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On the lattice point theory of multidimensional ellipsoids

by

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To Professor K. Mahler
on the occasion of his 70th birthday

1. Introduction. Let $r \geq 2$ be a natural number and let $Q(u) = Q(u_j) = \sum_{j,l=1}^{r} a_{jl} u_j u_l$ be a positive definite quadratic form with a symmetric coefficient matrix of determinant D. For $x > 0$, denote by $A(x)$ the number of lattice points in the region $Q(u) \leq x$ and by $V(x) = \pi^{\frac{r}{2}} x^{\frac{r}{2}} / \Gamma\left(\frac{r}{2}+1\right)\sqrt{D}$ the volume of this region. Finally, let

$$P(x) = A(x) - V(x) \tag{1}$$

be the usual lattice remainder term. O- and Ω-estimates of the function $P(x)$ were investigated in many papers (see e.g. [1]–[6] and the references given there).

In the following, we shall restrict ourselves to the special forms

$$Q(u) = a_1 Q_1(u_1, \ldots, u_{r_1}) + a_2 Q_2(u_{r_1+1}, \ldots, u_{r_1+r_2}) + \ldots + a_\sigma Q_\sigma(u_{r_1+r_2,\ldots,r_{\sigma-1}+1}, \ldots, u_r), \tag{2}$$

where a_j are positive real numbers, Q_j are positive definite quadratic forms with integral coefficients, r_j and σ are natural numbers, $j = 1, 2, \ldots \ldots, \sigma$ and $r = r_1 + r_2 + \ldots + r_\sigma$. These forms were studied by Jarník in a series of papers. A basic result is given in [4]: if $\sigma = 2$, $r_1 \geq 4$ and $r_2 \geq 4$ then

$$P(x) = O(x^{\frac{r}{2}-1-\frac{1}{\gamma}+\varepsilon}), \quad P(x) = \Omega(x^{\frac{r}{2}-1-\frac{1}{\gamma}-\varepsilon}) \tag{3}$$

for each $\varepsilon > 0$. Here (and in the following), $\gamma = \gamma(a_1, a_2, \ldots, a_\sigma)$ denotes the supremum of all numbers $\beta > 0$, for which the system of inequalities

$$\left|\frac{a_j}{a_1} q - p_j\right| < q^{-\beta}, \quad j = 1, 2, \ldots, \sigma$$

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has infinitely many solutions in natural numbers $q, p_1, p_2, \ldots, p_\sigma$ (for $\gamma = +\infty$ we put $\frac{1}{\gamma} = 0$). A generalization of this result to the case $\sigma > 2$ made some difficulties. Jarník shows in [5] a very fine theorem:

If we put

$$f_Q = \limsup_{x\to+\infty} \frac{\log|P(x)|}{\log x},$$

then it holds

$$(4) \qquad \frac{r}{2} - \sigma \leqslant f_Q \leqslant \frac{r}{2} - 1$$

for each Q of the form (2), where $r_j \geqslant 4$ for $j = 1, 2, \ldots, \sigma$. On the other hand, Jarník shows that to an arbitrary value f, $\frac{r}{2} - \sigma \leqslant f \leqslant \frac{r}{2} - 1$ there exists a form Q of the type (2) for which $f_Q = f$ and he even determines the Hausdorff dimension of all these forms. The proof of the last assertion was a purely existential one. For a concrete form (2) with $r_j \geqslant 4$, $j = 1, 2, \ldots, \sigma$, it only follows from the Jarník's paper [5] that $f_Q = \frac{r}{2} - 1$ for $\gamma = +\infty$ and, in general, the inequality $f_Q \geqslant \frac{r}{2} - 1 - \frac{1}{\gamma}$. Also $f_Q = \frac{r}{2} - \sigma$ for almost all forms Q in the sense of Lebesgue measure in R_σ.

In 1968, the first of the authors succeded in extending the validity of (3) to the case $\sigma > 2$. It was, however, necessary to assume that $r_j \geqslant \frac{2(\gamma+1)}{\gamma}$, $j = 1, 2, \ldots, \sigma$ [1]. In the same year, Jarník published his last paper on the lattice point theory in which, in conformity with known result by K. Chandrasekharan and R. Narasimhan, he turned the whole subject into another direction.

Put $P_0(x) = P(x)$ and let for $\varrho > 0$ be

$$(5) \qquad P_\varrho(x) = \frac{1}{\Gamma(\varrho)} \int_0^x P(t)(x-t)^{\varrho-1}\,dt.$$

From a series of results in [6] we shall state only the following two:

If $\varrho > \frac{r}{2} - \frac{1}{2}$, then for each form (not only of the type (2))

$$(6) \qquad P_\varrho(x) = O(x^{\frac{r}{4} - \frac{1}{4} + \frac{\varrho}{2}}), \qquad P_\varrho(x) = \Omega(x^{\frac{r}{4} - \frac{1}{4} + \frac{\varrho}{2}}).$$

[1] As a matter of fact, this result was proved in the quoted paper for the case Q_j are sums of squares only. It is easily seen that it holds in our general case also.

If $0 \leqslant \varrho < \frac{r}{2} - 2$ and if Q has integral coefficients, then

$$P_\varrho(x) = O(x^{\frac{r}{2}-1}), \qquad P_\varrho(x) = \Omega(x^{\frac{r}{2}-1}).$$

For $\frac{r}{2} - 2 \leqslant \varrho \leqslant \frac{r}{2} - \frac{1}{2}$ there are no definitive results known, not even for integer forms; for $\varrho = 0$ we obtain actually the classical unsolved cases $r = 2, 3$ and 4 (for a more detailed discussion see [6], pp. 141–142).

