$\chi = \chi(n)$ is the non-principal primitive real character to the modulus D. Since by Siegel's theorem [2] we have $L(1, \chi) > c_2 D^{-\tau}$ with any $\tau > 0$, it follows from $\varepsilon \leqslant x$ and (1) that

$$D < c_3 (\log x)^{2/(1-2\delta-2\tau)},$$

and we get (3).

It seems that modern theory of the distribution of real zeros of L-functions gives the possibility to prove that

$$\sum_{D<y} \mu^2(D) L^{-\lambda}(1, \chi) < c_4 y (\log y)^\mu,$$

where λ, μ and c_4 are positive constants (μ and c_4 depend on λ). If that is true, the estimate (3) may be improved to

$$E_\delta(x) < c_5 (\log x)^{2/(1-2\delta)} (\log\log x)^\mu,$$

where μ depends only on δ.

References

[1] H. Davenport, *Multiplicative Number Theory*, Chicago 1967.

[2] C. L. Siegel, *Über die Classenzahl quadratischer Zahlkörper*, Acta Arith. 1 (1935), pp. 83–86.

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