converges in distribution, and also in probability. Hence by Kolmogorov's three series criterion (see for example Doob [2], Theorem 2.5, pp. 111–114) we deduce that the following series are convergent:

$$\sum_{|f(p)|>1} \frac{1}{p-1}, \qquad \sum_{|f(p)|\leqslant 1} \frac{f(p)}{p-1}, \qquad \sum_{|f(p)|\leqslant 1} \frac{f^2(p)}{p-1}.$$

This completes the proof of the theorem.

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Received on 15. 6. 1972 (294)

ACTA ARITHMETICA
XXV (1974)

Odd perfect numbers are divisible by at least seven distinct primes

by

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If n is a positive integer, we let $\sigma(n)$ be the sum of the positive divisors of n. n is said to be *perfect* if $\sigma(n) = 2n$. It is well-known that if 2^k-1 is prime, then $2^{k-1}(2^k-1)$ is perfect and that all even perfect numbers are of this form. No odd perfect numbers are known, but neither has any proof of their non-existence ever been discovered.

If n is a positive integer and if $p_1^{a_1}p_2^{a_2}\ldots p_k^{a_k}$ is the unique prime factorization of n, we shall call $p_1^{a_1}, p_2^{a_2}, \ldots, p_k^{a_k}$ the *components* of n.

The modern work on the subject was begun by Sylvester. He proved that an odd perfect number (o.p.n.) has at least five components [16] (also proved by Dickson [4] and Kanold [11]) and that an o.p.n. not divisible by 3 has at least eight components [17]. Sylvester claimed he could prove that an o.p.n. has at least six components [18]. Sylvester [18] and Kanold [9] have been the only researchers on the subject aware of 1.8. However, Sylvester's proof of 1.8 is incorrect. A neat proof of this much-proved theorem may be found in Artin [1]. 1.8 is originally due to Bang [2], Birkhoff–Vandiver [3], and Zsigmondy [21].

Gradstein [6], Kühnel [12], and Webber [20] have each independently proved that an o.p.n. has at least six components. Kanold [10] proved that an o.p.n. not divisible by 3 has at least nine components. Tuckerman [19] proved that any o.p.n. is greater than 10^{36}. Hagis [7] proved that any o.p.n. is greater than 10^{50}. Recently Stubblefield [15] announced he could prove any o.p.n. is greater than 10^{100}.

In this paper, I will prove that any o.p.n. has at least seven components. In light of the result mentioned above by Gradstein, Kühnel, and Webber, all I need prove is that every odd number with exactly six components is not perfect.

* This paper is the author's doctoral dissertation which was submitted in June 1972 and directed by Dr. John Tate of Harvard University.

If n is a positive integer, we let $h(n) = \sigma(n)/n$. We shall see in 1.3 that if $m|n$, $m < n$, then $h(m) < h(n)$. Dickson [4] defined a primitive abundant number n as a number for which $h(n) \geq 2$ and whenever $m|n$, $m < n$ then $h(m) < 2$. It is clear then that any perfect number is also primitive abundant. Dickson went on to prove that for any k there are only finitely many odd primitive abundant numbers with precisely k components. Then Dickson proved the corollary (also proved by Gradstein [6]) that for any k there are at most finitely many o.p.n.'s with precisely k components. Hence a potential search procedure for o.p.n.'s is to list all of the odd primitive abundant numbers with a given number of components and check each one to see if it is perfect. Dickson [4] employed this method to show there were not any four component o.p.n.'s. Even though it is theoretically a simple procedure to locate all of the odd primitive abundant numbers with k components, when k is large (say ≥ 6), there are so many of them and the primes involved are so large that even computer techniques would be impractical.

We look then for alternative methods in examining this finite but large set. Among these methods are results that go back to Euler [5] and new results proved here for the first time. One class of possible theorems about o.p.n.'s is:

(*) An o.p.n. is divisible by j distinct primes $> N$.

Kanold [11] proved this result for $j = 1$ and $N = 60$. His proof is short and elementary and we use the result in the present paper. Since this paper was written, Hagis and McDaniel [8] announced they had proven the result for $j = 1$, $N = 11200$. Their proof involved extensive work on a computer. Since the present paper relies only marginally on a few computer factorizations due to Tuckerman [19] and since the Hagis–McDaniel result would shorten the present proof only slightly, it was decided not to rewrite the paper incorporating this result.

A result that would be of significant help would be to establish (*) for some $j \geq 2$ for even a relatively small N, say 500. The principle motivation behind Section 3 in this paper is to prove (*) for $j = 2$ and $N = 1000$ in a special case (cf. 5.1, 5.2).

The main result proved in this paper has been independently and simultaneously obtained by Robbins [14]. His proof is quite similar to this proof, the chief difference being the results obtained here in Section 3. I wish to thank Dr. Robbins for advising me of the factorization of $\sigma(3^{24})$, due to Muskat, which saved some lines in 4.6. Also Dr. Robbins persuaded me to incorporate Kanold's result (1.15), and he advised me of the similarity of my 1.5 to a result of Gradstein [6].

1. Preliminary results

1.1. Euler [5] proved that if n is an o.p.n. then n can be written in the form $p_1^{a_1}p_2^{a_2}\ldots p_k^{a_k}$ where $p_1, p_2, \ldots, p_k$ are distinct odd primes, $p_1 \equiv a_1 \equiv 1\ (4)$, and $a_2 \equiv \ldots \equiv a_k \equiv 0\ (2)$. We shall call p_1 the *special prime*, usually denoting it by π and its exponent by m. If p is a prime divisor of an o.p.n. we shall sometimes denote its exponent by $\exp p$. This will not be confused with the usual notation $\exp x = e^x$.

1.2. We shall denote by F_d the dth cyclotomic polynomial. If p is a prime and a is a positive integer, then

$$\sigma(p^a) = p^a + p^{a-1} + \ldots + 1 = \frac{p^{a+1}-1}{p-1} = \prod_{\substack{d|a+1\\ d>1}} F_d(p).$$

Hence if $n = \prod_{i=1}^{k} p_i^{a_i}$ is an o.p.n. then

$$2\prod_{i=1}^{k} p_i^{a_i} = \sigma\Big(\prod_{i=1}^{k} p_i^{a_i}\Big) = \prod_{i=1}^{k}\sigma(p_i^{a_i}) = \prod_{i=1}^{k}\prod_{\substack{d|a_i+1\\ d>1}} F_d(p_i).$$

Then for each $p_i|n$, there is a $p_j|n$ and a $d > 1$, $d|a_j+1$ such that $p_i|F_d(p_j)$. Furthermore if $p_i|n$ is non-special, then for each $d > 1$, $d|a_i+1$ we have $F_d(p_i)|n$; and if p_i is the special prime then for each $d > 2$, $d|a_i+1$ we have $F_d(p_i)|n$ and $\frac{1}{2}F_2(p_i)|n$. We note that $\frac{1}{2}F_2(p_i) = \dfrac{p_i+1}{2}$. That is, if π is the special prime of an o.p.n. n, then $\dfrac{\pi+1}{2}\Big| n$.

1.3. We have already mentioned the multiplicative function $h(n) = \dfrac{\sigma(n)}{n}$ in the introduction. If p is a prime then $h(p^a) = 1 + p^{-1} + \ldots + p^{-a}$ increases with a and $\lim_{a\to\infty} h(p^a) = \dfrac{p}{p-1}$. Hence we shall write $h(p^\infty) = h(\overline{p}) = \dfrac{p}{p-1}$. h is multiplicative in this extended sense, so if $x = p_1^{a_1}\ldots p_k^{a_k}$ where $p_1, \ldots, p_k$ are distinct primes and $a_1, \ldots, a_k$ are non-negative integers or ∞, then

$$h(x) = h(p_1^{a_1})\ldots h(p_k^{a_k}).$$

If $p > q$ are odd primes, a is a non-negative integer or ∞, and b is a positive integer or ∞, then

$$h(p^a) \leq h(\overline{p}) = \frac{p}{p-1} < \frac{q+1}{q} = h(q) \leq h(q^b).$$

These remarks yield the following general result:

Suppose $x = p_1^{a_1} \dots p_k^{a_k}$ *where* $p_1, \dots, p_k$ *are distinct odd primes,* $a_1, \dots, a_k$ *are non-negative integers or* ∞, $y = q_1^{b_1} \dots q_k^{b_k}$ *where* $q_1, \dots, q_k$ *are distinct odd primes, and* $b_1, \dots, b_k$ *are non-negative integers or* ∞. *Furthermore suppose*

1) *if* $p_i = q_i$ *then* $a_i \leqslant b_i$,
2) *if* $p_i \neq q_i$ *then* $p_i > q_i$ *and* $b_i \neq 0$.

Then $h(x) \leqslant h(y)$ *where equality holds only if* $x = y$. (What we mean by $x = y$ is that for each i, $p_i = q_i$ and $a_i = b_i$.)

In the above notation, if $h(y) < 2$, we define $g(y) = \frac{2}{2-h(y)}$. It then follows that $g(x) \leqslant g(y)$.

1.4. In this section we shall prove two lemmas concerning the function g.

Let $x = p_1^{a_1} \dots p_k^{a_k}$, $y = xq_1^{b_1} \dots q_m^{b_m}$ *where* $p_1, \dots, p_k, q_1, \dots, q_m$ *are distinct primes,* $a_1, \dots, a_k$ *are non-negative integers or* ∞, *and* $b_1, \dots, b_m$ *are non-negative integers. Further suppose that* $h(x) < 2 \leqslant h(y)$. *Then if* $q = \min\{q_1, \dots, q_m\}$, *then* $q < mg(x)$.

Proof. 1.3 implies

$$2 \leqslant h(y) = h(x) \prod_{i=1}^{m} h(q_i^{b_i}) < h(x) \left(\frac{q}{q-1}\right)^m.$$

Let $H = h(x)$. Then $2H^{-1} < \left(\frac{q}{q-1}\right)^m$, so $(2H^{-1})^{1/m} < \frac{q}{q-1}$ and

$$q < \frac{(2H^{-1})^{1/m}}{(2H^{-1})^{1/m}-1} = \frac{2H^{-1} + (2H^{-1})^{(m-1)/m} + \dots + (2H^{-1})^{1/m}}{2H^{-1}-1}$$
$$= \frac{2 + 2^{(m-1)/m}H^{1/m} + \dots + 2^{1/m}H^{(m-1)/m}}{2-H} \leqslant \frac{2m}{2-H}$$

since $H < 2$. But $\frac{2m}{2-H} = mg(x)$. ■

For our second lemma, *suppose* m *is a positive integer,* $h(m) < 2$, *and* p *is a prime with* $p \nmid m$ *and* $p < g(m) - 1$. *Then* $h(mp) > 2$.

Proof. $h(mp) = h(m)h(p) = h(m)\frac{p+1}{p} > h(m)\frac{g(m)}{g(m)-1} = 2$. ■

1.5. The following result is similar to Theorem V in Gradstein [6]. However there are several misprints in his statement and proof.

Let $x = p_1^{a_1} \dots p_k^{a_k}$, $y = q_1^{b_1} \dots q_m^{b_m}$, $z = q_1^{\infty} \dots q_m^{\infty}$ *where* $p_1, \dots, p_k, q_1, \dots, q_m$ *are distinct primes and* $a_1, \dots, a_k, b_1, \dots, b_m$ *are positive integers. Then*

$$h(xy) \geqslant h(xz) - h(xz) \sum_{i=1}^{m} \frac{1}{q_i^{b_i+1}}.$$

Proof. We make use of the following well-known inequality: if $0 < e_i < 1$ for $i = 1, \dots, m$, then

$$\prod_{i=1}^{m} (1-e_i) \geqslant 1 - \sum_{i=1}^{m} e_i.$$

This inequality may be proved easily by an induction argument on m. Now

$$1 - \frac{1}{q^{b+1}} = \frac{q^{b+1}-1}{q^{b+1}} = \frac{h(q^b)}{h(\overline{q})} \quad \text{for any prime } q.$$

Hence

$$\frac{h(y)}{h(z)} = \prod_{i=1}^{m} \frac{h(q_i^{b_i})}{h(\overline{q}_i)} = \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{b_i+1}}\right) \geqslant 1 - \sum_{i=1}^{m} \frac{1}{q_i^{b_i+1}},$$

$$h(y) \geqslant h(z) - h(z) \sum_{i=1}^{m} \frac{1}{q_i^{b_i+1}},$$

and our conclusion follows since h is multiplicative. ■

We make use of this lemma in the following situation: we would like to prove $h(xy) > 2$, but this is too difficult to evaluate. Instead we find $h(xz) > 2 + \varepsilon$ and then use the lemma to show $h(xy) > 2$ provided ε is large enough and $\sum_{i=1}^{m} \frac{1}{q_i^{b_i+1}}$ is small enough.

1.6.

$$v_q(\sigma(p^a)) = \begin{cases} v_q(p^{o_q(p)}-1) + v_q(a+1), & \textit{if} \quad o_q(p) \mid a+1 \textit{ and } o_q(p) > 1, \\ v_q(a+1), & \textit{if} \quad o_q(p) = 1, \\ 0, & \textit{otherwise}. \end{cases}$$

Here p and q are distinct primes, $q \geqslant 3$, v_q is the q-valuation and $o_q(p)$ is the order of $p \bmod q$. This result follows easily from Theorems 94 and 95 in Nagell [13] (pp. 164–166) when we notice that

$$\sigma(p^a) = \frac{p^{a+1}-1}{p-1} = \prod_{\substack{d \mid a+1 \\ d>1}} F_d(p).$$

We also remark that if q is a prime, $q \nmid k$, then $o_q(k) \mid q-1$.

1.7. *If* q *is a Fermat prime (i.e., a prime* 1 *greater than a power of* 2) *and if* p^a *is a component of an o.p.n., then*

$$v_q(\sigma(p^a)) = \begin{cases} v_q(a+1), & \textit{if } p \equiv 1\ (q), \\ v_q(p+1) + v_q(a+1), & \textit{if } p \equiv -1\ (q) \textit{ and } p \textit{ is the special prime}, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. This follows from 1.1, 1.6, and the fact that $o_q(p) \mid q-1$, a power of 2. ■

COROLLARY. *If a Fermat prime q divides an o.p.n., then so does a prime $\equiv \pm 1\ (q)$.*

1.8. (Bang [2], Birkhoff–Vandiver [3], Sylvester [18], Zsigmondy [21].) *$F_d(m)$ is divisible by a prime p with $o_p(m) = d$ whenever m is an integer $\geqslant 2$ and d is an integer $\geqslant 1$, except for $m = 2$ and $d = 1$ or 6 and for m a Mersenne number and $d = 2$.* (A *Mersenne number* is a number 1 less than a power of 2.)

Note. For our purposes, m will always be an odd prime. Also whenever $d = 2$ we will have $m \equiv 1\ (4)$, so m will not be a Mersenne number. Hence the exceptional cases will be irrelevant.

1.9. *If p^a is a component of an o.p.n., and if $d \mid a+1$, $d > 1$, then some prime q divides this o.p.n. with $o_q(p) = d$. In particular $q \equiv 1\ (d)$.*

Proof. This result follows when 1.8 is applied to 1.2. ∎

1.10. *If n is a positive integer and p is a prime, let $\omega(n, p) =$ the number of distinct primes q dividing n such that $q \neq p$, $p \not\equiv \pm 1\ (q)$, and q is not Fermat. Let $\tau(n) =$ the number of distinct positive divisors of n. Suppose p^a is a component of an o.p.n. n. Then*

1) if p^a is non-special, then $\tau(a+1) - 1 \leqslant \omega(n, p)$,

2) if p^a is special, then $\tau(a+1) - 2 \leqslant \omega(n, p)$.

Proof. This follows when 1.9 is applied to 1.2 and 1.7. ∎

1.11. Let p^a be a non-special component of a 6 component o.p.n. n. Let q be a prime dividing n such that $p \equiv 1\ (q)$. Then if $v_q(\sigma(p^a)) = k$, then 1.6 implies $v_q(a+1) = k$. (We note that the assumptions q is Fermat and $k > 0$ would force $p \equiv 1\ (q)$ by 1.7.) Then 1.10 implies $k \leqslant 4$, where if $k \geqslant 2$, then $a+1 = q^k$. Furthermore 1.9 implies that n is divisible by distinct primes $p_1, \ldots, p_k$ different from p such that $p_i \equiv 1\ (q^i)$ for $1 \leqslant i \leqslant k$. Also n is divisible by $F_q(p), \ldots, F_{q^k}(p)$.