Thus, these results essentially formulate a new problem. Namely, to determine the dependence of the value

$$f_Q(\varrho) = \limsup_{x\to+\infty} \frac{\log|P_\varrho(x)|}{\log x}$$

not only on the form Q but also on the parameter ϱ. The aim of this paper is to investigate this question for forms Q of the type (2), or more exactly, to extend the result (3) to the case $\varrho > 0$. We shall prove the following

Main Theorem. *Let Q be a form of the type* (2), *where $Q_1, Q_2, \ldots, Q_\sigma$ are integral forms, put $\gamma = \gamma(a_1, a_2, \ldots, a_\sigma)$ and let also $\frac{r}{2} - 2 > \varrho \geqslant 0$ and $r_j \geqslant \frac{2(\varrho+1)(\gamma+1)}{\gamma}$, $j = 1, 2, \ldots, \sigma$. Then*

$$f_Q(\varrho) = \frac{r}{2} - 1 - \frac{\varrho+1}{\gamma}.$$

In paper [7] (Theorem 4, p. 273) it is shown that if there exist sequences $\{q_n\}_{n=1}^\infty$, $\{p_{jn}\}_{n=1}^\infty$, $j = 1, 2, \ldots, \sigma$, of natural numbers such that $q_n \to \infty$ and

$$\left|\frac{a_j}{a_1} q_n - p_{jn}\right| < \frac{1}{q_n^\beta \log^\alpha q_n}, \qquad j = 1, 2, \ldots, \sigma, \ n = 1, 2, \ldots,$$

where $\alpha \geqslant 0$ and $\beta > 0$ are constants, then for the forms (2) always holds

$$P_\varrho(x) = \Omega(x^{\frac{r}{2} - 1 - \frac{\varrho+1}{\beta}} \log^{\frac{\alpha(\varrho+1)}{\beta}} x).$$

If $\sigma = 1$, then by the same paper (Theorem 1, p. 266) we always have

$$P_\varrho(x) = \Omega(x^{\frac{r}{2}-1}).$$

In view of the mentioned results in [1] it follows from this at once that for the proof of the Main Theorem it will be sufficient to prove the following two statements.

THEOREM 1. *Let Q be of type (2), where $Q_1, Q_2, \ldots, Q_\sigma$ are integral forms. If $0 < \varrho < \frac{r}{2} - 2$, then*

$$(7) \qquad P_\varrho(x) = O(x^{\frac{r}{2}-1}).$$

THEOREM 2. *Let the assumptions of the Main Theorem be satisfied. If $\gamma < +\infty$ and $\sigma \geqslant 2$, then for each $\varepsilon > 0$*

$$(8) \qquad P_\varrho(x) = O(x^{\frac{r}{2}-1-\frac{\varrho+1}{\gamma}+\varepsilon}).$$

For the proof of these two statements we shall use the Jarník's method (see [3], [4]) in combination with a metric lemma from [1].

2. Proof of Theorems 1 and 2. We shall have always $\varrho > 0$ and Q will be a form of the type (2) with $Q_1, Q_2, \ldots, Q_\sigma$ having integral coefficients. We shall denote by the letter c (in general, different) positive constants, depending only on ϱ and Q, x will be sufficiently large, $x > c$. For $\varepsilon > 0$, by $c(\varepsilon)$ we shall denote positive constants which again, in general, will be not the same at each occurrence and may depend only on ϱ, Q and ε. Instead of $|A| \leqslant cB$ we shall write shortly $A \ll B$. If $\delta > 0$ and $\mathfrak{B}$ is an interval with endpoints α, β, $\alpha < \beta$, then $\delta\mathfrak{B}$ means the interval with endpoints $\delta\alpha$ and $\delta\beta$.

For a complex s with $\operatorname{Re} s > 0$ put

$$(9) \qquad \Theta(s) = \Theta_Q(s) = \sum_{m_1, m_2, \ldots, m_r = -\infty}^{\infty} e^{-sQ(m_1, m_2, \ldots, m_r)}.$$

The function (9) is obviously holomorphic in the half plane $\operatorname{Re} s > 0$ and, as it is known, for each $a > 0$ is

$$(10) \qquad P_\varrho(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{F(s)e^{xs}}{s^{\varrho+1}}\, ds,$$

where

$$F(s) = \Theta(s) - \frac{\pi^{r/2}}{\sqrt{D} s^{r/2}}$$

and the integration path is the line $\operatorname{Re} s = a$. Let us remark that by s^β, $\beta > 0$, we mean that branch of the function s^β which is positive for positive values of s.

Now, we shall make use of known transformation properties of the function (9) (see e.g. [7], Lemma 1, p. 264).

