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Note on a paper by T. Nagell

by

K. Szymiczek (Katowice)

T. Nagell [1] proved the following

Theorem. *Let d be a square-free positive integer. The equation*

$$x^2+y^2+z^2 = 0 \tag{1}$$

has a non-zero solution in the quadratic field $Q(\sqrt{(-d)})$ if and only if $d \not\equiv -1 \pmod 8$.

He gave two proofs, each of them demanding seperate consideration of all possible residues of d modulo 8.

We will show here how to get the result in a few lines. Our proof is effective, we show explicitly a non-trivial solution of (1) if it exists. One of the two basic arguments will be the identity:

$$(u^2+v^2)(p^2+q^2) = (up+vq)^2+(uq-vp)^2. \tag{2}$$

First show the sufficiency. If $d \not\equiv -1 \pmod 8$ then, by Gauss's classical theorem, d can be written as a sum of three squares of rational integers, $d = a^2+b^2+c^2$, say. Then $(\sqrt{(-d)})^2+a^2+b^2+c^2 = 0$ and multiplying this by b^2+c^2 and using (2) we get the following non-zero solution of (1) in the field $Q(\sqrt{(-d)})$:

$$(ab+c\sqrt{(-d)})^2+(ac-b\sqrt{(-d)})^2+(b^2+c^2)^2 = 0.$$

Now the proof of necessity. If (1) has a non-trivial solution in $Q(\sqrt{(-d)})$ then we may assume that

$$w^2+(u+p\sqrt{(-d)})^2+(v+q\sqrt{(-d)})^2 = 0,$$

where w, u, v, p, q are rational integers. Hence we get $w^2+u^2+v^2 = d(p^2+q^2)$ and $up+vq = 0$. Multiplying the first equality by p^2+q^2 and using (2) we obtain

$$(wp)^2+(wq)^2+(uq-vp)^2 = d(p^2+q^2)^2.$$

Hence by Gauss's theorem $d(p^2+q^2)^2$ is not of the form $4^a(8b+7)$ and consequently $d \not\equiv -1 \pmod{8}$.

We remark here that the above theorem can be used for determining the stufe s of any quadratic field. By definition, the stufe $s = s(k)$ of a field k is the smallest positive integer n such that -1 is a sum of n squares over the field k (or infinity, if such an n does not exist). It was proved first by C. L. Siegel that the stufe of an algebraic number field is always 1, 2, 4 or infinity.

For a non-real quadratic field $Q(\sqrt{(-d)})$ we have

(i) $s = 1$ if and only if $d = 1$,

(ii) $s = 2$ if and only if $1 \neq d \not\equiv -1 \pmod{8}$,

(iii) $s = 4$ if and only if $d \equiv -1 \pmod{8}$.

Here (i) is obvious and (ii) follows from the above theorem. If d is any positive integer then by Lagrange's theorem $d = p^2+q^2+r^2+t^2$ for some rational integers p, q, r, t and so

$$\left(\left(\sqrt{(-d)}\right)/t\right)^2+(p/t)^2+(q/t)^2+(r/t)^2 = -1.$$

Hence for any quadratic field $s \leqslant 4$ and now (i) and (ii) imply (iii).

Reference

[1] T. Nagell, *Sur la résolubilité de l'équation $x^2+y^2+z^2 = 0$ dans un corps quadratique*, Acta Arith. 21 (1972), pp. 35–43.

INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
Katowice

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