Let π^m be the special component of a 6 component o.p.n. n. Let q be a prime dividing n such that $\pi \equiv \pm 1\ (q)$. Suppose $v_q(\sigma(\pi^m)) = k > 0$ and $v_q(m+1) = j$. Then 1.6 implies $v_q(\pi+1) = k-j$. (We note that the assumptions that q is Fermat and $k > 0$ would force $\pi \equiv \pm 1\ (q)$ by 1.7.) Then 1.10 and the fact that $2 \mid m+1$ (1.1) imply $j \leqslant 2$ where if $j > 0$, then $m+1 = 2q^j$. If $j = 2$ then 1.9 implies n is divisible by distinct primes p_1, p_1', p_2, p_2' different from π such that $p_1 \equiv p_1' \equiv 1\ (q)$ and $p_2 \equiv p_2' \equiv 1\ (q^2)$. If $j = 1$ then only the p_1 and p_1' must occur. In either case, n is divisible by $F_{q^i}(\pi)$ and $F_{2q^i}(\pi)$ for $1 \leqslant i \leqslant j$.

1.12. *If 5^a is a component of an o.p.n. n with special component π^m, and if either $5^a \nmid \sigma(\pi^m)$ or if $\pi \equiv 1\ (5)$, then at least 2 primes divide n which are $\equiv 1\ (5)$ and one of them is $\geqslant 1381$.* (We note that if $\pi = 5$, then certainly $5^a \nmid \sigma(\pi^m)$.)

Proof. Since 5 is a Fermat prime, 1.7 and the conditions of the statement imply there is a component q^b of n with $q \equiv 1\ (5)$, $5 \mid \sigma(q^b)$, and $5 \mid b+1$. Then 1.2 implies $F_5(q) \mid n$, and 1.8 implies there is a prime $r \mid F_5(q)$ with $r \equiv 1\ (5)$. If $q \geqslant 1381$, the statement clearly follows since $r \neq q$. But for each choice of $q \equiv 1\ (5)$, $q < 1381$, Column H of Table 2 and 1.8 show that there is an $r \mid F_5(q)$ with $r \equiv 1\ (5)$ and $r \geqslant 1381$. ∎

1.13. *If 17^a is a component of an o.p.n. n with special component π^m, and if $17^a \nmid \pi+1$, then 2 primes divide n which are $\equiv 1\ (17)$. One of these primes is $\geqslant 103$, the other is $\geqslant 137$.*

Proof. Since 103 and 137 are the two smallest primes $\equiv 1\ (17)$ all we need show is that two primes $\equiv 1\ (17)$ occur. Now if $17^a \mid \sigma(\pi^m)$, then since $17^a \nmid \pi+1$ and 17 is Fermat, in the notation of 1.11 we have $v_{17}(m+1) = j > 0$ and so our conclusion follows from 1.11. If $17^a \nmid \sigma(\pi^m)$, then there is a non-special component q^b of n such that $17 \mid \sigma(q^b)$. Again our conclusion follows from 1.11. ∎

1.14. The following remarks are often useful:

If π is the special prime of an o.p.n. and k is an odd divisor of $\pi+1$, then $\pi \equiv 2k-1\ (4k)$. Indeed, $(4, k) = 1$, $\pi \equiv -1\ (k)$, and $\pi \equiv 1\ (4)$.

If p^a is a non-special component of an o.p.n. and if q is a prime divisor of $\sigma(p^a)$, then $o_q(p)$ is an odd divisor of $q-1$. Indeed, $o_q(p) \mid (a+1, q-1)$ (1.6) and $a+1$ is odd (1.1). If π^m is the special component of an o.p.n. and q is a prime divisor of $\sigma(\pi^m)$, then $o_q(\pi)$ is either an odd or singly even divisor of $q-1$. Indeed, $o_q(\pi) \mid (m+1, q-1)$ (1.6) and $m+1$ is singly even (1.1).

1.15. (Kanold [11].) *If n is an o.p.n. divisible by 3, then there is a prime $p \mid n$ such that $p \geqslant 61$.*

Note. It is not hard to show that the assumption $3 \mid n$ is superfluous, but we shall not need the result in this form.

1.16. *If p and q are primes, k is a positive integer, and $p \mid F_q(k)$, then $o_p(k) = 1$ or q. Furthermore $o_p(k) = 1$ if and only if $k \equiv 1\ (p)$ if and only if $p = q$.*

Proof. These facts follow from Theorems 94 and 95 in Nagell [13]. ∎

In the remainder of this paper, we will assume that an o.p.n. n with precisely 6 components exists.

2. 3 is the smallest prime occurring; the second smallest prime occurring is 5 or 7

2.1. *$3 \mid n$ and the second smallest prime occurring is 5, 7, or 11.*

Proof. If $3 \nmid n$, then 1.3 implies $h(n) < h(5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19) < 2$, a contradiction. If the second smallest prime occurring is $\geqslant 13$, then again by 1.3, we have $h(n) < h(3 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29) < 2$, a contradiction. ∎

In the remainder of this section we assume our o.p.n. $n = 3^a 11^b p^c q^d r^e s^f$ where $11 < p < q < r < s$ are primes.

2.2. $p = 13$ *and* $q = 17$ *or* 19.

Proof. If $p > 13$, then 1.3 and 1.15 imply

$$h(n) < h(\overline{3}\cdot\overline{11}\cdot\overline{17}\cdot\overline{19}\cdot\overline{23}\cdot\overline{61}) < 2,$$

a contradiction. Hence $p = 13$. If $q > 19$, then 1.3 implies

$$h(n) < h(\overline{3}\cdot\overline{11}\cdot\overline{13}\cdot\overline{23}\cdot\overline{29}\cdot\overline{31}) < 2,$$

a contradiction. ∎

2.3. $q = 17$.

Proof. Suppose not. Then we have just seen that $q = 19$. But $g(\overline{3}\cdot\overline{11}\cdot\overline{13}\cdot\overline{19}) < 18$, $g(\overline{3}\cdot\overline{11}\cdot\overline{13}\cdot\overline{19}\cdot\overline{23}) < 73$ so 1.4 implies $r < 36$ and $s < 73$. But 1.15 implies $s \geqslant 61$. If π is the special prime then 1.2 implies $\frac{\pi+1}{2}|n$, so $\pi = 37$, 53, or 61. Hence $\pi = s = 61$ and 31 also divides n. But 1.3 implies $h(n) < h(\overline{3}\cdot\overline{11}\cdot\overline{13}\cdot\overline{19}\cdot\overline{31}\cdot\overline{61}) < 2$, a contradiction. ∎

2.4. Here we conclude our proof that *the second smallest prime occurring is* 5 *or* 7.

Proof. If not, we have seen that $n = 3^a 11^b 13^c 17^d r^e s^f$ where $17 < r < s$ are primes. Since $g(\overline{3}\cdot\overline{11}\cdot\overline{13}\cdot\overline{17}) < 20$, 1.4 implies $r < 40$. Hence 1.13 implies $17^d | \pi+1$ where π is the special prime, which implies 17 is not special and d is even. Hence 1.14 implies $\pi \equiv 577$ (1156), so $\pi = s \geqslant 577$.

Now $r = 19$ for if not, $g(\overline{3}\cdot\overline{11}\cdot\overline{13}\cdot\overline{17}\cdot\overline{23}) < 139$ implies (1.4) that $s < 139$, contradicting $s \geqslant 577$. Similarly $a \geqslant 6$ since $g(3^2\cdot\overline{11}\cdot\overline{13}\cdot\overline{17}\cdot\overline{19}) < g(3^4\cdot\overline{11}\cdot\overline{13}\cdot\overline{17}\cdot\overline{19}) < 569$. Since $7|\sigma(11^2)$, $5|\sigma(11^4)$, we have $b \geqslant 6$. Also since 13, 17, and 19 are not special, we have $c \geqslant 2$, $d \geqslant 2$, $e \geqslant 2$. Now $h(\overline{3}\cdot\overline{11}\cdot\overline{13}\cdot\overline{17}\cdot\overline{19}) > 2.0047$, so 1.3 and 1.5 imply

$$h(n) > h(3^6\cdot 11^6\cdot 13^2\cdot 17^2\cdot 19^2)$$

$$> 2.0047 - 2.0047\left(\frac{7}{10^4} + \frac{1}{10^7} + \frac{5}{10^4} + \frac{3}{10^4} + \frac{2}{10^4}\right) > 2. \ ∎$$

3. Two important lemmas. A prime p is said to have *property A* if either $F_p(3)$ is divisible by some prime $q > 1000$, $q \not\equiv 17$ (36) or if $F_p(3)$ is divisible by two primes $q_1, q_2 > 1000$. (We note that $k \equiv 17$ (36) if and only if $k \equiv 1$ (4) and $k \equiv -1$ (9).)

A prime p is said to have *property B* if either $F_p(5)$ is divisible by some prime $q > 1000$, $q \not\equiv 49$ (100) or if $F_p(5)$ is divisible by two primes q_1, $q_2 > 1000$. ($k \equiv 49$ (100) if and only if $k \equiv 1$ (4) and $k \equiv -1$ (25).)

In this section we shall prove (cf. 3.11, 3.12) that every prime $p \geqslant 7$ has both properties A and B except for 359 which might not have property B.

3.1. *If* $p > 500$, *then* p *has both property A and property B.*

Proof. Let p be a prime > 500. Then if $q|F_p(3)$, 1.16 implies $o_q(3) = p$ and we have $q \equiv 1$ (p) which implies $q \geqslant 2p+1 > 1000$. That is, every prime divisor of $F_p(3)$ is > 1000. Hence if $F_p(3)$ is divisible by two distinct primes then clearly p has property A. But if $F_p(3) = q^a$ where $a \geqslant 1$, then $q^a \equiv 4$ (9) which implies $q \not\equiv -1$ (9) so $q \not\equiv 17$ (36).

The proof that p has property B is similar. ∎

3.2. *If* $F_p(3)$ $(F_p(5))$ *has no prime factors* < 1000, *then* p *has property A (property B).*

Proof. This was essentially proven in 3.1. ∎

3.3. *Suppose* $F_p(3)$ *is divisible by no prime* < 1000 *except for* r *and* $r \equiv 3$ (4) *with the exponent* a *of* r *being odd. Then* p *has property A.*

Proof. For any positive integer k, $(3^k)^2 + 3(3^{k-1})^2 \equiv 0$ (4). Suppose p has the above conditions. Then letting $b = (p-1)/2$, we have

$$F_p(3) = [(3^b)^2 + 3(3^{b-1})^2] + [(3^{b-1})^2 + 3(3^{b-2})^2] + \ldots + [3^2+3] + 1 \equiv 1 \ (4).$$

But $r^a \equiv 3$ (4), so $r^{-a}F_p(3) \equiv 3$ (4) which implies some prime $q \equiv 3$ (4) divides $r^{-a}F_p(3)$. Then $q > 1000$, $q \not\equiv 17$ (36), so p has property A. ∎

3.4. *Suppose* $F_p(3)$ *is divisible by no prime* < 1000 *except for* r *with exponent* a *such that* $r^a \not\equiv 4$ *or* 5 (9). *Then* p *has property A.*

Proof. Suppose p has the above conditions. Since $F_p(3) \equiv 4$ (9), we have $r^{-a}F_p(3) \not\equiv \pm 1$ (9). Therefore some prime $q \not\equiv \pm 1$ (9) divides $r^{-a}F_p(3)$. But then $q > 1000$, $q \not\equiv 17$ (36), so p has property A. ∎

Corollary. *If* $F_p(3)$ *is divisible by no prime* < 1000 *except* **possibly** *for* r *and* $r \equiv \pm 1$ (9), *then* p *has property A regardless of whether* r *divides* $F_p(3)$ *or whatever the exponent of* r *is.*

3.5. *Suppose* $F_p(5)$ *is divisible by no prime* < 1000 *except for* r *with exponent* a *and* $pr^a \equiv 3$ (4). *Then* p *has property B.*

Proof. Suppose p has the above conditions. Then since $F_p(5) \equiv p$ (4), we have $r^{-a}F_p(5) \equiv r^a F_p(5) \equiv r^a p \equiv 3$ (4). Hence some prime $q \equiv 3$ (4) divides $r^{-a}F_p(5)$. Then $q > 1000$, $q \not\equiv 49$ (100), so p has property B. ∎

Corollary. *If* $p \equiv 3$ (4) *and if* $F_p(5)$ *is divisible by no prime* < 1000 *except possibly for* $r \equiv 1$ (4), *then* p *has property B.*

3.6. *Suppose* $F_p(5)$ *is divisible by no prime* < 1000 *except for* r *with exponent* a *and* $r^a \not\equiv 6$ *or* 19 (25). *Then* p *has property B.*

Proof. Suppose p has the above conditions. Since $F_p(5) \equiv 6$ (25), we have $r^{-a}F_p(5) \not\equiv \pm 1$ (25). Hence some prime q divides $r^{-a}F_p(5)$ with $q \not\equiv \pm 1$ (25). Then $q > 1000$, $q \not\equiv 49$ (100), so p has property B. ∎

COROLLARY. *If $F_p(5)$ is divisible by no prime < 1000 except possibly for $r = 1, 7, 18$, or $24\ (25)$, then p has property B.*

Proof. Modulo 25, the set $\{1, 7, 18, 24\}$ is closed under multiplication. Hence $r^a \neq 6$ or $19\ (25)$. ∎

3.7. *Suppose $p \neq q$ are primes, $p \neq 2$. If $q \mid F_p(m)$, where m is an integer, then $q \equiv 1\ (p)$ and $\left(\frac{m}{q}\right) = 1$. Conversely, if $\left(\frac{m}{q}\right) = 1$, $m \not\equiv 1\ (q)$, and $q = 2p+1$, then $q \mid F_p(m)$.*

Proof. Suppose $p \neq q$ are primes, $p \neq 2$. If $q \mid F_p(m)$ then 1.16 implies $o_q(m) = p$ which implies $q \equiv 1\ (p)$. Now $F_p(m) = \frac{m^p-1}{m-1}$ and so $q \mid m^p - 1$. Then $1 = \left(\frac{m^p}{q}\right) = \left(\frac{m}{q}\right)^p = \left(\frac{m}{q}\right)$ since p is odd.

Suppose now $\left(\frac{m}{q}\right) = 1$, $m \not\equiv 1\ (q)$, and $q = 2p+1$. Then some y exists with $y^2 \equiv m\ (q)$. Then $1 \equiv y^{q-1} \equiv y^{2p} \equiv m^p\ (q)$, so $q \mid m^p - 1$. But $q \nmid m-1$, so $q \mid \frac{m^p-1}{m-1} = F_p(m)$. ∎

3.8. *Suppose p, q are primes, $p > 3$. If $q \mid F_p(3)$, then $q \equiv 1\ (p)$, $q \equiv \pm 1\ (12)$, and $\frac{q-1}{p} \equiv 0, 2$, or $10\ (12)$. If $q = 2p+1$, then $q \mid F_p(3)$.*

Proof. Suppose p, q are primes, $p > 3$. Suppose $q \mid F_p(3)$. Since p is odd we have $F_p(3)$ odd, so q is odd. Hence since $o_q(3) \neq 1$ ($3 \not\equiv 1\ (q)$), 1.16 implies $q \neq p$. Hence applying 3.7 we have $q \equiv 1\ (p)$ and $\left(\frac{3}{q}\right) = 1$. Now if $q \equiv 1\ (4)$ then $1 = \left(\frac{3}{q}\right) = \left(\frac{q}{3}\right)$ which implies $q \equiv 1\ (3)$ and so $q \equiv 1\ (12)$. If $q \equiv 3\ (4)$ then $1 = \left(\frac{3}{q}\right) = -\left(\frac{q}{3}\right)$ which implies $q \equiv 2\ (3)$ and so $q \equiv -1\ (12)$. The fact that $\frac{q-1}{p} \equiv 0, 2$, or $10\ (12)$ follows from the fact that $p \mid q-1$, $q \equiv \pm 1\ (12)$, and $p \equiv \pm 1$ or $\pm 5\ (12)$.