LEMMA 1. *Let h be an integer and k a natural number, g.c.d. $(h, k) = 1$, $\operatorname{Re} s > 0$ and let the form Q have integral coefficients. Then*

$$(11) \qquad \Theta_Q(s) = \frac{\pi^{r/2}}{\sqrt{D} k^r \left(s - \frac{2\pi i h}{k}\right)^{r/2}} \sum_{m_1, m_2, \ldots, m_r = -\infty}^{\infty} S_{h,k,(m)}\, e^{-\frac{\pi^2 \overline{Q}(m_1, m_2, \ldots, m_r)}{k^2\left(s - \frac{2\pi i h}{k}\right)}}$$

where $\overline{Q}$ is the form conjugate to Q and $S_{h,k,(m)} = S_{h,k,(m_1, m_2, \ldots, m_r)}$ are generalized Gaussian sums defined by

$$S_{h,k,(m)} = \sum_{b_1, b_2, \ldots, b_r = 1}^{k} e^{\frac{2\pi i h}{k} Q(b_1, b_2, \ldots, b_r) + \frac{2\pi i}{k} \sum_{j=1}^{r} m_j b_j}.$$

For these sums always holds the estimate

$$(12) \qquad S_{h,k,(m)} \ll k^{r/2}.$$

Let us mention that for $h = 0$ and $k = 1$ the relation (11) is true for any positive definite form Q.

If now we have $h = 0$, $k = 1$, $s = \frac{1}{x} + it$, $t \ll x^{-1/2}$, then by (11)

$$\frac{F(s)}{s^{\varrho+1}} = \frac{\pi^{r/2}}{\sqrt{D} s^{\frac{r}{2}+\varrho+1}} \sum{}' e^{-\frac{\pi^2 \overline{Q}(m_1, m_2, \ldots, m_r)}{s}}$$

$$\ll \frac{x^{\frac{r}{2}+\varrho+1}}{(1+x^2t^2)^{\frac{r}{4}+\frac{\varrho+1}{2}}} \sum{}' e^{-\frac{\pi^2 x \overline{Q}(m_1, m_2, \ldots, m_r)}{1+x^2t^2}} \ll \frac{x^{\frac{r}{2}+\varrho+1}}{(1+x^2t^2)^{\frac{r}{4}+\frac{\varrho+1}{2}}} e^{-\frac{cx}{1+x^2t^2}},$$

since $\overline{Q}(m_1, m_2, \ldots, m_r) \gg m_1^2 + m_2^2 + \ldots + m_r^2$ and from $t \ll x^{-1/2}$ it follows that $x \gg 1 + x^2t^2$ ($\sum'$ denotes the sum extended over all integral numbers $m_1, m_2, \ldots, m_r$, $m_1^2 + m_2^2 + \ldots + m_r^2 > 0$). For $A = c$ (it will be useful to choose $A = \max_j \frac{2\pi}{a_j}$) we have thus

(13)

$$\frac{1}{2\pi i} \int_{\frac{1}{x} - i\frac{A}{\sqrt{x}}}^{\frac{1}{x} + i\frac{A}{\sqrt{x}}} \frac{F(s)e^{xs}}{s^{\varrho+1}}\, ds \ll x^{\frac{r}{4}+\frac{\varrho+1}{2}} \int_0^{\frac{A}{\sqrt{x}}} \left(\frac{x}{1+x^2t^2}\right)^{\frac{r}{4}+\frac{\varrho+1}{2}} e^{-\frac{cx}{1+x^2t^2}}\, dt \ll x^{\frac{r}{4}+\frac{\varrho}{2}},$$

since $\xi^c e^{-c\xi} \ll 1$ on the interval $[0, +\infty)$.

By (10) and (13), it will suffice for the proofs of both theorems to estimate the corresponding integrals

$$J = \frac{1}{2\pi i}\int_{\frac{1}{x}+i\frac{A}{\sqrt{x}}}^{\frac{1}{x}+i\infty} \frac{F(s)e^{xs}}{s^{\varrho+1}}\,ds, \qquad \bar{J} = \frac{1}{2\pi i}\int_{\frac{1}{x}-i\infty}^{\frac{1}{x}-i\frac{A}{\sqrt{x}}} \frac{F(s)e^{xs}}{s^{\varrho+1}}\,ds.$$

J and $\bar{J}$ are obviously complex conjugate numbers. Thus, for the proof of Theorem 1 it will suffice to show that under the assumptions of Theorem 1 holds

(14) $$J \ll x^{\frac{r}{2}-1}$$

(since $0<\varrho<\frac{r}{2}-2$ implies $\frac{r}{4}+\frac{\varrho}{2}<\frac{r}{2}-1$), and for the proof of Theorem 2 it will be sufficient to show that for each $\varepsilon>0$ (under the assumptions of Theorem 2)

(15) $$J \ll x^{\frac{r}{2}-1-\frac{\varrho+1}{\gamma}+\varepsilon}$$

(the assumptions $\sigma \geqslant 2$ and $r_j \geqslant \frac{2(\varrho+1)(\gamma+1)}{\gamma}$ imply $r \geqslant r_1+r_2 \geqslant \frac{4(\varrho+1)(\gamma+1)}{\gamma}$ and hence $\frac{r}{2}-1-\frac{\varrho+1}{\gamma} \geqslant \frac{r}{4}+\varrho > \frac{r}{4}+\frac{\varrho}{2}$).

