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On the upper asymptotic density of $(0, r)$-primitive sequences

by

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1. In this paper A will denote a subsequence of the sequence of positive integers. For a set V we denote by $A(V) = A(V, A)$ the number of elements of $A \cap V$. Moreover we put

$$\underline{d}A = \liminf \frac{A([1, n])}{n} \quad \text{and} \quad \bar{d}A = \limsup \frac{A([1, n])}{n}$$

for the lower and upper asymptotic density of A; if $\underline{d}A = \bar{d}A$ we write dA for the asymptotic density of A.

A sequence $A = (a_i)$ is called *primitive* if $a_i \not\equiv 0 \pmod{a_j}$ if $i \neq j$. For a survey of the theory of primitive sequences we refer to [5], chapter V and [4]. We only state here three well-known results, see [5], p. 244–245.

THEOREM 1. *If A is a primitive sequence, then $\bar{d}A < \frac{1}{2}$.*

THEOREM 2. (Behrend [1].) *For every primitive sequence, $\underline{d}A = 0$.*

THEOREM 3. (Besicovitch [2].) *Corresponding to every $\varepsilon > 0$, there exists a primitive sequence A, depending on ε, such that $\bar{d}A > \frac{1}{2} - \varepsilon$.*

Let r be a positive integer. We will call in this paper a sequence $A = (a_i)$ *$(0, r)$-primitive* if $a_i \not\equiv 0, r \pmod{a_j}$ if $i \neq j$. In the following sections we give estimations for $\bar{d}A$ of $(0, r)$-primitive sequences, similar to the Theorems 1 and 3.

2. In this section we study $(0, r)$-primitive sequences with r odd.

THEOREM 4. *Let r be an odd positive integer. If A is a $(0, r)$-primitive sequence then $\bar{d}A \leqslant \frac{1}{4}$.*

Proof. Let n be a positive integer and $a_1, \ldots, a_t$ the elements of A not exceeding n. Let a_i' $(1 \leqslant i \leqslant t)$ denote the greatest odd divisor of a_i and $A' = (a_i')_{i=1}^{t}$. Since $a_i' = a_j'$ implies $a_i | a_j$ or $a_j | a_i$ all numbers a_i' are distinct.

We construct a one-to-one correspondance between the odd integers in $[1, \frac{1}{2}n]$ and the odd integers in $(\frac{1}{2}n + r, n + r]$. To every odd integer c in $[1, \frac{1}{2}n]$ there exists exactly one integer of the form $2^k c$ in $(\frac{1}{2}n, n]$ and therefore exactly one odd integer of the form $2^k c+r$ in $(\frac{1}{2}n+r, n+r]$. Put $f(c) = 2^k c+r$. If c_1 and c_2 are distinct odd integers in $[1, \frac{1}{2}n]$ then $f(c_1) \neq f(c_2)$ and the relation between c and $f(c)$ is one-to-one.

We prove that from a pair $c, f(c)$ at most one occurs in A'. Suppose $c \in A'$ and $f(c) = 2^k c+r \in A'$. Then $c \in A'$ implies that there is an element $a_i \in A$ with $a_i = 2^h c$ where $0 \leqslant h \leqslant k$. On the other hand $f(c) \in A'$ implies, since $f(c) > \frac{1}{2}n$, that $f(c) \in A$. However, $f(c) = 2^k c+r \equiv r \pmod{a_i}$, which is a contradiction. This proves the theorem.

THEOREM 5. *Let r be an odd positive integer and ε a positive real number. There exists a $(0, r)$-primitive sequence A such that $\bar{d}A > \frac{1}{4}-\varepsilon$.*

Proof. According to Theorem 3 there exists a primitive sequence $A_0 = (a_i)$ such that $\bar{d}A_0 > \frac{1}{2}-2\varepsilon$. Then $A = (2a_i)$ is a $(0, r)$-primitive sequence satisfying the condition of the theorem.

3. If r is even the situation is more complicated and we have not succeeded in solving the problem entirely. First we prove a result similar to Theorem 3 and Theorem 5.

THEOREM 6. *Let r be an even integer and ε a positive real number, then there exists a $(0, r)$-primitive sequence A, such that $\bar{d}A > \frac{7}{24}-\varepsilon$.*

The proof is based on two lemmas.

LEMMA 1. *Let r be an even positive integer and T an arbitrary positive integer. There exists a $(0, r)$-primitive sequence A_T in $[T, 3T]$ such that $A([T, 3T], A_T) > \frac{7}{24}\cdot 3T-2r$.*

Proof. Let A_T consist of the integers in $[T, 2T]$ which are modulo $2r$ congruent to one of the numbers $0, 1, 2, \ldots, r-1$ and of the integers in $(2T, 3T]$ which are modulo $2r$ congruent to one of the odd numbers $1, 3, 5, \ldots, r-1$ or modulo $4r$ to one of the even numbers $3r, 3r+2, \ldots$ $\ldots, 4r-2$.