Suppose now $q = 2p+1$. Since q is prime, we have $p \equiv 2\ (3)$ and hence $q \equiv 2\ (3)$. Then $\left(\frac{q}{3}\right) = -1$. But $q \equiv 3\ (4)$, so $\left(\frac{3}{q}\right) = 1$ and we may apply 3.7. ∎

3.9. *Suppose p, q are primes, $p > 2$. If $q \mid F_p(5)$, then $q \equiv 1\ (p)$ and $q \equiv \pm 1\ (10)$. If $p \equiv 9\ (10)$ and $q = 2p+1$, then $q \mid F_p(5)$.*

Proof. Suppose p, q are primes, $p > 2$. If $q \mid F_p(5)$, then as in 3.8, $q \neq p$. Now 3.7 implies $q \equiv 1\ (p)$ and $\left(\frac{5}{q}\right) = 1$. Then $1 = \left(\frac{5}{q}\right) = \left(\frac{q}{5}\right)$ which implies $q \equiv \pm 1\ (5)$, so $q \equiv \pm 1\ (10)$.

If $p \equiv 9\ (10)$ and $q = 2p+1$ then $q \equiv 9\ (10)$ and $\left(\frac{5}{q}\right) = \left(\frac{q}{5}\right) = \left(\frac{4}{5}\right) = 1$. Hence 3.7 implies $q \mid F_p(5)$. ∎

3.10. *Every prime $p \geqslant 7$ which does not appear in column I of Table 3 has both property A and property B.*

Proof. An examination of Table 4 shows that for $7 \leqslant p \leqslant 37$, p has property A, and for $7 \leqslant p \leqslant 23$, p has property B. 3.1 implies we need not worry if $p > 500$. Now every other prime $p \geqslant 7$ missing from column I of Table 3 has the property that there is no prime $q < 1000$ with $q \equiv 1\ (p)$ and either $q \equiv \pm 1\ (12)$ or $q \equiv \pm 1\ (10)$. Hence 3.8 and 3.9 imply respectively that $F_p(3)$, $F_p(5)$ have no prime divisors < 1000. Hence 3.2 completes the proof. ∎

3.11. *Every prime $p \geqslant 7$ has property A.*

Proof. 3.2, 3.8 and 3.10 imply we need only look at those primes p in column I of Table 3 for which either the J or K entry contains a prime with exponent $\neq 0$. Using 3.3 we deduce that 41, 83, 113, 131, 173, 191, 239, 281, 293, 419, 443, and 491 have property A. Using the corollary to 3.4 we deduce that the rest of the primes in column I have property A. ∎

3.12. *Every prime $p \geqslant 7$ except possibly 359 has property B.*

Proof. 3.2, 3.9, and 3.10 imply we need only look at those primes p in column I of Table 3 for which either the L or M entry contains a prime with exponent $\neq 0$. Using 3.6 and its corollary, we deduce that 29, 37, 83, 89, 97, 179, 239, and 419 have property B. Using 3.5 and its corollary we deduce that the rest of the primes in column I except possibly 359 have property B.

3.13. *$F_{359}(5)$ is divisible by some prime $q > 1000$ with $5^4 \nmid q+1$.*

Proof. Table 3 shows that $719^{-1} F_{359}(5)$ is divisible by no prime < 1000. Now $719 \equiv 94\ (5^4)$ and $F_{359}(5) \equiv 156\ (5^4)$. Hence $719^{-1} F_{359}(5) \not\equiv \pm 1\ (5^4)$. Hence some prime $q \mid 719^{-1} F_{359}(5)$ with $5^4 \nmid q+1$. As we have already noted that $q > 1000$, the proof is complete. ∎

4. 5 is the second smallest prime occurring. Throughout this section we shall assume the second smallest prime occurring is not 5 and hence, in light of Section 2, the second smallest prime occurring is 7. We shall assume that we have an o.p.n. $n = 3^a 7^b p^c q^d r^e s^f$, where $7 < p < q < r < s$ are primes.

4.1. $p = 11$ *or* 13.

Proof. Assume $p > 13$. Since $13 \mid F_3(3)$, $11 \mid F_5(3)$ we have $3, 5 \nmid a+1$. Section 3 tells us that some prime > 1000 occurs as a divisor of $\sigma(3^a) = \frac{3^{a+1}-1}{3-1}$. Then $p \leqslant 19$, for if not, $h(n) < h(\overline{3}\cdot\overline{7}\cdot\overline{23}\cdot\overline{29}\cdot\overline{31}\cdot\overline{1009}) < 2$ noting that 1009 is the smallest prime > 1000).

Suppose $p = 19$. Then $q = 23$, $r = 29$, for if not, $h(n) < h(\overline{3}\cdot\overline{7}\cdot\overline{19}\cdot\overline{23}\cdot\overline{31}\cdot\overline{1009}) < 2$. Hence $\pi = s > 1000$ (recalling that π stands for the special prime) since the only other prime occurring that is $\equiv 1\ (4)$ is 29 and $5 \mid F_2(29)$. Hence $3 \nmid e+1$ for if not $127 \mid F_3(19)$ would occur. Also $9 \nmid b+1$ for if not $37 \mid F_9(7)$ would occur. Using 1.7 and the fact that $a \geqslant 6$, we have $3^5 \mid \sigma(s^f)$. Using 1.11 and the fact that $s = \pi$, we have $3^3 \mid s+1$. But from Section 3, we may assume our prime > 1000 is either $\not\equiv 1\ (4)$ or $\not\equiv -1\ (9)$. Hence we have too many primes occurring and $p \neq 19$.

Suppose $p = 17$. 1.4 and the fact that $g(\overline{3}\cdot\overline{7}\cdot\overline{17}\cdot\overline{19}\cdot\overline{1009}) < 57$ imply that only one prime > 57 occurs. Hence 1.13 implies 17 is not special. In fact 1.13 implies that $17^c \mid \pi+1$, and since no prime < 57 has this property we must have $\pi = s$. But then $9 \nmid b+1$ for $1063 \mid F_9(7)$ would occur and $1063 \not\equiv 1\ (4)$. Since q and r are both $\geqslant 19$ and < 57, then neither $3 \mid \sigma(q^d)$ nor $3 \mid \sigma(r^e)$ for if not $127 \mid F_3(19)$, $331 \mid F_3(31)$, $67 \mid F_3(37)$, or $631 \mid F_3(43)$ would occur. Hence, as in the preceding paragraph, $a \geqslant 6$, $3^5 \mid \sigma(s^f)$, $3^3 \mid s+1$ and we get a contradiction. ∎

4.2. *If* $p = 11$, *then* $q \neq 13$.

Proof. Suppose $p = 11$, $q = 13$. Then $d = 1$, for if not, either $h(n) > h(3^2\cdot 7^4\cdot 11^2\cdot 13^2) > 2$ or $h(n) > h(3^2\cdot 7^2\cdot 11^2\cdot 13^2\cdot 19^2) > 2$ (noting that if $b = 2$ then $19 \mid F_3(7)$ occurs). Hence 13 is the special prime. Also $a = 2$, for if not, $h(n) > h(3^4\cdot 7^2\cdot 11^2\cdot 13) > 2$. We get $r > 179$ using 1.4 and the fact that $g(3^2\cdot 7^2\cdot 11^2\cdot 13) > 180$. Hence $3 \nmid b+1$, $3, 5 \nmid c+1$ since $19 \mid F_3(7)$, $19 \mid F_3(11)$, $5 \mid F_5(11)$. Then $b \geqslant 4$, $c \geqslant 6$ and since $g(3^2\cdot 7^4\cdot 11^6\cdot 13) > 522$, 1.4 implies that $r > 521$.

We have $r \equiv s \equiv 1\ (3)$. Indeed, $3^2 \mid \sigma(r^e s^f)$. If $3 \mid \sigma(r^e)$, $3 \mid \sigma(s^f)$ then since r and s are non-special we have $r \equiv s \equiv 1\ (3)$ (cf. 1.7). Suppose $9 \mid \sigma(r^e)$. Then $r \equiv 1\ (3)$ and some prime $\equiv 1\ (9)$ occurs. But this prime must be s, so again $r \equiv s \equiv 1\ (3)$. The same argument applies if $9 \mid \sigma(s^f)$, so in any case $r \equiv s \equiv 1\ (3)$.

Since $o_{11}(7) = 10$ we have $11 \nmid \sigma(7^b)$ and $11^c \mid \sigma(r^e s^f)$. Since $c \geqslant 6$, we have either $11^3 \mid \sigma(r^e)$ or $11^3 \mid \sigma(s^f)$. Suppose $11^3 \mid \sigma(r^e)$. Since r is non-special, by 1.14 either $o_{11}(r) = 1$ or $o_{11}(r) = 5$. In the first case we would have $11^3 \mid e+1$ implying 3 primes other than r are $\equiv 1\ (11)$, an absurdity. Hence $o_{11}(r) = 5$. Hence $5 \mid e+1$, and if $11^3 \nmid F_5(r)$, then $55 \mid e+1$ implying 2 primes other than r are $\equiv 1\ (11)$, also an absurdity. Our conclusion is that $11^3 \mid F_5(r)$. Similarly, if $11^3 \mid \sigma(s^f)$, then $11^3 \mid F_5(s)$.

Now if t is a prime and $11^3 \mid F_5(t)$, then $t \equiv 124, 632, 735$, or 1170 (11^3). The smallest such $t \equiv 1\ (3)$ is 9949. But $g(3^2\cdot\overline{7}\cdot\overline{11}\cdot 13) = 540$, so 1.4 implies $r < 1080$. Then $11^3 \nmid \sigma(r^e)$, so $11^3 \mid \sigma(s^f)$, $5 \mid f+1$, and $s \geqslant 9949$. Since $g(3^2\cdot\overline{7}\cdot\overline{11}\cdot 13\cdot\overline{9949}) < 571$, 1.4 and the first paragraph of this proof imply that $521 < r < 571$. Since $r \equiv 1\ (3)$ we have $r \in \{523, 541, 547\}$.

Now $3 \mid \sigma(r^e)$, for if not, $9 \mid \sigma(s^f)$ which implies $9 \mid f+1$. But we already have $5 \mid f+1$, so $45 \mid f+1$, contradicting 1.10. But $3 \nmid \sigma(523^e)$ for $7027 \mid F_3(523)$ would occur and $7027 < 9949$. Similarly $3 \nmid \sigma(547^e)$ for $163 \mid F_3(547)$ would occur. Also $3 \nmid \sigma(541^e)$ since $13963 \mid F_3(541)$ would occur and $13963 \equiv 653 \not\equiv 124, 632, 735$, or 1170 (11^3). ∎

4.3. *If* $p = 11$, *then* $q \neq 17, 19$, *or* 23.

Proof. If $q = 17$ or 19, then $a \geqslant 4$, for if $a = 2$, then $13 \mid F_3(3)$ would occur. Also $b \geqslant 2$, $c \geqslant 2$, $d \geqslant 1$. Then $h(n) > h(3^4\cdot 7^2\cdot 11^2\cdot 19) > 2$, so $q \neq 17$ or 19.

If $q = 23$, then as above $a \geqslant 4$. Also $b \geqslant 4$, $c \geqslant 4$ since $19 \mid F_3(7)$, $F_3(11)$. Since $d \geqslant 2$ we have $h(n) > h(3^4\cdot 7^4\cdot 11^4\cdot 23^2) > 2$. ∎

4.4. *If* $p = 11$, *then* $a \neq 6$.

Proof. Suppose $a = 6$. Then $1093 = F_7(3)$ does occur. Now $3 \nmid 1 + \exp 1093$, for if not, the prime $398581 \mid F_3(1093)$ would appear and we would have $h(n) < h(3^6\cdot\overline{7}\cdot\overline{11}\cdot\overline{29}\cdot\overline{1093}\cdot\overline{398581}) < 2$ (using the fact that 4.2, 4.3 imply $q \geqslant 29$). Also $3 \nmid b+1$ for otherwise $19 \mid F_3(7)$ would occur contradicting $q \geqslant 29$.

Hence the 2 undetermined components (other than 3, 7, 11, 1093) must account for 6 factors of 3. Then (cf. 1.7 and 1.9) since 3, 7, 11, $1093 \not\equiv 1\ (9)$, we must have one of the undetermined components be the special component π^m where $\pi \equiv -1\ (3)$ and so we can not get more than 1 factor of 3 from the other undetermined component. Thus $3^5 \mid \sigma(\pi^m)$, and since $18 \nmid m+1$ (cf. 1.10), we have $3^4 \mid \pi+1$. Then $\pi \geqslant 809$. But then $h(n) < h(3^6\cdot\overline{7}\cdot\overline{11}\cdot\overline{29}\cdot\overline{809}\cdot\overline{1093}) < 2$. ∎

4.5. *If* $p = 11$, *then* $a = 4$.

Proof. We can not have 2 different primes > 1000 occurring, since if we did they would be $\geqslant 1009, 1013$ and we would have $h(n) < h(\overline{3}\cdot\overline{7}\cdot\overline{11}\cdot\overline{29}\cdot\overline{1009}\cdot\overline{1013}) < 2$.

Suppose now $a \neq 4$. Since $13 = F_3(3)$, 4.2 tells us that $3 \nmid a+1$, so $a \neq 2, 8$. 4.4 says $a \neq 6$. Hence we would have $a \geqslant 10$. Now $3 \nmid b+1$ for 19 would otherwise appear. From the arguments of 1.7, 1.9, and 1.11 we have $3^6 \mid \sigma(\pi^m)$ and $3^5 \mid \pi+1$. Hence $\pi > 1000$.

Hence we must have $a+1$ a power of 5, for if not some prime $\geqslant 7$ would divide $a+1$ and Section 3 would imply the existence of a prime

>1000 and $\not\equiv 17$ (36) and hence distinct from π. But we just noticed that there can not be more than 1 prime occurring >1000.

Hence we may assume $25\mid a+1$. But $F_{25}(3) = 8951\cdot 391151$, so two primes >1000 occur, a contradiction. ∎

4.6. *If* $p = 11$, *then* $q \leqslant 43$.

Proof. In view of 1.4, the fact that $g(3^4\cdot\overline{7}\cdot\overline{11}) < 25$ implies $q < 75$. Suppose $q \geqslant 47$. Since $g(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{47}\cdot\overline{53}) < 559$, 1.4 again implies that $s < 559$. Now $3\nmid c+1$ for otherwise $19\mid F_3(11)$ would appear; $5\nmid c+1$ for otherwise $5\mid F_5(11)$ would appear; $7\nmid c+1$ for otherwise $43\mid F_7(11)$; and $11\nmid c+1$ for otherwise the prime $1806113\mid F_{11}(11)$ would appear contradicting $s < 559$. Hence $c \geqslant 12$.

Then $11^{10}\mid\sigma(q^d r^e s^f)$. Then for $(t, k) = (q, d)$, (r, e), or (s, f) we have $11^4\mid\sigma(t^k)$. Since there are not more than 2 primes $\neq t$ which are $\equiv 1$ (11) we have $o_{11}(t) = 2$, 5, or 10 and $11^3\mid F_2(t)$, $11^3\mid F_5(t)$, or $11^3\mid F_{10}(t)$. In the first case $t \equiv 1330$ (11^3); in the second case $t \equiv 124$, 632, 735, or 1170 (11^3); and in the third case $t \equiv 161$, 596, 699, or 1207 (11^3). But in any case we would have $t > 559$, a contradiction. ∎

4.7. *If* $p = 11$, *then* $q \neq 29$.

Proof. Suppose $q = 29$. Since $g(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{29}) < 139$, 1.4 implies $r < 278$. Also if $r \geqslant 139$ then 1.4 and the fact that $g(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{29}\cdot\overline{139}) < 29975$ imply we would have $s < 29975$.

Now $11\nmid\sigma(29^d)$ for otherwise, since $o_{11}(29) = 10$, we would have $10\mid d+1$ which would imply 29 is special and so $5\mid F_2(29)$ would appear. Then $11^{c-2}\mid\sigma(r^e s^f)$. Now $11^3\nmid\sigma(r^e)$. Indeed, if $11^3\mid\sigma(r^e)$, then since outside of r the only candidate for a prime $\equiv 1$ (11) is s, we would have $11^3\mid F_2(r)$, $F_5(r)$, or $F_{10}(r)$. But as in the proof of 4.6 this is impossible with $r < 278$. Hence $11^{c-4}\mid F_2(s)$, $F_5(s)$, or $F_{10}(s)$.