Let us now consider Farey fractions corresponding to $\sqrt{x}$, i.e. all numbers of the form $\frac{h}{k}$, where h and k are relatively prime integral numbers and $0<k\leqslant\sqrt{x}$. With each such fraction we associate the interval

$$\mathfrak{B}_{h,k} = \left[2\pi\frac{h+h_1}{k+k_1},\, 2\pi\frac{h+h_2}{k+k_2}\right),$$

where $\frac{h_1}{k_1}<\frac{h}{k}<\frac{h_2}{k_2}$ is a triple of consecutive Farey fractions under consideration. As it is well known (see e.g. [6], p. 152),

$$\mathfrak{B}_{h,k} = \left[2\pi\frac{h}{k}-\frac{\vartheta_1}{k\sqrt{x}},\, 2\pi\frac{h}{k}+\frac{\vartheta_2}{k\sqrt{x}}\right),$$

where $\pi\leqslant\vartheta_1,\vartheta_2\leqslant 2\pi$. Hence, if $t\in\mathfrak{B}_{h,k}$, then $\left|t-\frac{2\pi h}{k}\right| \ll \frac{1}{k\sqrt{x}}$, and if $t\notin\mathfrak{B}_{h,k}$, then $\left|t-\frac{2\pi h}{k}\right| \gg \frac{1}{k\sqrt{x}}$. Let us mention also that for $s = \frac{1}{x}+it$, $t\in\mathfrak{B}_{h,k}$ and $h>0$ is $\frac{h}{k}\ll|s|\ll\frac{h}{k}$. We have further obviously

$$[w, +\infty) = \bigcup_{h>0,\,0<k\leqslant\sqrt{x}} \mathfrak{B}_{h,k},$$

for $w = \frac{2\pi}{[\sqrt{x}]+1}$, where the union is disjoint.

For simplicity, we shall always assume that the numbers h and k are natural and relatively prime and $k\leqslant\sqrt{x}$ (similarly, the same about the pairs $h_1, k_1; h_2, k_2; \ldots; h_\sigma, k_\sigma$). We shall make use of the above mentioned facts without further reference.

If now Q is a form (2), then obviously

(16) $$\Theta_Q(s) = \prod_{j=1}^{\sigma}\Theta_{Q_j}(a_j s).$$

To prove (14), let us notice that

$$F(s) \ll \sum_{j=1}^{\sigma}|\Theta_{Q_j}(a_j s)|^{r/r_j}+|s|^{-r/2}.$$

Because of

(17) $$e^{x\left(\frac{1}{x}+it\right)} \ll 1$$

and

$$\int_{A/\sqrt{x}}^{\infty}\frac{dt}{\left|\frac{1}{x}+it\right|^{\frac{r}{2}+\varrho+1}} = x^{\frac{r}{2}+\varrho}\int_{A\sqrt{x}}^{\infty}\frac{du}{(1+u^2)^{\frac{r}{4}+\frac{\varrho+1}{2}}} \ll x^{\frac{r}{4}+\frac{\varrho}{2}}$$

it will suffice for the proof of (14) to estimate the integrals ($s=\frac{1}{x}+it$)

(18) $$I_j = \int_{A/\sqrt{x}}^{\infty}\frac{|\Theta_{Q_j}(a_j s)|^{r/r_j}}{s^{\varrho+1}}\,dt.$$

By Lemma 1, we have for $t\in\frac{1}{a_j}\mathfrak{B}_{h,k}$, $s=\frac{1}{x}+it$

(19) $$\Theta_{Q_j}(a_j s) \ll \frac{x^{r_j/2}}{k^{r_j/2}\left(1+x^2\left|t-\frac{2\pi h}{a_j k}\right|^2\right)^{r_j/4}}$$

and thus (here we can see why we made the choice $A = \max_j \frac{2\pi}{a_j}$)

$$I_j \ll \sum_{h,k} \int_{\frac{1}{a_j}\mathfrak{B}_{h,k}} \frac{x^{r/2}}{k^{r/2}\left(1+x^2\left|t-\frac{2\pi h}{a_j k}\right|^2\right)^{r/2}} \left(\frac{k}{h}\right)^{\varrho+1} dt$$

$$\ll x^{r/2} \sum_{h,k} \frac{1}{h^{\varrho+1}} \frac{1}{k^{r/2-\varrho-1}} \int_0^{\frac{c}{k\sqrt{x}}} \frac{du}{(1+x^2u^2)^{r/4}} \ll x^{r/2-1} \sum_{k\leqslant\sqrt{x}} \frac{1}{k^{r/2-\varrho-1}} \ll x^{r/2-1},$$

since $\frac{r}{2}-\varrho-1>1$. This proves the relation (14) and thus completes the proof of Theorem 1.

For the proof of (15) we shall need a finer estimate of the function $F(s)$ than we have used above. Let pairs h_j, k_j, $j = 1, 2, \ldots, \sigma$ be given. Let $s = \frac{1}{x}+it$, $t \in \frac{1}{a_j}\mathfrak{B}_{h_j,k_j}$ (thus $\left|t-\frac{2\pi h_j}{a_j k_j}\right| \leqslant \frac{1}{k_j\sqrt{x}}$), $j = 1, 2, \ldots, \sigma$. Using (16) and (19) we obtain

$$\Theta_Q(s) \ll \frac{x^{r/2}}{\prod_{j=1}^{\sigma}\left(k_j\left(1+x\left|t-\frac{2\pi h_j}{a_j k_j}\right|\right)\right)^{r_j/2}}$$