Obviously A_T is a $(0, r)$-primitive sequence. Moreover

$$A([T, 2T], A_T) \geqslant \left(\frac{T}{2r}-1\right) r$$

and

$$A\big((2T, 3T], A_T\big) \geqslant \left(\frac{T}{2r}-1\right)\cdot \tfrac{1}{2}r + \left(\frac{T}{4r}-1\right)\cdot \tfrac{1}{2}r$$

and Lemma 1 follows.

For a sequence A we denote by $B(A)$ the set consisting of all distinct positive multiples of elements of A. It holds (for a proof see [5], p. 256):

LEMMA 2. (Erdös [3].) *Let S_T denote the set of integers lying in the interval $(T, 2T]$. Then*

$$\lim_{T\to\infty} dB(S_T) = 0.$$

Writing $[T, 3T] = [T, 2T] \cup [\frac{3}{2}T, 3T]$ we see that the same result holds with S_T replaced by the set U_T of all integers lying in $[T, 3T]$.

Proof of Theorem 6. Let $0 < \varepsilon < \frac{1}{4}$ and put $\varepsilon_k = \frac{3}{4}(\frac{1}{2})^{k+1}\varepsilon$ $(k = 1, 2, \ldots)$. In view of Lemma 2 we may choose an infinite sequence $T_1, T_2, \ldots$ of positive integers satisfying

$$d_k = dB(U_{T_k}) < \varepsilon_k$$

and

$$T_{k+1} > (3T_k)!.$$

In $[T_k, 3T_k]$ $(k = 1, 2, \ldots)$ we take the set of integers A_{T_k} from Lemma 1. Let G_0 be the union of these sets A_{T_k}, then by Lemma 1,

$$A([1, 3T_k], G_0) > \tfrac{7}{24}\cdot 3T_k-2r.$$

Let G be the $(0, r)$-primitive sequence obtained from G_0 by removing from A_{T_k} $(k = 1, 2, \ldots)$ all those integers which belong to $B_{k-1} = \bigcup_{i=1}^{k-1} B(A_{T_i})$ and to $r+B_{k-1}$. We observe that $B(A_{T_i})$ can be represented as the union of a number of congruence classes to the modulus $(3T_i)!$. Thus, since $T_k > (3T_i)!$ $(1 \leqslant i \leqslant k-1)$, the set $[T_k, 3T_k]$ contains at most

$$2dB(A_{T_i})\cdot 2T_k < 4d_i T_k < 4\varepsilon_i T_k$$

members of $B(A_{T_i})$ $(1 \leqslant i \leqslant k-1)$. Hence

$$A([1, 3T_k], G) > \tfrac{7}{24}\cdot 3T_k-2r-\sum_{i=1}^{k-1} 8\varepsilon_i T_k > \tfrac{7}{24}\cdot 3T_k-2r-3\varepsilon T_k.$$

From this inequality Theorem 6 follows.

We continue by proving two theorems similar to the Theorems 1 and 4.

THEOREM 7. *Let $r = 2a$, where a is an odd positive integer. If A is a $(0, r)$-primitive sequence, then $\bar{d}A \leqslant \frac{5}{16}$.*

Proof. Let n be a positive integer. We divide the elements of A in $[1, n]$ into two classes $E = E_n$ and $O = O_n$, the set of even and the set of odd elements. Let for $a_i \in E$ the integer a_i' denote the greatest odd divisor of a_i. As in the proof of Theorem 4 the elements a_i' are distinct and differ from the elements in O. Put $E' = (a_i')$, where $a_i \in E$ and $A' = A_n' = E' \cup O$.

Similar to the proof of Theorem 4 we make a one-to-one correspondence between the odd integers in $[1, \frac{1}{4}n]$ and the odd integers in $(\frac{1}{4}n+a, \frac{1}{2}n+a]$, such that if $v(c)$ in $[1, \frac{1}{4}n]$ and c in $(\frac{1}{4}n+a, \frac{1}{2}n+a]$ are corresponding odd integers then $c = 2^{k(c)}v(c)+a$ for some integer $k(c)$.

Let c be an odd integer in $(\frac{1}{4}n+a, \frac{1}{2}n]$ and let c belong to E'. We will show that then $v(c) \notin A'$. Since $c > \frac{1}{4}n$, obviously $2c \in A$. Suppose

$v(c)\in A'$, then there exists an element $a_i\in A$ with $a_i = 2^h v(c)$ $(0\leqslant h \leqslant k(c)+1)$ and then

$$2c = 2\left(2^{k(c)}v(c)+a\right) \equiv 2a \pmod{a_i},$$

which is a contradiction.