Suppose $r \geqslant 139$. We have previously noted that $3, 5\nmid c+1$. Now also $7\nmid c+1$ since $43\mid F_7(11)$ and $19\nmid c+1$ for $37\mid F_{19}(11)$. Since $s < 29975$ we have $11\nmid c+1$ since the prime $1806113\mid F_{11}(11)$, $13\nmid c+1$ since the prime $3158528101\mid F_{13}(11)$, and $17\nmid c+1$ since 50544702849929377, a prime, is $= F_{17}(11)$. Hence $c \geqslant 22$. But then either $11^{18}\mid F_2(s)$, $F_5(s)$, or $F_{10}(s)$. We have $F_2(s) < F_{10}(s) < F_5(s) < (s+1)^4$, so $11^{18} < (s+1)^4$ which implies $11^{4\frac{1}{2}} < s+1$, so $s > 40000$, a contradiction.

Hence we have $r \leqslant 137$. Now $h(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{29}\cdot\overline{137}) > 2.00014$. We noticed above that $11^{c-4}\mid F_2(s)$, $F_5(s)$, or $F_{10}(s)$, and $c-4 \geqslant 2$. Hence $s \equiv 120$, 3, 9, 27, 81, 40, 94, 112, or 118 (121). Then $b \neq 4$, for otherwise $F_5(7) = 2801 = s$ and $2801 \equiv 18$ (121). Hence $b \geqslant 6$. Also $d \neq 2$ since $13\mid F_3(29)$. Then since $d \neq 1$, we have $d \geqslant 4$. Using 1.5 we have $h(n) > h(3^4\cdot 7^6\cdot 11^6\cdot 29^4\cdot 137) > 2.00014 - 2.00014\left(\dfrac{2}{10^6} + \dfrac{1}{10^7} + \dfrac{6}{10^6} + \dfrac{6}{10^5}\right) > 2$. ∎

4.8. *If* $p = 11$, *then* $q \neq 31$.

Proof. Suppose $q = 31$. Then from 1.4, $g(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{31}) < 106$ implies $r < 212$.

Suppose $r \geqslant 107$. Since $11\nmid\sigma(31^d)$ (for otherwise since $o_{11}(31) = 5$, we would have $5\mid d+1$ and 5 would occur) we may apply the arguments in 4.7 and get $s > 40000$. But $g(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{31}\cdot\overline{107}) < 6048$, so $s < 6048$, a contradiction.

Hence we have $r \leqslant 103$. Now $h(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{31}\cdot\overline{103}) > 2.0004$. Again applying the arguments of 4.7 we have $b \geqslant 6$. Also $d \geqslant 4$, for if $d = 2$ then $s = 331\mid F_3(31)$ and $331 \equiv 89$ (121). Hence

$$\begin{aligned} h(n) &> h(3^4\cdot 7^6\cdot 11^6\cdot 31^4\cdot 103) \\ &> 2.0004 - 2.0004\left(\frac{2}{10^6} + \frac{1}{10^7} + \frac{4}{10^8} + \frac{1}{10^4}\right) > 2. \ ∎\end{aligned}$$

4.9. *If* $p = 11$, *then* $q \neq 37$.

Proof. Suppose $q = 37$. Since $g(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{37}) < 68$, 1.4 implies $r < 136$.

Suppose $r \geqslant 71$. Then since $g(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{37}\cdot\overline{71}) < 1315$, 1.4 implies $s < 1315$. Since $o_{11}(37) = 5$ and the prime $4271\mid F_5(37)$, we have $11\nmid\sigma(37^d)$. Now if $11^2\mid\sigma(r^e)$, then, as in 4.7, $11^2\mid F_2(r)$, $F_5(r)$, or $F_{10}(r)$ which implies $r \geqslant 233$, contradicting $r < 136$. Hence, since $c \geqslant 6$, we have $11^3\mid\sigma(s^f)$ which implies $11^3\mid F_2(s)$, $F_5(s)$, or $F_{10}(s)$, where in the first and third possibilities we also have $s \equiv 1$ (4). Then $s \geqslant 3361$, contradicting $s < 1315$.

Hence we have $r \leqslant 67$. Suppose $r = 67$. Now $h(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{37}\cdot\overline{67}) > 2.00018$. As in 4.7, $b \geqslant 6$, $c \geqslant 6$, $d \geqslant 2$, $e \geqslant 2$ and

$$\begin{aligned} h(n) &> h(3^4\cdot 7^6\cdot 11^6\cdot 37^2\cdot 67^2) \\ &> 2.00018 - 2.00018\left(\frac{2}{10^6} + \frac{1}{10^7} + \frac{3}{10^5} + \frac{4}{10^6}\right) > 2.\end{aligned}$$

Thus $r \neq 67$. But then

$$h(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{37}\cdot\overline{61}) > 2.003$$

and

$$h(n) > h(3^4\cdot 7^6\cdot 11^6\cdot 37^2\cdot 61) > 2.003 - 2.003\left(\frac{2}{10^6} + \frac{1}{10^7} + \frac{3}{10^5} + \frac{3}{10^4}\right) > 2. \ ∎$$

4.10. *If* $p = 11$, *then* $q \neq 41$.

Proof. Suppose $q = 41$. Since $g(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{41}) < 58$, 1.4 implies $r < 118$. If $r \geqslant 59$ then we get the same sort of contradiction as in 4.9 since $g(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{41}\cdot\overline{59}) < 1793$ implies $s < 1793$ and $11\nmid\sigma(41^d)$ (since $o_{11}(41) = 10$ and if $11\mid\sigma(41^d)$, then $10\mid d+1$ and 5 would occur).

Hence we have $r \leqslant 53$. Now $h(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{41}\cdot\overline{53}) > 2.0027$. Then as in 4.7, $b \geqslant 6$, $c \geqslant 6$ and

$$h(n) > h(3^4\cdot7^6\cdot11^6\cdot41\cdot53) > 2.0027-2.0027\left(\frac{2}{10^6}+\frac{1}{10^7}+\frac{7}{10^4}+\frac{4}{10^4}\right) > 2. \ \blacksquare$$

4.11. *If $p = 11$, then $q \neq 43$.*

Proof. Suppose $q = 43$. Since $g(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{43}) < 54$, 1.4 implies $r < 108$. If $r \geqslant 59$, then $s < 582$ since $g(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{43}\cdot\overline{59}) < 582$. But since $o_{11}(43) = 2$ and 43 can not be special, we have $11\nmid\sigma(43^d)$ and we get the same sort of contradiction as in 4.9. Hence $r = 47$ or 53.

Suppose $r = 53$ and $e = 1$. Then the fact that $g(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{43}\cdot 53) < 8178$ implies $s < 8178$. We have s non-special and $11^{c-2}\,|\,\sigma(s^f)$. Hence $11^4\,|\,F_5(s)$ since $c \geqslant 6$. But no prime < 8178 has this property. (In fact, the only prime $t < 8178$ for which $11^3\,|\,F_5(t)$ is 6779. But $11^4\nmid F_5(6779)$.)

Hence for $r = 47$ or 53 we have $e \geqslant 2$. Also as in 4.7, $b \geqslant 6$, $c \geqslant 6$. We also have $d \geqslant 2$. Then since

$$h(3^4\cdot\overline{7}\cdot\overline{11}\cdot\overline{43}\cdot\overline{53}) > 2.00046$$

we have

$$h(n) > h(3^4\cdot7^6\cdot11^6\cdot43^2\cdot53^2) > 2.00046-2.00046\left(\frac{2}{10^6}+\frac{1}{10^7}+\frac{2}{10^5}+\frac{1}{10^5}\right) > 2. \ \blacksquare$$

Parts 4.2–4.11 have shown that $p \neq 11$. Hence in the remaining parts of Section 4, we may assume $p = 13$. Since $11\,|\,F_5(3)$, we shall also assume $5\nmid a+1$.

4.12. *If $p = 13$ and $q = 17$, then $a = 2$.*

Proof. Suppose $q = 17$. Then since $g(3^2\cdot7^2\cdot13\cdot17) > 23$, we have $3\nmid b+1$, for otherwise $19\,|\,F_3(7)$ would occur, contradicting 1.4. Then if $a \neq 2$, we have $a \geqslant 6$. Also either $c \geqslant 2$ or $d \geqslant 2$. But $h(3^6\cdot7^4\cdot13^2\cdot17) > 2$ and $h(3^6\cdot7^4\cdot13\cdot17^2) > 2$. $\blacksquare$

4.13. *If $p = 13$ and $q = 19$, then $a = 2$.*

Proof. Assume $q = 19$, $a \neq 2$. Suppose $c = 1$. Then $113 < g(3^6\cdot7^2\cdot13\cdot19^2) < g(\overline{3}\cdot\overline{7}\cdot13\cdot\overline{19}) < 188$ implies $112 < r < 376$ by 1.4. Then $9\nmid b+1$ since $37\,|\,F_9(7)$ and $9\nmid d+1$ since $523, 29989\,|\,F_9(19)$. Therefore by 1.7, $3^{a-2}\,|\,\sigma(r^e s^f)$. By 1.11, $v_3(\sigma(r^e)), v_3(\sigma(s^f)) \leqslant 3$, so $a \leqslant 8$. If $a = 8$, then we note that $13\,|\,\sigma(3^8)$ and the prime $757 = F_9(3)$ occurs. Since $13\,|\,F_3(757)$, if $3\,|\,1+\exp 757$ we would have 2 factors of 13 contradicting $c = 1$. Hence $v_3(\sigma(s^f)) = 0$ and $v_3(\sigma(r^e)) \geqslant 6$, an impossiblity. If $a = 6$ then $s = 1093$

$= F_7(3)$ and $v_3(\sigma(s^f)) = 0$ since otherwise $398581\,|\,F_3(1093)$ would occur. But then $v_3(\sigma(r^e)) \geqslant 4$, an impossibility. Hence $c \neq 1$.

Suppose $b = 2$. Then $294 < g(3^6\cdot7^2\cdot13^2\cdot19^2) < g(\overline{3}\cdot7^2\cdot\overline{13}\cdot\overline{19}) < 428$ implies $293 < r < 856$ by 1.4. Then $3\nmid c+1$, $d+1$ for otherwise $61\,|\,F_3(13)$, $127\,|\,F_3(19)$ would respectively occur. Suppose $r \leqslant 421$. Then $h(\overline{3}\cdot7^2\cdot\overline{13}\cdot\overline{19}\cdot\overline{421}) > 2.000073$. If $a = 6$ or $a = 8$ then either $1093 = F_7(3)$ or $757 = F_9(3)$ occurs. But since $h(3^6\cdot7^2\cdot13^4\cdot19^4\cdot421\cdot1093) > 2$, we have $a \geqslant 10$. Then

$$h(n) > h(3^{10}\cdot7^2\cdot13^4\cdot19^4\cdot421) > 2.000073-2.000073\left(\frac{9}{10^6}+\frac{3}{10^6}+\frac{5}{10^7}+\frac{6}{10^6}\right) > 2.$$

Hence we would have $r \geqslant 431$. Now $g(\overline{3}\cdot7^2\cdot\overline{13}\cdot\overline{19}\cdot\overline{431}) < 54669$ implies by 1.4 that $s < 54669$. If $a = 6$, then (since $r < 856$) $s = 1093 = F_7(3)$. Also $3\nmid f+1$ for otherwise $398581\,|\,F_3(1093)$ would occur. As we have already noted that $3\nmid c+1, d+1$, we have $3^5\,|\,\sigma(r^e)$. Hence $\pi = r$ and by 1.11, $3^4\,|\,\pi+1$, and hence $\pi = 809$. But then $5\,|\,F_2(809)$ would occur, so $a \neq 6$. If $a = 8$, then $757 = F_9(3)$ occurs and $g(\overline{3}\cdot7^2\cdot\overline{13}\cdot\overline{19}\cdot\overline{757}) < 982$ implies by 1.4 that $s < 982$. We have $3\nmid 1+\exp 757$ for otherwise the prime $14713\,|\,F_3(757)$ would occur. Then $3^7\,|\,\sigma(\pi^m)$, $3^6\,|\,\pi+1$, and $\pi > 982$. Hence $a \neq 8$. Also $a \neq 10, 12$ for $23\,|\,F_{11}(3)$, $797161\,|\,F_{13}(3)$. Hence $a \geqslant 16$ which implies $3^{15}\,|\,\sigma(r^e s^f)$, so $3^{12}\,|\,\sigma(\pi^m)$ and $3^{11}\,|\,\pi+1$. But $3^{11} > 54669$. Hence $b \neq 2$.

Now $h(3^6\cdot\overline{7}\cdot\overline{13}\cdot\overline{19}) > 2.00024$. Since $g(3^6\cdot7^2\cdot13^2\cdot19^2) > 294$, 1.4 implies $c, d \geqslant 4$ for $61\,|\,F_3(13)$, $127\,|\,F_3(19)$. Hence

$$h(n) > h(3^6\cdot7^4\cdot13^4\cdot19^4) > 2.00024-2.00024\left(\frac{7}{10^5}+\frac{3}{10^6}+\frac{5}{10^7}\right) > 2. \ \blacksquare$$

4.14. *If $p = 13$, then $a = 2$.*

Proof. We assume $a \neq 2$. From 4.12, 4.13, we have $q \geqslant 23$. Since $5\nmid a+1$, we have either $9\,|\,a+1$ or some prime $\geqslant 7$ divides $a+1$. Applying Section 3 we have occurring a prime $\geqslant F_9(3) = 757$ dividing $\sigma(3^a)$ which is not $\equiv 17\ (36)$. Then $g(\overline{3}\cdot\overline{7}\cdot\overline{13}\cdot\overline{23}\cdot\overline{757}) < 131$ implies by 1.4 that $r < 131$, $s \geqslant 757$ and $s \not\equiv 17\ (36)$.

We have $q \leqslant 31$, for if not $h(n) < h(\overline{3}\cdot\overline{7}\cdot\overline{13}\cdot\overline{37}\cdot\overline{41}\cdot\overline{757}) < 2$. Suppose $a \geqslant 10$. Now $v_3(\sigma(7^b)) = 0$ for otherwise 19 would occur. Also if 13 were special we would have $v_3(\sigma(13^c)) = 0$ since $F_6(13) = 157$, a prime. Then in this case we would have $a \leqslant v_3(d+1)+v_3(e+1)+v_3(f+1) \leqslant 3+2+2 < 10$ (noting that $3, 7, 13, q \not\equiv 1\ (9)$). Hence either r or s is special, and if the special prime were $\equiv 1\ (3)$, we would have $a \leqslant v_3(c+1)+v_3(d+1)+v_3(e+1)+v_3(f+1) \leqslant 3+3+2+1 < 10$. Hence $\pi \equiv -1\ (3)$, and $a \leqslant 2+$

$+2+1+1+v_3(\pi+1)$ which implies $3^4 | \pi+1$. (We note that we have been using the arguments in 1.11 repeatedly.) Then $\pi \geqslant 809$, $\pi = s$, $\pi \equiv 17\ (36)$, a contradiction.

Hence $a = 6$ or 8. If $a = 8$ then $F_9(3) = 757 = s$. Then $v_3(\sigma(s^f)) = 0$ for $14713 | F_3(757)$. Also $v_3(\sigma(13^c)) \leqslant 1$ since some prime > 757 divides $F_9(13)$. Also $v_3(\sigma(q^d)) = 0$, for if $q \equiv 1\ (3)$, the n$q = 31$ and $331 | F_3(31)$, but $r < 131$. Hence $3^7 | \sigma(\pi^m)$ which implies by 1.11 that $3^6 | \pi+1$ and hence $\pi > s = 757$, a contradiction.

Hence $a = 6$. Then $F_7(3) = 1093 = s$. Hence $v_3(\sigma(s^f)) = 0$ since $398581 | F_3(1093)$. Also $v_3(\sigma(13^c)) \leqslant 1$ since we already noted $F_9(13)$ is divisible by a large prime $\equiv 1\ (9)$ and $1093 \not\equiv 1\ (9)$, contradicting $r < 131$. Also as above, $v_3(\sigma(q^d)) = 0$. Hence $3^5 | \sigma(\pi^m)$, $3^4 | \pi+1$, and $\pi \geqslant 809$. But $\pi \neq 1093$, so again we get a contradiction, since $r < 131$. ∎

4.15. *If* $p = 13$, *then* $q \neq 17$.