($|1+ixu| \geqslant 1+x|u|$). Since we have also

$$|s| \geqslant \frac{h_j}{k_j} \gg x^{-1/2} \gg \frac{k_j+\sqrt{x}}{x} \gg \frac{k_j\left(1+x\left|t-\frac{2\pi h_j}{a_j k_j}\right|\right)}{x}$$

for our s and $j = 1, 2, \ldots, \sigma$, we obtain finally for $t \in \bigcap_{j=1}^{\sigma} \frac{1}{a_j}\mathfrak{B}_{h_j,k_j}$

$$F(s) \ll x^{r/2} \prod_{j=1}^{\sigma}\left(k_j^{-1}\min\left(1, \frac{1}{x\left|t-\frac{2\pi h_j}{a_j k_j}\right|}\right)\right)^{r_j/2}. \tag{20}$$

We shall decompose the integration path $A/\sqrt{x} \leqslant t < +\infty$ into intersections of the intervals $\frac{1}{a_j}\mathfrak{B}_{h_j,k_j}$, $j = 1, 2, \ldots, \sigma$. With regard to the factor $\left|t-\frac{2\pi h_j}{a_j k_j}\right|$ in (20) it will be useful to decompose these intersections yet further. For that purpose, let us introduce the following notations. If we are given numbers h_j, k_j (remember that these are still pairs of natural relatively prime numbers, $k_j \leqslant \sqrt{x}$) and non-negative integers $n_1, n_2, \ldots, n_\sigma$, let us denote by

$$\mathfrak{M}(h, k, n) = \mathfrak{M}(h_1, h_2, \ldots, h_\sigma, k_1, k_2, \ldots, k_\sigma, n_1, n_2, \ldots, n_\sigma)$$

the set of all t, which lie in the intersection of the intervals $\frac{1}{a_j}\mathfrak{B}_{h_j,k_j}$ and satisfy the inequalities

$$\frac{1}{2^{n_j+1}k_j\sqrt{x}} < \left|\frac{a_j}{2\pi}t-\frac{h_j}{k_j}\right| \leqslant \frac{1}{2^{n_j}k_j\sqrt{x}}, \quad j = 1, 2, \ldots, \sigma. \tag{21}$$

Obviously, the union of all the sets $\mathfrak{M}(h, k, n)$ covers the whole interval $[A/\sqrt{x}, +\infty)$ with the exception of countably many pairs $\frac{2\pi h_j}{a_j k_j}$, $j = 1, 2, \ldots, \sigma$. For $s = \frac{1}{x}+it$ with $t \in \mathfrak{M}(h, k, n)$, we have by (20) and (21)

$$F(s) \ll \prod_{j=1}^{\sigma} \min{}^{r_j/2}\left(\frac{x}{k_j}, 2^{n_j}\sqrt{x}\right). \tag{22}$$

We shall split the system of the sets $\mathfrak{M}(h, k, n)$ into classes as follows. Let integral, non-negative numbers $l, m_1, m_2, \ldots, m_\sigma, n_1, n_2, \ldots, n_\sigma$ be given. We shall say that the set $\mathfrak{M}(h, k, n)$ belongs to the class $\mathfrak{N}(l, m, n) = \mathfrak{N}(l, m_1, \ldots, m_\sigma, n_1, \ldots, n_\sigma)$ if $2^l \leqslant h_1 < 2^{l+1}$ and $2^{m_j} \leqslant k_j < 2^{m_j+1}$ for $j = 1, 2, \ldots, \sigma$. If $\mathfrak{M}(h, k, n)$ is a set of the class $\mathfrak{N}(l, m, n)$, then for $s = \frac{1}{x}+it$ and $t \in \mathfrak{M}(h, k, n)$ we have by (22)

$$F(s) \ll \prod_{j=1}^{\sigma} \min{}^{r_j/2}\left(\frac{x}{2^{m_j}}, 2^{n_j}\sqrt{x}\right). \tag{23}$$

Since $k_j \leqslant \sqrt{x}$, we may restrict ourselves to those m_j for which $2^{m_j} \leqslant \sqrt{x}$, $j = 1, 2, \ldots, \sigma$. By (21), the measure of a set $\mathfrak{M}(h, k, n)$ of the class $\mathfrak{N}(l, m, n)$ is at most $c2^{-m_1-n_1}x^{-1/2}$, and if t lies in such a set, then

$$|s| \gg 2^{l-m_1}. \tag{24}$$

Also, we shall make use of the following lemma from [1] (p. 135).

Lemma 2. *Let $\varepsilon > 0$ be given. Then the number of non-empty sets $\mathfrak{M}(h, k, n)$ of the class $\mathfrak{N}(l, m, n)$ is at most*

$$U(l, m, n) = c(\varepsilon)2^{(l+m_2+\ldots+m_\sigma)(1+\varepsilon)+\frac{m_1+\ldots+m_{\sigma-1}-n_\sigma}{\beta}} x^{-\frac{1}{2\beta}}$$

where $\beta = \gamma+\varepsilon$.