Consider now two odd integers c and $c-r$ in $(\frac{1}{4}n+a, \frac{1}{2}n]$ with corresponding $v(c)$ and $v(c-r)$ in $[1, \frac{1}{2}n]$. We will show that at least one of them does not occur in A'. Suppose that they all belong to A'. Then $c\in E'$ or $c\in O$. If $c\in E'$, then $v(c)\notin A'$; contradiction. If, on the other hand, $c\in O$, then since A is a $(0, r)$-primitive sequence $c-r\notin O$, therefore $c-r\in E'$, which implies $v(c-r)\notin A'$; contradiction.

From this we get

$$\limsup \frac{A([1, \frac{1}{2}n], A'_n)}{n} \leqslant \frac{3}{4}\cdot\frac{1}{4}. \tag{1}$$

If c is an odd integer in $(\frac{1}{2}n, n]$ and $c\in A'$, then obviously $c\in A$. Therefore from a pair of odd integers c, $c+r$ in $(\frac{1}{2}n, n]$ at most one occurs in A'. Hence

$$\limsup \frac{A\left((\frac{1}{2}n, n], A'_n\right)}{n} \leqslant \frac{1}{8}. \tag{2}$$

From (1) and (2) follows Theorem 7.

THEOREM 8. *Let $r = 4b$, where b is a positive integer. If A is a $(0, r)$-primitive sequence, then*

$$\bar{d}A \leqslant \begin{cases} \dfrac{21}{64} & \text{if } b \text{ is odd},\\[2mm] \dfrac{43}{128} & \text{if } b \text{ is even}.\end{cases}$$

Proof. Let n be a positive integer. We define E, O, E' and A' as in the proof of Theorem 7. As above we see that (2) holds.

Consider the interval $(\frac{1}{4}n, \frac{1}{2}n]$. Let c be an odd integer in $(\frac{1}{4}n, \frac{1}{2}n]$, such that $c\in E'$. Then, obviously $2c\in A$. This implies $2c-r\notin A$ and $c-2b\notin A$. Therefore the odd integer $c-2b$ does not occur in A'.

We will show now that from a set of four odd integers $c-6b$, $c-4b$, $c-2b$, c in $(\frac{1}{4}n, \frac{1}{2}n]$ at least one does not occur in A'. Suppose that the four integers are elements from A'. Then $c\in E'$ or $c\in O$. If $c\in E'$ then $c-2b\notin A'$; contradiction. If, on the other hand, $c\in O$, then $c-4b\notin O$, thus $c-4b\in E'$ and $c-6b\notin A'$; contradiction. Hence

$$\limsup \frac{A\left((\frac{1}{4}n, \frac{1}{2}n], A'_n\right)}{n} \leqslant \frac{3}{4}\cdot\frac{1}{8}. \tag{3}$$

For an estimation of $A([1, \frac{1}{4}n], A')$ we distinguish two cases: 1° b odd; 2° b even.

ad 1. Similar to the proof of Theorem 4 we make a one-to-one correspondence between the odd integers $v(c)$ in $[1, \frac{1}{8}n]$ and the odd integers c in $(\frac{1}{8}n+b, \frac{1}{4}n+b]$, such that $c = 2^{k(c)}v(c)+b$ for some integer $k(c)$.

Let c be an odd integer in $(\frac{1}{8}n+b, \frac{1}{4}n]$ with $c\in E'$. Then $2c\in A$ or $4c\in A$. Now $2c\in A$ implies $2c-4b\notin A$ and $c-2b\notin A$, therefore $c-2b\notin A'$. On the other hand, $4c\in A$ gives $v(c)\notin A'$, since $v(c)\in A'$ implies that there exists an $a_i\in A$ with $a_i = 2^h v(c)$ $(0\leqslant h\leqslant k+2)$ and then

$$4c \equiv 4\left(2^{k(c)}v(c)+b\right) \equiv 4b \pmod{a_i},$$

which is a contradiction. Hence if c is odd in $(\frac{1}{8}n+b, \frac{1}{4}n]$ and $c\in E'$, then $c-2b\notin A'$ or $v(c)\notin A'$.

Consider the four odd integers $c-6b$, $c-4b$, $c-2b$, c in $(\frac{1}{8}n+b, \frac{1}{4}n]$ with corresponding elements in $(1, \frac{1}{8}n]$. With a similar argument as above we see that at least one of them does not occur in A'. Therefore

$$\limsup \frac{A([1, \frac{1}{4}n], A'_n)}{n} \leqslant \frac{7}{8}\cdot\frac{1}{8}. \tag{4}$$

ad 2. We divide $[1, \frac{1}{4}n]$ into two parts, $[1, \frac{1}{8}n]$ and $(\frac{1}{8}n, \frac{1}{4}n]$. If c is an odd integer in $(\frac{1}{8}n, \frac{1}{4}n]$ and $c\in E'$, then $2c\in A$ or $4c\in A$. If $4c\in A$ then, as above, $c-b\notin A'$. If, on the other hand, $2c\in A$, then $2c-4b\notin A$ and $c-2b\notin A$. Then $c-2b\in A'$ if and only if $4c-8b\in A$; in the last case, however, $c-3b\notin A'$. Therefore $2c\in A$ implies $c-2b\notin A'$ or $c-3b\notin A'$. Hence $c\in E'$ implies that at least one of the elements $c-b$, $c-2b$ and $c-3b$ does not occur in A'.