Proof. Assume $q = 17$. Since $a = 2$ and $g(3^2\cdot\overline{7}\cdot\overline{13}\cdot\overline{17}) < 34$, 1.4 implies $r < 68$. Hence 1.13 implies $\pi \neq 17$, $17^d | \pi+1$, and $d \geqslant 2$. Hence $\pi \geqslant 577$. Now since $h(3^2\cdot\overline{7}\cdot\overline{13}\cdot\overline{17}\cdot\overline{37}\cdot\overline{577}) < 2$, we have $r \leqslant 31$. But $h(3^2\cdot\overline{7}\cdot\overline{13}\cdot\overline{17}\cdot\overline{31}) > 2.004$. The work in 4.12 shows 19 does not occur, so $b \geqslant 4$. Also since $\pi \geqslant 577$, we have $c, d, e \geqslant 2$. Then

$$h(n) > h(3^2\cdot 7^4\cdot 13^2\cdot 17^2\cdot 31^2) > 2.004 - 2.004\left(\frac{7}{10^5}+\frac{5}{10^4}+\frac{3}{10^4}+\frac{4}{10^5}\right) > 2. \ ∎$$

4.16. *If* $p = 13$ *and* $q = 19$, *then* $r \geqslant 29$.

Proof. If not, then $p = 13$, $q = 19$, $r = 23$. Since $g(3^2\cdot 7^2\cdot 13\cdot 19^2\cdot 23^2) > 558$, 1.4 implies $s > 557$. Hence, $3 \nmid c+1$, $d+1$, $e+1$ for otherwise $61 | F_3(13)$, $127 | F_3(19)$, $79 | F_3(23)$ would respectively occur. So $d, e \geqslant 4$ and either $c = 1$ or $c \geqslant 4$. Suppose $b = 2$ and $c = 1$. Then $639 < g(3^2\cdot 7^2\cdot 13\cdot 19^4\cdot 23^4) < g(3^2\cdot 7^2\cdot 13\cdot\overline{19}\cdot\overline{23}) < 640$, so 1.4 implies $638 < s < 640$, an impossibility since 639 is not prime. Hence either $b \neq 2$ or $c \neq 1$ which implies either $b \geqslant 4$ or $c \geqslant 4$. But $h(3^2\cdot 7^4\cdot 13\cdot 19^2\cdot 23^2) > 2$ and $h(3^2\cdot 7^2\cdot 13^4\cdot 19^2\cdot 23^2) > 2$. ∎

4.17. *If* $p = 13$, *then* $q \neq 19$.

Proof. Assume $q = 19$. Using 1.4 and 4.16 we have $g(3^2\cdot\overline{7}\cdot\overline{13}\cdot\overline{19}\cdot\overline{29}) < 484$ implying $s < 484$. Also $g(3^2\cdot\overline{7}\cdot\overline{13}\cdot\overline{19}) < 28$ implies $r < 56$.

Suppose $3 | e+1$. Then $s = 61$. Since $c \geqslant 2$ we have at least 2 factors of 13 to account for, and one of them comes from $\sigma(3^2)$. Suppose a prime t is $\leqslant 61$ and $o_{13}(t) = 3$. Then if $t > 3$, we have $t = 29$ or 61. But $67 | F_3(29)$, $97 | F_3(61)$. Suppose $o_{13}(t) = 1$. Then $t = 53$. But this is the only prime $\leqslant 61$ which is $\equiv 1\ (13)$ and if we were to obtain a factor of 13 here we would be contradicting 1.11. Finally suppose $o_{13}(t) = 2$ or 6 and $t \equiv 1\ (4)$. Then if $t \leqslant 61$ we have $t = 17$, but 17 can not occur. Hence $3 \nmid e+1$.

Now $e \neq 4$ since $F_5(13) = 30941 > 484 > s$. $e \neq 5$ since $3 | 5+1$. If $e \geqslant 6$ we have $13^5 | \sigma(s^f)$. Indeed, if $13 | \sigma(r^e)$, then the above paragraph shows that $r = 29$, $3 | e+1$. But then $s = 67$ and $13^4 | \sigma(67^f)$, an impossibility since $o_{13}(67) = 12$. Hence $13 \nmid \sigma(r^e)$ and $13^5 | \sigma(s^f)$. Since we do not have 2 primes $\neq s$ which are $\equiv 1\ (13)$ then either $13^5 | F_2(s)$, $13^5 | F_3(s)$, or $13^5 | F_6(s)$. But $F_2(s) < F_6(s) < F_3(s) < (s+1)^2$ which implies $s+1 > 13^2\cdot 3 = 507 > 484 > s$, a contradiction. Hence $e \ngeqq 6$. Our conclusion is that $e = 1$.

Since $g(3^2\cdot\overline{7}\cdot 13\cdot\overline{19}\cdot\overline{29}) < 126$, 1.4 implies $s < 126$. Then $v_3(\sigma(7^b)) \leqslant 1$ since $1063 | F_9(7)$. Since $v_7(\sigma(13^c)) = 1$, we have an odd number of factors of 7 and at least 1 factor of 3 to place in $\sigma(r^e s^f)$. Since 13 is the special prime and since $29 \leqslant r < 56$, if $3 | \sigma(r^e)$, then $r = 31, 37$, or 43. But $331 | F_3(31)$ and $631 | F_3(43)$ both contradicting $s < 126$. Also $67 | F_3(37)$ and $h(3^2\cdot\overline{7}\cdot\overline{13}\cdot\overline{19}\cdot\overline{37}\cdot\overline{67}) < 2$. Hence $3 \nmid \sigma(r^e)$. So $3 | \sigma(s^f)$. But Table 1 shows the only choices for $s \geqslant 31$, < 126 such that $3 | F_3(s)$ and there is no prime $t | F_3(s)$ with $t > s$ are $s = 67$ and 79. But $7^2 | F_3(67)$, $7^2 | F_3(79)$ so we still have at least 1 factor of 7 to place in $\sigma(r^e s^f)$. Now clearly $7^3 \nmid \sigma(s^f)$, for if not, $21 | f+1$, which would imply that 2 primes $\neq s$ are $\equiv 1\ (7)$. Hence $7 | \sigma(r^e)$. Now $o_7(r) = 3$ since r is non-special and if $o_7(r) = 1$, then $7 | f+1$ which would imply $s \equiv 1\ (7)$ contradicting $s = 67$ or 79. But if $s = 67$, then $r = 31 | F_3(67)$ and $o_7(31) \neq 3$, and if $s = 79$, then $r = 43 | F_3(79)$ and $o_7(43) \neq 3$ ∎

4.18. $p \neq 13$.

Proof. Suppose not. Parts 4.12–4.17 imply we may assume $a = 2$, $q \geqslant 23$. Since $g(3^2\cdot\overline{7}\cdot\overline{13}\cdot\overline{23}\cdot\overline{29}) < 87$, 1.4 implies $s < 87$. But 1.15 implies $s \geqslant 61$.

Since $g(3^2\cdot\overline{7}\cdot\overline{13}\cdot\overline{23}\cdot\overline{61}) < 34$, 1.4 implies $r = 29$ or 31. Hence, since $331 | F_3(31)$, we have $9 | \sigma(13^c s^f)$. Now $9 \nmid \sigma(13^c)$ since Table 1 shows a large prime dividing $F_9(13)$. If $3 | \sigma(13^c)$, then $s = 61 | F_3(13)$, and $97 | F_3(61)$ also occurs, a contradiction. Hence $9 | \sigma(s^f)$. Since $61 \leqslant s < 87$, if $s \equiv 1\ (3)$, then Table 1 shows large primes dividing $F_9(s)$. Hence $s \equiv -1\ (3)$ and s is special. But there is no such s (a prime $\equiv 5\ (12)$) $\geqslant 61$ and < 87. ∎

Section 4 has shown that 7 is not the second smallest prime, so from Section 2 we see that the second smallest prime dividing n must be 5. We divide this case into two subcases: 5 is special, 5 is not special.

5. The special prime is 5. We have already deduced that our o.p.n. n is in the form $3^a 5^b p^c q^d r^e s^f$ where $5 < p < q < r < s$ are primes. In this section we shall assume that 5 is not special, i.e., that b is even. From this assumption we shall obtain a contradiction and hence prove the title of this section.

We recall that π denotes the special prime and $\exp\pi = m$.

5.1. $\pi > 1000$, $7 \nmid n$, $11 \nmid n$, $5 \nmid a+1$, $5 \nmid b+1$, *and not both* $3 \mid a+1$, $3 \mid b+1$.

Proof. First we note that if $3^5 \mid \pi+1$ or if $5^3 \mid \pi+1$, then $\pi > 1000$. Indeed, using 1.14, if $3^5 \mid \pi+1$, then $\pi \equiv 485\ (972)$ and if $5^3 \mid \pi+1$ then $\pi \equiv 249\ (500)$. Next we note that if either $a \geqslant 12$ or if $b \geqslant 12$, then $\pi > 1000$. Indeed, if $a \geqslant 12$ then we can not get more than 9 factors of 3 from the non-special components (cf. 1.11) which implies $\pi \equiv -1\ (3)$ and we can not get more than 6 factors of 3 from the non-special components. Thus $3^5 \mid \pi+1$ and $\pi > 1000$. Similarly by applying the same arguments from 1.11 to the case $b \geqslant 12$ we get $5^5 \mid \pi+1$ and $\pi > 1000$.

$11 \nmid n$ since if $a = 2$ then 13 appears and $h(3^2\cdot 5^2\cdot 11^2\cdot 13) > 2$ and if $a \neq 2$ then $h(3^4\cdot 5^2\cdot 11^2) > 2$. Since $11 \mid F_5(3), F_5(5)$, we have $5 \nmid a+1, b+1$.

Suppose both $3 \mid a+1$ and $3 \mid b+1$. Then both $13 = F_3(3)$ and $31 = F_3(5)$ occur. 13 is not special since $7 \mid F_2(13)$ and $h(3^2\cdot 5\cdot 7^2) > 2$. But $h(3^2\cdot 5^2\cdot 13^2\cdot 31^2) > 2$. Hence either $3 \nmid a+1$ or $3 \nmid b+1$.

Suppose $b \geqslant 6$. To prove $\pi > 1000$, we may assume $a = 2, 6, 8$, or 10. Now if a non-special prime $\neq 3$ or 5 and $\not\equiv 1\ (5)$ occurs, then from 1.11 we get $5^3 \mid \pi+1$ and hence $\pi > 1000$. But if $a = 2$ or 8, then $13 = F_3(3)$ occurs and we have already noticed that 13 can not be special. If $a = 6$ then $1093 = F_7(3)$ occurs and if 1093 is special, then $547 \mid F_2(1093)$ occurs which is non-special. If $a = 10$ then $23 \mid F_{11}(3)$ occurs and 23 can not be special.

Hence to prove $\pi > 1000$, we may assume $b = 2$ and $a = 6$ or 10. But if $a = 10$ then $23 \mid F_{11}(3)$ occurs and since $23 \not\equiv 1\ (3)$ the arguments in 1.11 show that $3^7 \mid \pi+1$ and hence $\pi > 1000$. Thus we may assume $a = 6$.

Since $g(3^6\cdot 5^2\cdot \overline{31}\cdot \overline{1093}) < 26$, 1.4 implies some prime < 52 occurs. Therefore not both $3 \mid 1+\exp 31$ and $3 \mid 1+\exp 1093$ since $331 \mid F_3(31)$ and $398581 \mid F_3(1093)$ would both appear. Also $9 \nmid 1+\exp 31$ for both 331 and a prime $\geqslant 739$ and $\equiv 1\ (9)$ would occur (cf. Table 1). And $9 \nmid 1+\exp 1093$ for both 398581 and a prime $\geqslant 73$ and $\equiv 1\ (9)$ would occur (using Table 1 and the fact that $1093 \equiv 10\ (19)$, $1093 \equiv 20\ (37)$). Hence between the 31 and 1093 components we get no more than 1 factor of 3; from another non-special component we get no more than 2 factors of 3 (since $31, 1093 \not\equiv 1\ (9)$), so we have $\pi \equiv -1\ (3)$. Then $3^4 \mid \sigma(\pi^m)$ and $3^3 \mid \pi+1$ which implies $\pi \geqslant 53$. But some prime < 52 occurs, so $3 \nmid 1+\exp 31$, $3 \nmid 1+\exp 1093$. Then $3^4 \mid \pi+1$. Hence if $\pi < 1000$, then $\pi = 809$. But $3^5 \mid \sigma(\pi^m)$ and $v_3(810) = 4$. Then $3 \mid 1+\exp 809$. But $7 \mid F_3(809)$ and we have already noticed that 7 can not appear. Hence $\pi > 1000$. ■

5.2. *There is a non-special prime occurring which is* > 1000.

Proof. We make use of the two lemmas of Section 3. Then if $9 \mid \pi+1$

and if some prime $t \mid a+1$ with $t \neq 2, 3, 5$ or if $25 \mid \pi+1$ and if some prime $t \mid b+1$ with $t \neq 2, 3, 5, 359$, then our statement follows.

Suppose $359 \mid b+1$. Then from 1.11 we have $5^{352} \mid \pi+1$. But 3.13 implies that a prime $t \mid F_{359}(5)$ with $t > 1000$ and $5^4 \nmid t+1$. Hence t is non-special.

Suppose $a \geqslant 12$ and $a+1$ is not a power of 3. Then as in 5.1, $3^5 \mid \pi+1$. But since 5.1 implies $5 \nmid a+1$, our conclusion follows from the first paragraph. Similarly, if $b \geqslant 12$ and $b+1$ is not a power of 3, then we may apply the above two paragraphs.

Since from 5.1, not both $3 \mid a+1$, $3 \mid b+1$ we are left with the following 3 cases: $a+1 = 3^k$ and $b \in \{6, 10\}$, $b+1 = 3^k$ and $a \in \{6, 10\}$, and $a, b \in \{6, 10\}$.

Suppose $a+1 = 3^k$ and $b \in \{6, 10\}$. Then $13 = F_3(3)$ occurs and since $b \geqslant 6$, 1.11 implies $5^3 \mid \pi+1$. But either $7 \mid b+1$ or $11 \mid b+1$, so Section 3 gives our result.

Suppose $b+1 = 3^k$ and $a \in \{6, 10\}$. If $a = 6$, we note that the last paragraph of the proof of 5.1 shows that if $a = 6$ and if $b = 2$ then $3^4 \mid \pi+1$. Hence since $7 \mid a+1$, we may use Section 3. If $a = 6$ and if $b \geqslant 8$, then since $1093 = F_7(3)$ occurs, 1.11 implies $5^5 \mid \pi+1$ and hence 1093 is non-special. If $a = 10$, then $3851 \mid F_{11}(3)$ occurs and $3851 \not\equiv 1\ (4)$.

Finally suppose $a, b \in \{6, 10\}$. Then either $23 \mid F_{11}(3)$ or $1093 = F_7(3)$ occurs and 1.11 implies $5^3 \mid \pi+1$. Hence we may use Section 3. ■

5.3. $p = 13, 17, 19, 23$, *or* 29.

Proof. 5.1 and 5.2 show that 2 primes > 1000 occur and the 2 smallest such primes are 1009 and 1013. Then since $g(\overline{3}\cdot\overline{5}\cdot\overline{1009}\cdot\overline{1013}) < 17$, 1.4 implies $p < 34$. But $h(\overline{3}\cdot\overline{5}\cdot\overline{31}\cdot\overline{37}\cdot\overline{1009}\cdot\overline{1013}) < 2$. Hence $p \leqslant 29$. In 5.1 we have already seen that $p \neq 7$ or 11. ■

5.4. $p \neq 29$.

Proof. Suppose $p = 29$. Then since $g(\overline{3}\cdot\overline{5}\cdot\overline{29}\cdot\overline{1009}\cdot\overline{1013}) < 37$, we have $q = 31$. Now $a \geqslant 6$ since if $a = 2$, $13 = F_3(3)$ would occur. Also $b \geqslant 6$, since if $b = 2$, $h(\overline{3}\cdot 5^2\cdot\overline{29}\cdot\overline{31}\cdot\overline{1009}\cdot\overline{1013}) < 2$. Also $c \geqslant 2$, since 5.1 implies 29 is not special, and $d \geqslant 2$. Then since $h(\overline{3}\cdot\overline{5}\cdot\overline{29}\cdot\overline{31}) > 2.0066$, we have

$$h(n) > h(3^6\cdot 5^6\cdot 29^2\cdot 31^2) > 2.0066 - 2.0066\left(\frac{7}{10^4} + \frac{2}{10^5} + \frac{5}{10^5} + \frac{4}{10^5}\right) > 2. \ ■$$

5.5. $p \neq 23$.