Let $\mathfrak{A} = \mathfrak{A}(l, m, n)$ be the union of all sets $\mathfrak{M}(h, k, n)$ of the class $\mathfrak{N}(l, m, n)$ and $s = \frac{1}{x} + it$. Because of symmetry, it suffices to estimate the integral

$$\int_{\mathfrak{A}} \frac{F(s)e^{xs}}{s^{\varrho+1}}\, dt$$

only for systems of numbers $l, m_1, \ldots, m_\sigma, n_1, \ldots, n_\sigma$ satisfying

(25) $$2^{m_1+n_1} \geqslant 2^{m_2+n_2} \geqslant \ldots \geqslant 2^{m_\sigma+n_\sigma}$$

and, by what we have said above,

(26) $$2^{m_1} \leqslant \sqrt{x}, \quad 2^{m_2} \leqslant \sqrt{x}, \quad \ldots, \quad 2^{m_\sigma} \leqslant \sqrt{x}.$$

Let us choose now a positive, sufficiently small number $\varepsilon > 0$, $\varepsilon < c$ (an explicite value of this constant can easily be determined at the course of the proof). By what has been said above, by (17), (23) and (24), we have for $s = \frac{1}{x} + it$

(27)
$$\int_{\mathfrak{A}} \frac{F(s)e^{xs}}{s^{\varrho+1}}\, dt \ll \frac{1}{2^{m_1+n_1}\sqrt{x}}\, 2^{(m_1-l)(\varrho+1)}\, U(l, m, n) \prod_{j=1}^{\sigma} \min{}^{r_j/2}\left(\frac{x}{2^{m_j}}, 2^{n_j}\sqrt{x}\right).$$

Clearly, it suffices to us to restrict ourselves only to such systems $l, m_1, \ldots$ $\ldots, m_\sigma, n_1, \ldots, n_\sigma$ for which $U(l, m, n) \geqslant 1$, i.e.

(28) $$2^{-l(1+\varepsilon)} \ll 2^{(m_2+\ldots m_\sigma)(1+\varepsilon)+\frac{m_1+\ldots+m_{\sigma-1}-n_\sigma}{\beta}}\, x^{-\frac{1}{2\beta}}.$$

For given m_j and n_j, $j = 1, 2, \ldots, \sigma$ the summation of (27) over the corresponding l does yield us by (28) for the integral J an estimate of the form

(29) $$c(\varepsilon)\, x^{\frac{r}{4}-\frac{1}{2}-\frac{1}{2\beta}\left(\frac{\varrho-\varepsilon}{1+\varepsilon}+1\right)} \sum_{\substack{m_1,\ldots,m_\sigma\\ n_1,\ldots,n_\sigma}} M_1 M_2 \ldots M_\sigma$$

where

$$M_1 = 2^{-m_1\left(\frac{r_1}{2}-\varrho-\frac{\varrho+1}{\beta(1+\varepsilon)}\right)-n_1} \min{}^{r_1/2}(\sqrt{x}, 2^{m_1+n_1}),$$

$$M_j = 2^{m_j\left(\varrho+1+\frac{1}{\beta}\frac{\varrho+1}{1+\varepsilon}-\frac{r_j}{2}\right)} \min{}^{r_j/2}(\sqrt{x}, 2^{m_j+n_j}), \quad j = 2, 3, \ldots, \sigma-1,$$

and

$$M_\sigma = 2^{m_\sigma\left(\varrho+1-\frac{r_\sigma}{2}\right)-\frac{n_\sigma}{\beta}\left(1+\frac{\varrho-\varepsilon}{1+\varepsilon}\right)} \min{}^{r_\sigma/2}(\sqrt{x}, 2^{m_\sigma+n_\sigma}).$$

In the following computations, it will be convenient to use the following convention: if we have a positive function of the variable ε defined on certain interval $0 < \varepsilon < c$ and having limit zero for $\varepsilon \to 0+$, then instead of its explicite expression we shall write usually simply μ. Thus, for example, we can write μ instead of ε^2 or εc, we can write $\frac{\varrho}{2\gamma} - \mu$ instead of $\frac{\varrho-\varepsilon}{2\beta}$, etc. This (somewhat unusual) notation will simplify a little bit our successive results. Let us mention also that we assume $r_j \geqslant \frac{2(\varrho+1)(\gamma+1)}{\gamma}$, and thus $r_j > \frac{2(\varrho+1)(\beta+1)}{\beta}$, $j = 1, 2, \ldots, \sigma$.

For $j = 2, \ldots, \sigma$ we put

$$S_j = \sum_{n_j, m_j} M_j = \sum_{2^{n_j+m_j} \leqslant \sqrt{x}} M_j + \sum_{2^{n_j+m_j} > \sqrt{x}} M_j = S_{j1} + S_{j2},$$

where the summation runs over all natural numbers m_j, n_j for which (25) and (26) holds. Thus, we have

(30) $$J \ll x^{\frac{r}{4}-\frac{1}{2}-\frac{\varrho+1}{2\gamma}+\mu} \sum_{m_1, n_1} M_1 S_2 \ldots S_\sigma.$$

First, let $j = \sigma$. Then (we omit the condition (25))

$$S_{\sigma 1} \ll \sum_{2^{m_\sigma+n_\sigma} \leqslant \sqrt{x}} 2^{m_\sigma(\varrho+1)-n_\sigma\left(\frac{\varrho+1}{(1+\varepsilon)\beta}-\frac{r_\sigma}{2}\right)} \ll \sum_{2^{n_\sigma} \leqslant \sqrt{x}} \left(\frac{\sqrt{x}}{2^{n_\sigma}}\right)^{\varrho+1} 2^{n_\sigma\left(\frac{r_\sigma}{2}-\frac{\varrho+1}{(1+\varepsilon)\beta}\right)}$$
$$\ll x^{\frac{\varrho+1}{2}+\mu} \sum_{2^{n_\sigma} \leqslant \sqrt{x}} 2^{n_\sigma\left(\frac{r_\sigma}{2}-\frac{(\varrho+1)(\beta+1)}{\beta}+\mu\right)} \ll x^{\frac{r_\sigma}{4}-\frac{\varrho+1}{2\gamma}+\mu},$$