Proof. Suppose $p = 23$. Then since $g(\overline{3}\cdot\overline{5}\cdot\overline{23}\cdot\overline{1009}\cdot\overline{1013}) < 56$, we have $q < 56$. As in 5.4, $a \geqslant 6$. Then $b \geqslant 6$ since $h(3^6\cdot 5^2\cdot 23^2\cdot 31^2) > 2$. Then from $h(\overline{3}\cdot\overline{5}\cdot\overline{23}\cdot\overline{47}) > 2.0028$, if $q \leqslant 47$, then

$$h(n) > h(3^6\cdot 5^6\cdot 23^2\cdot 47^2) > 2.0028 - 2.0028\left(\frac{7}{10^4} + \frac{2}{10^5} + \frac{9}{10^5} + \frac{1}{10^5}\right) > 2.$$

Therefore, $q = 53$. Now $23, 53 \not\equiv 1\,(3)$ or (5) so 1.11 implies $3^6\cdot 5^6 \mid \pi+1$ which gives $\pi > 10^7$. But $g(\overline{3}\cdot\overline{5}\cdot\overline{23}\cdot\overline{53}\cdot\overline{1009}) < 21281$ which implies by 1.4 that all primes are < 21281, a contradiction. ∎

5.6. $p \neq 13$.

Proof. Suppose $p = 13$. Then $a = 2$ since $h(3^6\cdot 5^2\cdot 13^2) > 2$. Then from 5.1, $b \geqslant 6$. Since $g(3^2\cdot\overline{5}\cdot\overline{13}\cdot\overline{1009}\cdot\overline{1013}) < 50$, we have $q < 50$. Since $h(3^2\cdot 5^6\cdot 13^2\cdot 43^2) > 2$, we have $q = 47$.

Since $g(3^2\cdot\overline{5}\cdot\overline{13}\cdot\overline{47}) < 1371$, 1.4 implies $r < 2742$. Since $13, 47 \not\equiv 1\,(5)$, 1.11 implies $5^b \mid \pi+1$, and since $b \geqslant 6$, we have $\pi = s$. Hence $b \geqslant 10$, for if $b = 6$, $\pi = 19531 = F_7(5)$. Hence $5^{10} \mid \pi+1$ which implies $\pi+1 \geqslant 5^{10} = 9765625$. But if $r \geqslant 1373$ then $g(3^2\cdot\overline{5}\cdot\overline{13}\cdot\overline{47}\cdot\overline{1373}) < 746970$ implies $s < \pi$, a contradiction. Hence $r \leqslant 1367$. Also $r \geqslant g(3^2\cdot 5^2\cdot 13\cdot 47) - 1 > 65$.

Since $F_5(13) = 30941 \not\equiv -1\,(5)$, we have $c \geqslant 6$. (Clearly $c \neq 2$ for $61 \mid F_3(13)$ would occur.) Also $37 \mid F_3(47)$, so $d \geqslant 4$. Then, since $h(3^2\cdot\overline{5}\cdot\overline{13}\cdot\overline{47}\cdot\overline{1367}) > 2.0000037$, we have

$$h(n) > h(3^2\cdot 5^{10}\cdot 13^6\cdot 47^4\cdot 1367^2)$$
$$> 2.0000037 - 2.0000037\left(\frac{3}{10^8} + \frac{2}{10^8} + \frac{5}{10^9} + \frac{4}{10^{10}}\right) > 2. \quad ∎$$

5.7. *If $p = 19$, then $q = 97, 101$, or 103 and $3 \nmid a+1,\ b+1,\ c+1$.*

Proof. Suppose $p = 19$. As in 5.5, $3 \nmid a+1,\ b+1$. Since $g(\overline{3}\cdot\overline{5}\cdot\overline{19}\cdot\overline{1009}\cdot\overline{1013}) < 119$, we have $q < 119$. Then $3 \nmid c+1$ since $127 \mid F_3(19)$. Hence $a \geqslant 6$, $b \geqslant 6$, $c \geqslant 4$. But $h(\overline{3}\cdot\overline{5}\cdot\overline{19}\cdot\overline{89}) > 2.0016$ and if $q \leqslant 89$,

$$h(n) > h(3^6\cdot 5^6\cdot 19^4\cdot 89^2) > 2.0016 - 2.0016\left(\frac{7}{10^4} + \frac{2}{10^5} + \frac{5}{10^7} + \frac{2}{10^6}\right) > 2.$$

Hence $q \geqslant 97$. Now if $q \geqslant 107$, then $19, q \not\equiv 1\,(5)$ and 1.11 implies $5^6 \mid \pi+1$ which implies $\pi+1 \geqslant 5^6 = 15625$. But $g(\overline{3}\cdot\overline{5}\cdot\overline{19}\cdot\overline{107}\cdot\overline{1009}) < 11114$. Hence $q \leqslant 103$. ∎

5.8. *If $p = 19$, then $q \neq 97$.*

Proof. Suppose $q = 97$. Then since $g(\overline{3}\cdot\overline{5}\cdot\overline{19}\cdot\overline{97}) < 9217$, 1.4 implies $r < 18434$.

Since $19, 97 \not\equiv 1\,(5)$, 1.11 implies $5^b \mid \pi+1$. Since from 5.7, $b \geqslant 6$, we have $\pi \equiv 31249\ (62500)$ (cf. 1.14), so $\pi = s$.

Suppose $a = 6$. Then $r = 1093 = F_7(3)$ and since $h(3^6\cdot\overline{5}\cdot\overline{19}\cdot\overline{97}\cdot\overline{1093}) > 2.0006$, we have

$$h(n) > h(3^6\cdot 5^6\cdot 19^4\cdot 97^2\cdot 1093^2)$$
$$> 2.0006 - 2.0006\left(\frac{2}{10^5} + \frac{5}{10^7} + \frac{2}{10^6} + \frac{1}{10^9}\right) > 2.$$

Hence $a \neq 6$.

5.1 and 5.7 imply $a \neq 2, 4, 8$, or 14. $a \neq 10$ since $23 \mid F_{11}(3)$ and $a \neq 12$ since we would have $\pi = 797161 = F_{13}(3)$, contradicting $5^b \mid \pi+1$. Hence $a \geqslant 16$.

Also $b \neq 6$ since $F_7(5) = 19531$ and $19531 \not\equiv -1\,(5)$. 5.7 implies $b \neq 8$. Hence $b \geqslant 10$. Also $c \geqslant 6$ using 5.7 and the fact that $F_5(19) = 151\cdot 911$.

Suppose $d = 2$. Then $r = 3169 \mid F_3(97)$. Then since $h(\overline{3}\cdot\overline{5}\cdot\overline{19}\cdot\overline{97}\cdot\overline{3169}) > 2.0004$, we would have

$$h(n) > h(3^{16}\cdot 5^{10}\cdot 19^6\cdot 97^2\cdot 3169^2)$$
$$> 2.0004 - 2.0004\left(\frac{2}{10^8} + \frac{3}{10^8} + \frac{2}{10^9} + \frac{2}{10^6} + \frac{1}{10^9}\right) > 2.$$

Hence $d \geqslant 4$.

If $r \geqslant 9221$, then $g(\overline{3}\cdot\overline{5}\cdot\overline{19}\cdot\overline{97}\cdot\overline{9221}) < 16994305$, so 1.4 implies all primes are < 16994305. But $5^{10} \mid \pi+1$, so $\pi \equiv 19531249\ (39062500)$ (cf. 1.14), a contradiction. Hence $r \leqslant 9209$.

But $h(\overline{3}\cdot\overline{5}\cdot\overline{19}\cdot\overline{97}\cdot\overline{9209}) > 2.00000016$, so

$$h(n) > h(3^{16}\cdot 5^{10}\cdot 19^6\cdot 97^4\cdot 9209^2)$$
$$> 2.00000016 - 2.00000016\left(\frac{2}{10^6} + \frac{3}{10^8} + \frac{2}{10^9} + \frac{2}{10^{10}} + \frac{2}{10^{12}}\right) > 2. \quad ∎$$

5.9. *If $p = 19$, then $q \neq 101$.*

Proof. Suppose $q = 101$. Since $g(\overline{3}\cdot\overline{5}\cdot\overline{19}\cdot\overline{101}) = 1920$, 1.4 implies $r < 3840$. From 5.1, 5.7 we have $a \geqslant 6$, $b \geqslant 6$, and $3 \nmid c+1$. Hence since $3 \nmid \sigma(19^c)$, $101 \not\equiv 1\,(3)$, $19 \not\equiv 1\,(5)$, we have by 1.11 $3^4\cdot 5^3 \mid \pi+1$, so $\pi > 3840$ and $\pi = s$.

Suppose $a = 6$. Then $g(3^6\cdot\overline{5}\cdot\overline{19}\cdot\overline{101}\cdot\overline{1093}) < 15877$ implies (since $F_7(3) = 1093$) that $s < 15877$. But by 1.14 $\pi \equiv 20249\ (4\cdot 3^4\cdot 5^3)$, so $a \neq 6$.

Then, as in 5.8, $a \geqslant 16$, $b \geqslant 10$, $c \geqslant 6$.

Now if $r \geqslant 1931$, we have $g(3\cdot 5\cdot 19\cdot\overline{101}\cdot\overline{1931}) < 336873$ implying $s < 336873$. But by 1.11, $3^{16}\cdot 5^7 \mid \pi+1$, so $\pi > s$, a contradiction. Hence $r \leqslant 1913$. But $h(\overline{3}\cdot\overline{5}\cdot\overline{19}\cdot\overline{101}\cdot\overline{1913}) > 2.0000038$, so

$$h(n) > h(3^{16}\cdot 5^{10}\cdot 19^6\cdot 101^2\cdot 1913^2)$$
$$> 2.0000038 - 2.0000038\left(\frac{2}{10^8} + \frac{3}{10^8} + \frac{2}{10^9} + \frac{1}{10^6} + \frac{2}{10^{10}}\right) > 2. \quad ∎$$

5.10. *If $p = 19$, then $q \neq 103$.*

Proof. Assume $q = 103$. Since $g(\overline{3}\cdot\overline{5}\cdot\overline{19}\cdot\overline{103}) < 1399$, 1.4 implies $r < 2798$. Since $19, 103 \not\equiv 1\,(5)$, 1.11 implies $5^b \mid \pi+1$. From 5.1, 5.7,

$a \geqslant 6$, $b \geqslant 6$, $3 \nmid c+1$. Then $5^6 | \pi+1$, so $\pi > 1399$, which implies $\pi = s$. Since, by 1.14, $\pi \geqslant 31249 > 15877$, the argument in 5.9 implies $a \neq 6$. Hence as in 5.8, $a \geqslant 16$, $b \geqslant 10$, $c \geqslant 6$, and $\pi \geqslant 19531249$.

If $r \geqslant 1399$, then $g(\bar{3}\cdot\bar{5}\cdot\overline{19}\cdot\overline{103}\cdot\overline{1399}) = 13689216 > \pi$, a contradiction. Hence $r \leqslant 1381$. But $h(\bar{3}\cdot\bar{5}\cdot\overline{19}\cdot\overline{103}\cdot\overline{1381}) > 2.000018$, so

$$h(n) > h(3^{16}\cdot 5^{10}\cdot 19^6\cdot 103^2\cdot 1381^2)$$
$$> 2.000018-2.000018\left(\frac{2}{10^8}+\frac{3}{10^8}+\frac{2}{10^9}+\frac{1}{10^6}+\frac{4}{10^{10}}\right) > 2. \blacksquare$$

Sections 5.7, 5.8, 5.9, and 5.10 show that $p \neq 19$. In the remaining sections we establish the impossibility of the one remaining case: $p = 17$.

5.11. *If* $p = 17$, *then* $257 \leqslant q \leqslant 337$ *and* $3 \nmid a+1$, $b+1$.

Proof. Clearly $3 \nmid a+1$, since $p > 13 | F_3(3)$. Also, the work in 5.5 shows that $3 \nmid b+1$. Now since $g(\bar{3}\cdot\bar{5}\cdot\overline{17}\cdot\overline{1009}\cdot\overline{1013}) < 518$, 1.4 implies $q < 518$.

Suppose $q \leqslant 251$. Now from 5.1 and the above, we have $a \geqslant 6$, $b \geqslant 6$. Also $c \geqslant 4$ since $F_3(17) = 307 > 251$. If $a = 6$, then since $F_7(3) = 1093$ and since $h(3^6\cdot\bar{5}\cdot\overline{17}\cdot\overline{251}\cdot\overline{1093}) > 2.001$, we would have

$$h(3^6\cdot 5^6\cdot 17^4\cdot 251^2\cdot 1093) > 2.001-2.001\left(\frac{2}{10^5}+\frac{8}{10^7}+\frac{7}{10^8}+\frac{1}{10^6}\right) > 2.$$

Hence $a \neq 6$, so $a \geqslant 10$. Then $h(\bar{3}\cdot\bar{5}\cdot\overline{17}\cdot\overline{251}) > 2.00015$ implies

$$h(n) > h(3^{10}\cdot 5^6\cdot 17^4\cdot 251^2)$$
$$> 2.00015-2.00015\left(\frac{9}{10^6}+\frac{2}{10^5}+\frac{8}{10^7}+\frac{7}{10^8}\right) > 2.$$

Hence $q \geqslant 257$.

Suppose $q \geqslant 347$. Then $g(\bar{3}\cdot\bar{5}\cdot\overline{17}\cdot\overline{347}\cdot\overline{1009}) < 27532$ implies $\pi < 27532$ Since $c \geqslant 4$ $(F_3(17) = 307 < 347)$, 1.11 implies $\pi \equiv \pm 1\ (17)$. But $a, b \geqslant 6$, so 1.11 implies $3^3\cdot 5^3 | \pi+1$ which implies (1.14) that $\pi \equiv 6749\ (13500)$, so $\pi = 6749$ or 20249. But neither of these is $\equiv \pm 1\ (17)$. Hence $q \leqslant 337$. $\blacksquare$

5.12. *If* $p = 17$, *then* $a \neq 6$.

Proof. If $a = 6$, then $1093 = F_7(3)$ occurs. If $q \leqslant 283$, then since $h(3^6\cdot\bar{5}\cdot\overline{17}\cdot\overline{283}\cdot\overline{1093}) > 2.0001$ and since $b \geqslant 6$, $c \geqslant 4$ (since $q \leqslant 283 < 307 = F_3(17)$), we have

$$h(n) > h(3^6\cdot 5^6\cdot 17^4\cdot 283^2\cdot 1093)$$
$$> 2.0001-2.0001\left(\frac{2}{10^5}+\frac{8}{10^7}+\frac{5}{10^8}+\frac{1}{10^6}\right) > 2.$$

Hence $q \geqslant 293$, and $g(3^6\cdot\bar{5}\cdot\overline{17}\cdot\overline{293}\cdot\overline{1093}) < 26946$ implies $\pi < 26946$. Since $1093 \not\equiv 1\ (5)$, we have $5^6 | \pi+1$ so $\pi \geqslant 31249$ (cf. 1.14), a contradiction. $\blacksquare$

5.13. *If* $p = 17$, *then* $r < 2^{17} = 131072$, $a \geqslant 30$, $3^{27} | \pi+1$, *and* $\pi = s$.

Proof. From 1.4 we have

$$r < 2g(\bar{3}\cdot\bar{5}\cdot\overline{17}\cdot\bar{q}) = \frac{2^9(q-1)}{q-2^8} \leqslant \frac{2^9(257-1)}{257-2^8} = 2^{17},$$

since $q \geqslant 257$.

To prove $a \geqslant 30$, we note (using Table 4) that $a \neq 10$, 22, or 28 since respectively 23, 47, and 59 would appear. Since $3 \nmid a+1$ (5.11), $5 \nmid a+1$ (5.1), $a \neq 6$ (5.12), we have $a \geqslant 12$, so 1.11 implies $3^9\cdot 5^3 | \pi+1$ so $\pi > 131072$ and $\pi = s$. Then $a \neq 12$ or 18 since respectively $797161 \equiv 1\ (3)$ and $363889 \equiv 1\ (3)$ would occur. Also $a \neq 16$ since 1671 and 34511 would both appear. Hence $a \geqslant 30$. Then 1.11 implies $3^{27} | \pi+1$. $\blacksquare$

Definition. Let r_q be the largest prime $\leqslant g(\bar{3}\cdot\bar{5}\cdot\overline{17}\cdot\bar{q}) = \dfrac{2^8(q-1)}{q-2^8}$, and let R_q be the next prime larger than r_q.

5.14. *If* $p = 17$, *then* $r \leqslant r_q$.

Proof. First we prove that $g(\bar{3}\cdot\bar{5}\cdot\overline{17}\cdot\bar{q}\cdot\bar{R}_q) < 2^{33}$. Indeed, since $\dfrac{2^8q-2^8+1}{q-2^8} \leqslant R_q$, we have

$$h(\bar{3}\cdot\bar{5}\cdot\overline{17}\cdot\bar{q}\cdot\bar{R}_q) \leqslant \frac{2^8-1}{2^7}\cdot\frac{q}{q-1}\cdot\left(\frac{2^8q-2^8+1}{q-2^8}\Big/\frac{2^8q-2^8+1}{q-2^8}-1\right)$$
$$= \frac{(2^8-1)q}{2^7(q-1)}\cdot\frac{2^8q-2^8+1}{2^8q-q+1}.$$

Then

$$g(\bar{3}\cdot\bar{5}\cdot\overline{17}\cdot\bar{q}\cdot\bar{R}_q) \leqslant 2\Big/\left[2-\frac{(2^8q-q)(2^8q-2^8+1)}{(2^7q-2^7)(2^8q-q+1)}\right] = \frac{2^8(q-1)[q(2^8-1)+1]}{q-2^8}$$
$$\leqslant 2^8(q-1)[q(2^8-1)+1] \leqslant 256\cdot 336\,(337\cdot 255+1) < 2^{33}.$$

But 5.13 implies $3^{27} | \pi+1$, so $\pi \geqslant 3^{27}-1 > 2^{33}$. Then 1.4 implies $r < R_q$ which implies $r \leqslant r_q$. $\blacksquare$

5.15. *If* $p = 17$, *then* $b \geqslant 16$.

Proof. 5.1 and 5.11 imply $3, 5 \nmid b+1$. If $b = 6$, then $r = 19531 = F_7(5)$. Now $r_q \leqslant \dfrac{2^8(q-1)}{q-2^8}$. Hence if $q \geqslant 263$, then $r_q \leqslant r_{263} \leqslant \dfrac{2^8\cdot 262}{263-256} < 9582$. Then 1.4 implies (if $b = 6$) $q = 257$. But in this case $c \geqslant 4$ (since

$257 < 307 = F_3(17)$) and $h(\overline{3}\cdot 5^6\cdot\overline{17}\cdot\overline{257}\cdot\overline{19531}) > 2.00046$, so

$$h(n) > h(3^{30}\cdot 5^6\cdot 17^4\cdot 257^2\cdot 19531^2)$$
$$> 2.00046 - 2.00046\left(\frac{1}{10^{14}} + \frac{8}{10^7} + \frac{8}{10^8} + \frac{1}{10^{12}}\right) > 2.$$

Hence $b \neq 6$. Also $b \neq 10$ or 12 since, from Table 4, respectively 12207031 $\equiv 1\ (3)$ and $305175781 \equiv 1\ (3)$ would occur, and both are $> 2^{17}$, contradicting 5.13. Hence $b \geq 16$. ■

5.16. *If* $p = 17$, *then* $c \geq 8$.

Proof. Suppose $c = 2$. Then $q = 307 = F_3(17)$. Since $g(\overline{3}\cdot\overline{5}\cdot 17^2\cdot\overline{307}) < 1171$, 1.4 implies $r < 2342$. If $r \geq 1171$, then since $g(\overline{3}\cdot\overline{5}\cdot 17^2\cdot\overline{307}\cdot\overline{1171}) < 2082371$, we have $s < 2082371$, contradicting 5.13. If $r \leq 1163$, then $h(\overline{3}\cdot\overline{5}\cdot 17^2\cdot\overline{307}\cdot\overline{1163}) > 2.00001$ and hence

$$h(n) > h(3^{30}\cdot 5^{16}\cdot 17^2\cdot 307^2\cdot 1163^2)$$
$$> 2.00001 - 2.00001\left(\frac{1}{10^{14}} + \frac{8}{10^7} + \frac{4}{10^8} + \frac{1}{10^9}\right) > 2.$$

Hence $c \neq 2$.

If $c = 4$, then $r = 88741 = F_5(17)$. But $r_{307} \leq r_{263} < 9582$ (cf. work in 5.15), a contradiction in view of 5.14. Hence $c \neq 4$. For the same reason, $c \neq 6$, since $F_7(17) = 25646167 < 3^{27}$ is prime. ■

5.17. *If* $p = 17$, *then* $d \geq 4$.

Proof. Suppose $d = 2$. From 5.11, no prime may occur between 17 and 257. But $61 | F_3(257)$, $109 | F_3(281)$, $73 | F_3(283)$, $43 | F_3(307)$, $19 | F_3(311)$, $181 | F_3(313)$, $43 | F_3(337)$. Also if some prime occurs ≤ 17, it must be 3, 5, or 17. But $7 | F_3(263)$, $13 | F_3(269)$, $7 | F_3(277)$, $7 | F_3(317)$, $7 | F_3(331)$. Also if $q = 293$ or 271 we have respectively 86143, 24571 occurring. But $r_{293} \leq r_{271} \leq r_{263} < 9582$ (cf. work in 5.15), a contradiction in view of 5.13 and 5.14. Since we have examined every possible choice for q, we find that $d \neq 2$. ■

5.18. *If* $p = 17$, *then* $h(\overline{3}\cdot\overline{5}\cdot\overline{17}\cdot\overline{q}\cdot\overline{r}_q) > 2.0000000035$.

Proof. If $q = 257$, then

$$r_{257} = 65521 \quad\text{and}\quad h(\overline{3}\cdot\overline{5}\cdot\overline{17}\cdot\overline{257}\cdot\overline{65521}) > 2.0000000069.$$

If $q = 263$, then $r_q = 9551$ and $h(\overline{3}\cdot\overline{5}\cdot\overline{17}\cdot\overline{263}\cdot\overline{9551}) > 2.00000067$. Now for any q, we have $r \leq r_q \leq \dfrac{2^8q-2^8}{q-2^8}$ by 5.14. Now if $\dfrac{2^8q-2^8}{q-2^8}$ is an integer, it is even (since $q-2^8$ is odd and 2^8 is a divisor of the numerator) and so is not equal to r_q. Hence $r_q \leq \dfrac{2^8q-2^8-1}{q-2^8}$. Then

$$h(\overline{3}\cdot\overline{5}\cdot\overline{17}\cdot\overline{q}\cdot\overline{r}_q) \geq h(\overline{3}\cdot\overline{5}\cdot\overline{17}\cdot\overline{q})\left(\frac{2^8q-2^8-1}{q-2^8}\Big/\frac{2^8q-2^8-1}{q-2^8}-1\right)$$
$$= \frac{(2^8-1)q(2^8q-2^8-1)}{2^7(q-1)(2^8q-q-1)} = 2 + \frac{q-2^8}{2^7(q-1)[(2^8-1)q-1]}.$$

Hence, when $269 \leq q \leq 337$ we have

$$h(\overline{3}\cdot\overline{5}\cdot\overline{17}\cdot\overline{q}\cdot\overline{r}_q) \geq 2 + \frac{269-2^8}{2^7(337-1)[(2^8-1)337-1]} > 2.0000000035. \ ■$$

5.19. $p \neq 17$.

Proof. Using 5.13 through 5.18, if $p = 17$, then

$$h(n) > h(3^{30}\cdot 5^{16}\cdot 17^8\cdot q^4\cdot r^2)$$
$$> 2.0000000035 - 2.0000000035\left(\frac{1}{10^{14}} + \frac{2}{10^{12}} + \frac{9}{10^{12}} + \frac{9}{10^{13}} + \frac{1}{10^9}\right) > 2$$

(using $q \geq 257$, $r \geq 1009$), a contradiction. ■

6. There are no 6 component odd perfect numbers. The preceding sections have shown that if n is a 6 component o.p.n., then n is in the form $3^a5^bp^cq^dr^es^f$ where $5 < p < q < r < s$ are primes and b is odd. Hence throughout this section we shall assume that an o.p.n. of this form exists, and we shall obtain a contradiction, proving the title of the section.

We note that throughout this section 1.12 is applicable, so $s \geq 1381$ and 2 primes among p, q, r, s are $\equiv 1\ (10)$ with one of them ≥ 1381.

6.1. $b = 1$.

Proof. Since 5 is the special prime, 1.11 implies $b \leq 12$, so $b = 1, 5$, or 9. Suppose $b = 9$. Then $11 | F_5(5)$ occurs. If $a > 2$, then $h(n) > h(3^4\cdot 5^9\cdot 11^2) > h(3^4\cdot 5^2\cdot 11^2) > 2$. If $a = 2$, then $13 | F_3(3)$ occurs and $h(n) > h(3^2\cdot 5^2\cdot 11^2\cdot 13^2) > 2$. Therefore $b \neq 9$. If $b = 5$, then $7 | F_6(5)$ occurs. But $h(3^2\cdot 5\cdot 7^2) > 2$. ■

6.2. $a \leq 10$.

Proof. Since 5 is the special prime and $b = 1$, we have by 1.11, that $a \leq 13$. But either $5 | c+1$, $5 | d+1$, $5 | e+1$, or $5 | f+1$. From the corresponding component we get no more than 1 factor of 3 by 1.7 and 1.10. Hence $a \leq 11$ which implies $a \leq 10$. ■

6.3. $a \neq 10$.

Proof. Suppose $a = 10$. Then by 1.7 and 1.10, if there is one component among p^c, q^d, r^e, s^f from which we do not get a factor of 3, we must get a factor of 5 there. But $23 | F_{11}(3)$ and $23 \not\equiv 1\ (3)$, $23 \not\equiv 1\ (5)$. ■

6.4. $a \neq 8$.

Proof. Assume $a = 8$. Then $13 = F_3(3)$, $757 = F_9(3)$ occur. Also some prime $\geqslant 1381$ occurs and so $g(\overline{3}\cdot 5\cdot\overline{13}\cdot\overline{757}\cdot\overline{1381}) < 44$ implies by 1.4 that some prime $\equiv 1\ (5)$ and $\leqslant 43$ occurs. Since $h(3^8\cdot 5\cdot 11^2\cdot 13^2) > 2$ we have $p = 13$ and $q = 31$ or 41. Therefore $3\nmid e+1$ since $61\mid F_3(13)$. Hence $3^7\mid\sigma(q^d r^e s^f)$, no more than one 3 comes from the component from which we get a 5, no more than three 3's come from any component, so $27\mid 1+\exp 757$. Then $q\equiv 1\ (9)$, a contradiction. ∎

6.5. $a \neq 6$.

Proof. Suppose $a = 6$. Then $1093 = F_7(3)$ occurs. Since $g(3^6\cdot 5\cdot\overline{1093}\cdot\overline{1381}) < 11$, 1.4 implies $p < 22$. We now show that $p\equiv q\equiv r\equiv s\equiv 1\ (3)$. Indeed, suppose we had at most 3 primes $\equiv 1\ (3)$. Then no component can be responsible for more than two 3's, so we must get a 3 from every component whose prime is $\equiv 1\ (3)$. But if $3\mid 1+\exp 1093$, then 398581 occurs, and if $3\mid 1+\exp 398581$, then both 1621 and 32668561 occur, contradicting $p < 22$. Hence $p\equiv q\equiv r\equiv s\equiv 1\ (3)$.

Then $p = 13$ or 19. Then $g(3^6\cdot 5\cdot\overline{13}\cdot\overline{1093}\cdot\overline{1381}) < 42$ implies $q < 42$. Since p, $1093\not\equiv 1\ (5)$, we have $q\equiv 1\ (5)$ (in addition to $q\equiv 1\ (3)$) so $q = 31$. Then $3\nmid e+1$ since $61\mid F_3(13)$ and $127\mid F_3(19)$. Also $3\nmid d+1$ since $331\mid F_3(31)$. Also $5\nmid d+1$ since $11\mid F_5(31)$. Thus $45\mid f+1$, a contradiction in view of 1.10. ∎

6.6. *If* $a = 4$, *then* $p = 11$, $q\equiv r\equiv s\equiv 1\ (3)$, $\{v_3(d+1), v_3(e+1), v_3(f+1)\} = \{1\}$ *or* $\{0, 1, 2\}$, $e\geqslant 6$, *and* $q\in\{73, 79, 97, 103, 109, 127, 139\}$.

Proof. Suppose $a = 4$. Since $F_5(3) = 11^2$, we have $p = 11$. Since no other primes $\neq q, r, s$ are $\equiv 1\ (3)$ and since $3^3\mid\sigma(q^d r^e s^f)$, we have $q\equiv r\equiv s\equiv 1\ (3)$ and $\{v_3(d+1),\ v_3(e+1),\ v_3(f+1)\} = \{1\}$ or $\{0, 1, 2\}$. Now $e\geqslant 6$ since $7\mid F_3(11)$ and $3221\mid F_5(11)$ where $3221\not\equiv 1\ (3)$. Now since $71 < g(3^4\cdot 5\cdot 11^6) < g(3^4\cdot 5\cdot\overline{11}\cdot\overline{1381}) < 75$, 1.4 implies $70 < q < 150$. Since also $q\equiv 1\ (3)$ we have

$$q\in\{73, 79, 97, 103, 109, 127, 139\}. \ ∎$$

6.7. *If* $a = 4$, *then* $3\nmid d+1$.

Proof. Suppose $3\mid d+1$. If $q = 73$, then $g(3^4\cdot 5\cdot 11^6\cdot 73^2) > 2609$ implies by 1.4 that $r > 2608$. But $1801\mid F_3(73)$, so $q\neq 73$.

Noting that $7\mid F_3(79)$, $7\mid F_3(109)$, $13\mid F_3(139)$ we see that $q\neq 79, 109$, or 139.

Hence $q = 97$, 103, or 127. In each case $\frac{1}{3}F_3(q)$ is prime. Indeed, $\frac{1}{3}F_3(97) = 3169$, $\frac{1}{3}F_3(103) = 3571$, and $\frac{1}{3}F_3(127) = 5419$. Now $g(3^4\cdot 5\cdot\overline{11}\cdot\overline{97}\cdot\overline{3169}) < 287$, so $r < 287$ and $s = \frac{1}{3}F_3(q)$. The four primitive 5th roots of 1 mod 11^2 are 3, 9, 27, and 81. The smallest prime $\equiv 1\ (3)$ and $\equiv 3$, 9, 27, or 81 (11^2) is 487, so $11^2\nmid F_5(q)$, $11^2\nmid F_5(r)$. Also $s\not\equiv 3$, 4, 5, or 9 (11) (the primitive 5th roots of 1 mod 11) so $11\nmid F_5(s)$. Since $q\not\equiv 1\ (11)$ we have no more than 2 primes (possibly r and s) $\equiv 1\ (11)$. Then $v_{11}(\sigma(q^d))\leqslant 2$, $v_{11}(\sigma(r^e))\leqslant 1$, $v_{11}(\sigma(s^f))\leqslant 1$, so $e = 6$. But $45319\mid F_7(11)$, contradicting $s = \frac{1}{3}F_3(q)$. ∎

6.8. *If* $a = 4$, *then* $\{e+1, f+1\} = \{9, 15\}$ *and* $r\equiv s\equiv 1\ (30)$.

Proof. 6.6 and 6.7 imply $\{v_3(e+1), v_3(f+1)\} = \{1, 2\}$. Also 6.6 implies either $5\mid e+1$ or $5\mid f+1$. Then 1.10 implies $\{e+1, f+1\} = \{9, 15\}$.

Suppose now $f+1 = 15$. Then it is from the s component that we get a factor of 5 so $s\equiv 1\ (5)$ and $s\equiv 1\ (30)$. Also some prime occurs which is $\equiv 1\ (15)$ and this must be r, so $r\equiv 1\ (30)$. Similarly if $e+1 = 15$, then $r\equiv s\equiv 1\ (30)$. ∎

6.9. *If* $a = 4$, *then* $q = 73$ *or* 79.