$$S_{\sigma 2} \ll x^{\frac{r_\sigma}{4}} \sum_{2^{m_\sigma+n_\sigma} > \sqrt{x}} 2^{m_\sigma\left(\varrho+1-\frac{r_\sigma}{2}\right)-n_\sigma\frac{\varrho+1}{(1+\varepsilon)\beta}}$$
$$\ll x^{\frac{r_\sigma}{4}} \sum_{2^{m_\sigma} \leqslant \sqrt{x}} 2^{m_\sigma\left(\varrho+1-\frac{r_\sigma}{2}\right)} \left(\frac{2^{m_\sigma}}{\sqrt{x}}\right)^{\frac{\varrho+1}{(1+\varepsilon)\beta}}$$
$$\ll x^{\frac{r_\sigma}{4}-\frac{\varrho+1}{2\beta}+\mu} \sum_{2^{m_\sigma} \leqslant \sqrt{x}} 2^{-m_\sigma\left(\frac{r_\sigma}{2}-\frac{(\varrho+1)(\beta+1)}{\beta}+\mu\right)} \ll x^{\frac{r_\sigma}{4}-\frac{\varrho+1}{2\gamma}+\mu}.$$

It follows from these estimates that

(31) $$S_\sigma \ll x^{\frac{r_\sigma}{4}-\frac{\varrho+1}{2\gamma}+\mu}.$$

Secondly, let $1 < j < \sigma$. First of all, we have (again without (25))

$$S_{j1} \ll \sum_{2^{m_j+n_j} \leqslant \sqrt{x}} 2^{m_j\left(\varrho+1+\frac{1}{\beta}\frac{\varrho+1}{1+\varepsilon}\right)+n_j\frac{r_j}{2}} \ll \sum_{2^{m_j} \leqslant \sqrt{x}} 2^{m_j\left(\varrho+1+\frac{1}{\beta}\frac{\varrho+1}{1+\varepsilon}\right)} \left(\frac{\sqrt{x}}{2^{m_j}}\right)^{r_j/2}$$

$$\ll x^{r_j/4} \sum_{2^{m_j} \leqslant \sqrt{x}} 2^{-m_j\left(\frac{r_j}{2}-\frac{(\gamma+1)(\varrho+1)}{\gamma}+\mu\right)} \ll x^{r_j/4}.$$

For the estimate of S_{j2}, let us notice that by the definition of S_{j2} and by (25) we have $\sqrt{x} < 2^{m_j+n_j} \leqslant 2^{m_1+n_1}$. We obtain (with this summation region)

$$S_{j2} \ll x^{r_j/4} \sum 2^{m_j\left(\varrho+1+\frac{1}{\beta}\frac{\varrho+1}{1+\varepsilon}-\frac{r_j}{2}\right)} \ll x^{r_j/4} \sum_{2^{m_j} \leqslant \sqrt{x}} 2^{\varepsilon(m_1+n_1)} 2^{-m_j\left(\frac{r_j}{2}-\frac{(\varrho+1)(\gamma+1)}{\gamma}+\mu\right)}$$

$$\ll x^{\frac{r_j}{4}} 2^{\frac{\varepsilon(m_1+n_1)}{\sigma}},$$

since the number of those n_j for which $2^{m_j+n_j} \leqslant 2^{m_1+n_1}$ is for each $\varepsilon > 0$ at most $c(\varepsilon)2^{\varepsilon(m_1+n_1-m_j)}$. Together with (30) and (31), we obtain

$$(32) \qquad J \ll x^{\frac{r}{2}-\frac{1}{2}-\frac{\varrho+1}{\gamma}-\frac{r_1}{4}+\mu}\Bigg(\sum_{2^{m_1+n_1} \leqslant \sqrt{x}} 2^{m_1\left(\varrho+\frac{1}{\beta}+\frac{\varrho-\varepsilon}{\beta(1+\varepsilon)}\right)+n_1\left(\frac{r_1}{2}-1\right)} +$$

$$+ x^{r_1/4} \sum_{2^{m_1+n_1} > \sqrt{x}} 2^{m_1\left(-\frac{r_1}{2}+\varrho+\frac{1}{\beta}+\frac{\varrho-\varepsilon}{\beta(1+\varepsilon)}+\varepsilon\right)-n_1(1-\varepsilon)}\Bigg).$$

(The factor $2^{\varepsilon(m_1+n_1)}$ by the second summand comes from S_{j2}, $j = 2, \ldots, \sigma-1$.) Consequently, for the first sum on the right-hand side of (32) we get the estimate

$$\sum_{2^{n_1} \leqslant \sqrt{x}} \left(\frac{\sqrt{x}}{2^{n_1}}\right)^{\varrho+\frac{1}{\beta}+\frac{\varrho-\varepsilon}{\beta(1+\varepsilon)}} 2^{n_1\left(\frac{r_1}{2}-1\right)} \ll x^{\frac{\varrho}{2}+\frac{\varrho+1}{2\beta(1+\varepsilon)}} \sum_{2^{n_1} \leqslant \sqrt{x}} 2^{n_1\left(\frac{r_1}{2}-1-\varrho-\frac{\varrho+1}{\beta(1+\varepsilon)}\right)}$$