Proof. Suppose not, so $q\geqslant 97$ and $g(3^4\cdot 5\cdot\overline{11}\cdot\overline{97}\cdot\overline{1381}) < 325$ implies $r < 325$. Since $e+1 = 9$ or 15, we have $o_q(r) = 3$ or 9. (Indeed, if $e+1 = 9$ then $o_q(r) = 3$, $o_s(r) = 9$ or *vice versa*, and if $e+1 = 15$ then $o_{11}(r) = 5$, $o_s(r) = 15$, and $o_q(r) = 3$.) Also since $r\equiv 1\ (30)$, a quick examination of Table 1 shows that $r = 181$, $q = 139$. But $79\mid F_3(181)$, contradicting $q = 139$. ∎

6.10. *If* $a = 4$, *then* $q\neq 79$.

Proof. Suppose $q = 79$. Since $g(3^4\cdot 5\cdot\overline{11}\cdot\overline{79}) < 698$, 1.4 implies $r < 1396$. Now $79\not\equiv 1\ (9)$, $79\not\equiv 1\ (5)$, $79\not\equiv 1\ (15)$, so by 6.8 $o_{79}(r) = 3$. Therefore $r\equiv 23$ or 55 (79) in addition to $r\equiv 1\ (30)$. Hence $r = 181$ or 971. But $139\mid F_3(181)$ contradicting $s > r$ and $13\mid F_3(971)$ contradicting $q = 79$. ∎

6.11. *If* $a = 4$, *then* $q\neq 73$.

Proof. Assume $q = 73$. In 6.7 we saw $r > 2608$. Since $g(3^4\cdot 5\cdot\overline{11}\cdot\overline{73}) < 2628$, 1.4 implies $r < 5256$. Since $73\not\equiv 1\ (5)$, $73\not\equiv 1\ (15)$, 6.8 implies $o_{73}(r) = 3$ or 9, so $r\equiv 2$, 4, 8, 16, 32, 37, 55, or 64 (73) in addition to $r\equiv 1\ (30)$. Hence $r\in\{2851, 3001, 3121, 3301, 3541, 5101\}$. But $7\mid F_3(2851)$, $7\mid F_3(3301)$, $19\mid F_3(3541)$, $31\mid F_3(3001)$, $151\mid F_3(5101)$. Also $3121\equiv 55\ (73)$ so $e+1 = 9$. But $19\mid F_9(3121)$. ∎

6.6 through 6.11 have shown $a\neq 4$. We complete our proof that there are no 6 component o.p.n.'s by showing the impossibility of the last remaining case: $a = 2$.

6.12. $a\neq 2$.

Proof. Suppose $a = 2$. Then $13 = F_3(3)$ occurs. Now $p = 13$ since $h(3^2\cdot 5\cdot 11^2\cdot 13^2) > 2$. Since $g(3^2\cdot 5\cdot\overline{13}\cdot\overline{1381}) < 17$, 1.4 implies $q < 34$. But $h(3^2\cdot 5\cdot\overline{13}\cdot\overline{31}\cdot\overline{37}\cdot\overline{1381}) < 2$, so $q = 17$, 19, 23, or 29. Hence $r\equiv s\equiv 1\ (10)$.

If $q = 17$ then $d\geqslant 4$ (since if $d = 2$, then $F_3(17) = 307\not\equiv 1\ (10)$ would occur) and 1.11 provides a contradiction since no more than 2 primes are $\equiv 1\ (17)$.

If $q = 19$, then $104 < g(3^2\cdot 5\cdot 13^2\cdot 19^2) < g(3^2\cdot 5\cdot \overline{13}\cdot \overline{19}\cdot \overline{1381}) < 122$ implies by 1.4 that $103 < r < 122$, contradicting $r \equiv 1\ (10)$.

If $q = 23$, then $52 < g(3^2\cdot 5\cdot 13^2\cdot 23^2) < g(3^2\cdot 5\cdot \overline{13}\cdot \overline{23}\cdot \overline{1381}) < 57$ implies by 1.4 that $51 < r < 57$, providing the same contradiction.

If $q = 29$, then $35 < g(3^2\cdot 5\cdot 13^2\cdot 29^2) < g(3^2\cdot 5\cdot \overline{13}\cdot \overline{29}\cdot \overline{1381}) < 38$ implies again by 1.4 that $34 < r < 38$ and no r here is $\equiv 1\ (10)$. ∎

APPENDIX

While proving that a certain integer is prime is elementary in theory, the actual practice is often far from elementary. Thus the reader may wonder at the casualness with which I state in 4.7, for example, that $F_{17}(11) = 505447028499293771$ is prime! In fact, knowing the factorization of an $F_p(q)$ where p, q are primes is an important tool throughout the paper. Most of the "hard" factorizations, such as the above example and the entries in Table 4 are not my own work, but appear in a computer print out at the end of Tuckerman [19]. However, many of the other factorizations are my own. Included in this category are all of Table 1, almost all of Table 2, and all of Table 3. Also I verified that $F_7(17) = 25646167$ is prime (cf. 5.16).

Table 1

A	B	C	D	E
	2, 4	7	19	37, 1063
	3, 9	13	61	*
4, 5, 6, 9, 16, 17	7, 11	19	127	523, 29989
	5, 25	31	331	*
7, 9, 12, 16, 33, 34	10, 26	37	7, 67	73, 127, 92251
	6, 36	43	631	19, 181, 199, 3079
	13, 47	61	13, 97	19, *
	29, 37	67	7^2, 31	*
2, 4, 16, 32, 37, 55	8, 64	73	1801	19, 181, *
	23, 55	79	7^2, 43	397, *
	35, 61	97	3169	*
	46, 56	103	3571	127, *
16, 27, 38, 66, 75, 105	45, 63	109	7, 571	*
22, 37, 52, 68, 99, 103	19, 107	127	5419	37, *
	42, 96	139	13, 499	19, *
	32, 118	151	7, 1093	*
	12, 144	157	8269	19, 37, *
38, 40, 53, 85, 133, 140	58, 104	163	7, 19, 67	*
39, 43, 62, 65, 73, 80	48, 132	181	79, 139	37, *
	84, 108	193	7, 1783	*

Table 1 (cont.)

A	B	C	D	E
43, 58, 162, 175, 178, 180	92, 106	199	13267	19, *
	14, 196	211	13, 31, 37	*
	39, 183	223	16651	73, *
	94, 134	229	97, 181	37, *
	15, 225	241	19441	*
106, 125, 169, 178, 248, 258	28, 242	271	24571	19, 37, *
	116, 160	277	7, 19, 193	*
	44, 238	283	73, 367	19, *
46, 53, 93, 168, 274, 287	17, 289	307	43, 733	*
	98, 214	313	181^2	19, *
	31, 299	331	7, 5233	*
	128, 208	337	43, 883	*
	122, 226	349	19, 2143	37, *
	83, 283	367	13, 3463	19, 37, 73, *
	88, 284	373	7^2, 13, 73	*
84, 115, 180, 185, 234, 339	51, 327	379	61, 787	37, 163, 199, *
14, 79, 196, 286, 304, 312	34, 362	397	31, 1699	19, 73, *
	53, 355	409	55897	*
	20, 400	421	59221	*
27, 150, 153, 256, 296, 417	198, 234	433	37, 1693	127, *
	171, 267	439	31^2, 67	*
	133, 323	457	7, 9967	*
	21, 441	463	19, 3769	109, 379, *
41, 187, 220, 259, 362, 392	232, 254	487	7, 11317	*
	139, 359	499	7, 109^2	19, *
19, 94, 217, 361, 410, 468	60, 462	523	13, 7027	*
15, 214, 225, 312, 352, 505	129, 411	541	7, 13963	19, 109, *
	40, 506	547	163, 613	*
	109, 461	571	7, 103, 151	37, *
287, 321, 335, 384, 435, 540	213, 363	577	19, 5851	*
	24, 576	601	13, 9277	37, *
	210, 396	607	13, 9463	127, *
160, 318, 441, 467, 474, 592	65, 547	613	7, 17923	19, 379, *
	252, 366	619	19, 6733	*
32, 114, 376, 393, 485, 493	43, 587	631	307, 433	19, *
	177, 465	643	97, 1423	19, *

A – For each prime in C which is $\equiv 1\ (9)$, the primitive 9th roots of 1 mod that prime are listed.
B – For each prime in C, the primitive 3rd roots of 1 mod that prime are listed.
C – These are all the primes $\leqslant 643$ which are $\equiv 1\ (3)$.
D – For each prime p in C, these are the prime factors with correct exponents of $\frac{1}{3}F_3(p)$.
E – For each prime p in C, these are the prime factors with correct exponents of $\frac{1}{3}F_9(p)$. * means that every other prime divisor of $\frac{1}{3}F_9(p)$ is $\geqslant 739$.

Table 2

F	G	H
3, 4, 5, 9	11	3221
2, 4, 8, 16	31	11, 17351
10, 16, 18, 37	41	579281
9, 20, 34, 58	61	131, 21491
5, 25, 54, 57	71	11, 211, 2221
36, 84, 87, 95	101	31, 491, 1381
53, 58, 61, 89	131	61, 973001
8, 19, 59, 64	151	104670301
42, 59, 125, 135	181	11, *
39, 49, 109, 184	191	11, 1871, 13001
55, 71, 107, 188	211	1361, 292661
87, 91, 98, 205	241	61, *
20, 113, 149, 219	251	11^2, *
10, 100, 187, 244	271	251, *
86, 90, 153, 232	281	31, 271, 148961
6, 36, 52, 216	311	11, *
64, 124, 150, 323	331	37861, 63601
39, 72, 318, 372	401	11, 1231, 382861
252, 279, 354, 377	421	11, 181, 191, 16561
95, 116, 245, 405	431	71, 191, 510101
88, 114, 351, 368	461	41, 61, 151, 23971
101, 183, 316, 381	491	101, 191, 603791
25, 104, 396, 516	521	11, *
48, 124, 140, 228	541	101, *
106, 167, 387, 481	571	1831, *
32, 314, 423, 432	601	*
228, 242, 279, 512	631	11, 41, 1511, 46601

Table 2 (cont.)

F	G	H
357, 472, 531, 562	641	11, *
197, 247, 406, 471	661	*
89, 132, 149, 320	691	11, 61, *
89, 210, 464, 638	701	101, *
80, 392, 460, 569	751	11, *
67, 168, 602, 684	761	*
212, 339, 500, 570	811	*
51, 138, 161, 470	821	211, 241, 1789091
268, 286, 463, 744	881	*
19, 48, 361, 482	911	11, 701, 17884211
349, 364, 412, 756	941	*
65, 341, 732, 803	971	11^2, *
160, 197, 799, 825	991	*
589, 676, 802, 995	1021	11, 41, 1451, 332441
264, 518, 619, 660	1031	31, *
307, 413, 671, 710	1051	71, 241, *
220, 381, 655, 865	1061	11, *
93, 290, 786, 1012	1091	*
224, 334, 683, 1060	1151	31, 991, *
70, 216, 987, 1068	1171	11, *
81, 452, 656, 1172	1181	11, *
105, 216, 1018, 1062	1201	*
190, 401, 771, 1099	1231	*
319, 344, 855, 1063	1291	11^2, 821, *
163, 549, 870, 1019	1301	11, 61, *
133, 516, 735, 1257	1321	211, *
211, 309, 969, 1232	1361	*

F — For each prime in G, the primitive 5th roots of 1 mod that prime are listed.
G — These are the primes < 1381 which are $\equiv 1\ (5)$.
H — For each prime p in G, these are the prime factors with correct exponents of $\frac{1}{5}F_5(p)$. * means that every other prime divisor of $\frac{1}{5}F_5(p)$ is > 2000.

Note. The factorizations of $F_5(151)$, $F_5(331)$, and $F_5(911)$ come from Tuckerman [19].

Table 3

I	J	K	L	M
29	—	—	59^1	349^0, 929^0
31	—	—	*	311^0
37	—	—	*	149^a
41	83^1	*	*	739^0, 821^0
43	*	431^a, 947^0	*	431^0
47	*	659^0	*	659^0, 941^a
53	107^a	743^0	*	*
59	*	709^0, 827^a	*	709^a
61	*	733^0	*	*
67	*	*	*	269^a
71	*	853^0	*	569^a
73	*	877^0	*	439^0
83	167^1	997^0	*	499^a
89	179^a	*	179^1	*
97	*	971^a	*	389^1, 971^0
101	*	*	*	809^0
103	*	*	*	619^0
113	227^1	*	*	*
127	*	*	*	509^a
131	263^1	*	*	*
173	347^1	*	*	*
179	359^a	*	359^1	*
191	383^1	*	*	*
233	467^a	*	*	*
239	479^1	*	479^1	*
251	503^a	*	*	*
281	563^1	*	*	*
293	587^1	*	*	*
359	719^a	*	719^1	*
419	839^1	*	839^1	*
431	863^a	*	*	*
443	887^1	*	*	*
491	983^1	*	*	*

I — These are the primes p such that $29 \leqslant p \leqslant 499$ and for which there is a prime $q \equiv 1\ (p)$, $q < 1000$, and $q \equiv \pm 1\ (12)$ or $q \equiv \pm 1\ (10)$.

J — For $p \geqslant 41$ appearing in I, this is the prime $q = 2p+1$. * means $2p+1$ is not prime. A numerical exponent indicates that this is the exact power of q which divides $F_p(3)$. "a" indicates the irrelevance of the exponent.

K — For $p \geqslant 41$ appearing in I, this is a list of those primes $> 2p+1$ and < 1000 which are $\equiv 1\ (p)$ and $\equiv \pm 1\ (12)$. * means there are no such primes. The exponents are as in J.

L — For p in I, this is the prime $q = 2p+1$ where $p \equiv 9\ (10)$. * means that either $2p+1$ is not prime or $p \not\equiv 9\ (10)$. The exponents are as in J (5 replaces 3).

M — For p in I, this is a list of the primes $> 2p+1$ and < 1000 which are $\equiv 1\ (p)$ and $\equiv \pm 1\ (10)$. * means there are no such primes. The exponents are as in L.

Table 4

$F_7(3) = 1093$	$F_7(5) = 19531$
$F_{11}(3) = 23 \cdot 3851$	$F_{11}(5) = 12207031$
$F_{13}(3) = 797161$	$F_{13}(5) = 305175781$
$F_{17}(3) = 1871 \cdot 34511$	$F_{17}(5) = 409 \cdot 466344409$
$F_{19}(3) = 1597 \cdot 363889$	$F_{19}(5) = 191 \cdot 6271 \cdot 3981071$
$F_{23}(3) = 47 \cdot 1001523179$	$F_{23}(5) = 8971 \cdot 332207361361$
$F_{29}(3) = 59 \cdot 28537 \cdot 20381027$	
$F_{31}(3) = 683 \cdot 102673 \cdot 4404047$	
$F_{37}(3) = 13097927 \cdot 17189128703$	

This table is self-explanatory. The factorizations appear in Tuckerman [19]. All integers appearing in Table 4 are prime.

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Received on 1. 8. 1972 (310)

ACTA ARITHMETICA
XXV (1974)

О „большом решете"

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Основным неравенством „большого решета" Ю. В. Линника (см. [6]) в настоящее время называют неравенство типа:

$$\sum_{\substack{1\leqslant q\leqslant Q\\ q\in D}}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|S\left(\frac{a}{q}\right)\right|^2\leqslant \Delta(Q,x)\sum_{|n|\leqslant x}|a_n|^2\ (^1), \tag{1}$$

где D — произвольное множество натуральных чисел, a_n — произвольные комплексные, n — целые числа, $S(\alpha)=\sum_{|n|\leqslant x}a_ne^{2\pi i\alpha n}$, а

$$\Delta(Q,x)\leqslant\begin{cases}2\max(2x,Q^2) & (\text{см. } [3]),\\ Q^2+2\pi x & (\text{см. } [4]),\\ (Q+\sqrt{2x})^2 & (\text{см. } [1]).\end{cases}$$

Мы не касаемся аналогичного неравенства для $S(x_r)$, где x_r — иные последовательности точек из $[0,1]$.

Легко показать, что в некоторых предельных случаях естественно наличие в правой части (1) не только $2x$ или Q^2, но и постоянной, вообще говоря, большей 1 (см. [2]).

Левую часть (1) удобнее записывать в следующей — эквивалентной (1) — форме:

$$\sum_{\substack{1\leqslant q\leqslant Q\\ q|P}}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|S\left(\frac{a}{q}\right)\right|^2$$

где $P=P(Q)$ — наименьшее общее кратное чисел $\leqslant Q$ из D.

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(1) Условие $q\in D$ будем в дальнейшем опускать.