$$\ll x^{\frac{\varrho}{2}+\frac{\varrho+1}{2\beta(1+\varepsilon)}+\frac{r_1}{4}-\frac{1}{2}-\frac{\varrho}{2}-\frac{\varrho+1}{2\beta(1+\varepsilon)}} \ll x^{\frac{r_1}{4}-\frac{1}{2}},$$

and for the other sum we have

$$x^{\frac{r_1}{4}} \sum_{2^{m_1+n_1} > \sqrt{x}} 2^{m_1\left(-\frac{r_1}{2}+\varrho+\frac{\varrho+1}{\beta(1+\varepsilon)}+\varepsilon\right)-n_1(1-\varepsilon)}$$

$$\ll x^{\frac{r_1}{4}} \sum_{2^{m_1} \leqslant \sqrt{x}} 2^{m_1\left(-\frac{r_1}{2}+\varrho+\frac{\varrho+1}{\beta(1+\varepsilon)}+\varepsilon\right)} \left(\frac{2^{m_1}}{\sqrt{x}}\right)^{1-\varepsilon}$$

$$= x^{\frac{r_1}{4}-\frac{1}{2}+\frac{\varepsilon}{2}} \sum_{2^{m_1} \leqslant \sqrt{x}} 2^{-m_1\left(\frac{r_1}{2}-\frac{(\varrho+1)(\beta+1+\beta\varepsilon)}{\beta(1+\varepsilon)}\right)} \ll x^{\frac{r_1}{4}-\frac{1}{2}+\frac{\varepsilon}{2}}.$$

These estimates yield to us finally

$$J \ll x^{\frac{r}{2}-1-\frac{\varrho+1}{\gamma}+\mu}.$$

In view of the definition of J, this proves (15) and thus completes the proof of the Main Theorem.

Let us remark that using the same procedure it is possible to derive certain estimates of the remainder term $P_\varrho(x)$ also without the assumption $r_j \geqslant \dfrac{2(\varrho+1)(\gamma+1)}{\gamma}$ (analogous to Theorem 3, p. 138 in [1]), and to generalize Theorem 1 from [2] to $P_\varrho(x)$ ($\varrho > 0$) as well.

Finally, it is interesting to compare the Main Theorem with the following result (see [9], Theorem 3 or [10], p. 764):

Let r be a natural number and $\alpha_1, \alpha_2, \ldots, \alpha_r$ be real numbers. Let $Q(u) = Q(u_1, u_2, \ldots, u_r)$ be a positive definite quadratic form with integral coefficients and determinant D. Put

$$A(x) = \sum e^{2\pi i \sum_{j=1}^{r} \alpha_j m_j},$$

where the summation runs over all systems of integral numbers $m_1, m_2, \ldots, m_r$ satisfying $Q(m_1, m_2, \ldots, m_r) \leqslant x$. Let

$$P(x) = A(x) - \frac{\pi^{r/2} x^{r/2} \delta}{\sqrt{D}\,\Gamma\left(\frac{r}{2}+1\right)},$$

where $\delta = 1$ if all α_j are integers and $\delta = 0$ otherwise. Let us define $P_\varrho(x)$ as in (5), put $\gamma = \gamma(1, \alpha_1, \alpha_2, \ldots, \alpha_r)$ and assume $0 \leqslant \varrho < \dfrac{r}{2}-1-\dfrac{1}{\gamma}$. Then

$$\limsup_{x \to +\infty} \frac{\log|P_\varrho(x)|}{\log x} = \frac{r}{2}-1-\frac{1}{2(\gamma+1)}\left(\frac{r}{2}-1-\varrho\right).$$

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On a problem of Davenport and Schinzel

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We set for every integer l: $\mathbf{l} = \{0, 1, \ldots, l-1\}$.

Definition 1. A function $a\colon \mathbf{N}\to\mathbf{n}$ is said to be an *admissible n-sequence of length N* if $a_i \neq a_{i+1}$ for $i+1 < N$ (a_i is the value of the function at the place i).

We say that *a contains an alternating l-sequence* if there are numbers $b \neq c$ and $0 < i_0 < \ldots < i_{l-1} < N$ such that

$$\begin{cases} a_{i_{2s}} = c & \text{if} \quad 0 \leqslant 2s < l, \\ a_{i_{2s+1}} = b & \text{if} \quad 1 \leqslant 2s+1 < l. \end{cases} \tag{1}$$

Definition 2. $N_l(n) = \max\big(N$: there is an admissible n-sequence of length N not containing an alternating $(l+1)$-sequence$\big)$.

Remark. One can extend the notion of an admissible n-sequence of length N replacing in Definition 1 the set $\mathbf{n}$ by an arbitrary set of n elements. Clearly such an extension does not affect the definition of $N_l(n)$. A finite sequence $a\colon \mathbf{N}\to X$ will often be denoted by $\langle a_0, \ldots, a_{N-1}\rangle$ and the set of its elements by $\{a_0, \ldots, a_{N-1}\}$.

It is known from [1] and [2] that $N_l(n)$ exists for every l and n, and we have

$$N_3(n) = 2n-1, \tag{2}$$

$$N_l(n) > (l^2-4l+3)n - C(l) \tag{3}$$

if l is odd and $l > 3$,

$$N_l(n) > (l^2-5l+8)n - C(l) \tag{4}$$

if l is even and $l > 4$, where $C(l)$ is a constant depending on l only, and

$$N_4(n) \geqslant 5n-8, \quad \lim \frac{N_4(n)}{n} \geqslant 8. \tag{5}$$