From this we can prove as above that from the eight odd elements $c-tb$, $t = 0, 1, 2, \ldots, 7$, in $(\frac{1}{8}n, \frac{1}{4}n]$ at least one does not occur in A'. Hence

$$\limsup \frac{A\left((\frac{1}{8}n, \frac{1}{4}n], A'_n\right)}{n} \leqslant \frac{7}{8}\cdot\frac{1}{16}. \tag{5}$$

Moreover, trivially

$$\limsup \frac{A([1, \frac{1}{8}n], A'_n)}{n} \leqslant \frac{1}{16}. \tag{6}$$

If b is odd the theorem follows from (2), (3) and (4); if b is even we get Theorem 8 from (2), (3), (5) and (6).

4. For special values of r we can derive upper bounds for $\bar{d}A$ which are lower than the values given in Theorem 7 and Theorem 8. We treat here the case $r = 2$, for which we prove the following result.

THEOREM 9. *If A is a $(0,2)$-primitive sequence, then* $\bar{d}A \leqslant \frac{7}{24}+\frac{1}{144}$.

Proof. Let n be a positive integer and let E, O, E' and A' be defined as in the proof of Theorem 7. We will also use the one-to-one correspondence from the proof of Theorem 7 between the odd integers $v(c)$ in $[1, \frac{1}{4}n]$ and the odd integers c in $(\frac{1}{4}n+1, \frac{1}{2}n+1]$. We define three subsets of the set of odd integers in $[1, n]$:

1° the set C_1 of odd integers in $(\frac{1}{2}n, \frac{3}{4}n]$,

2° the set C_2 consisting of the odd integers in $(\frac{1}{3}n+1, \frac{1}{2}n+1]$ with the corresponding odd integers in $[1, \frac{1}{4}n]$;

3° the set C_3 consisting of the odd integers in $(\frac{1}{4}n+1, \frac{1}{3}n]$ with the corresponding odd integers in $[1, \frac{1}{4}n]$ and the set of odd integers in $(\frac{3}{4}n, n]$.

ad C_1. As in the proof of relation (2) we see that from a pair of odd integers c, $c+2$ in $(\frac{1}{2}n, \frac{3}{4}n]$ at most one occurs in A'. Therefore

$$\limsup \frac{A(C_1, A'_n)}{n} \leqslant \frac{1}{16}. \tag{7}$$

ad C_2. From the proof of relation (1) it follows that from a pair of odd integers c, $c+2$ in $(\frac{1}{3}n+1, \frac{1}{2}n]$ with corresponding odd integers $v(c)$, $v(c+2)$ in $[1, \frac{1}{4}n]$ at least one does not occur in A'. Hence

$$\limsup \frac{A(C_2, A'_n)}{n} \leqslant \frac{3}{4}\cdot\frac{1}{6}. \tag{8}$$

ad C_3. We recall that if c is an odd integer in $(\frac{1}{4}n+1, \frac{1}{3}n]$ and $c \in O$, then $c+2 \notin O$. Therefore we can divide the odd integers in $(\frac{1}{4}n+1, \frac{1}{3}n]$ into sets $\{c, c+2, \ldots, c+2s\}$ with

$$c, c+2, \ldots, c+2s-2 \notin O, \quad c+2s \in O \quad \text{and} \quad s \geqslant 1.$$

(It is possible that there remain two sets: a first one consisting of one single integer $c \in O$ and a last one $\{c, \ldots, c+2s\}$ with $c+2s \notin O$. It is easy to check, however, that they do not disturb our arguments below.)

Consider a set D which is the union of the following three sets: a) a set of $s+1$ consecutive odd integers $\{c, c+2, \ldots, c+2s\}$ in $(\frac{1}{4}n+1, \frac{1}{3}n]$ with $c, \ldots, c+2s-2 \notin O$, $c+2s \in O$; b) the set of corresponding integers $\{v(c), \ldots, v(c+2s)\}$ in $[1, \frac{1}{4}n]$; c) the set of $3s+3$ consecutive odd integers $\{3c-2, 3c, 3c+2, \ldots, 3c+6s-2, 3c+6s, 3c+6s+2\}$ in $(\frac{3}{4}n, n]$.

We derive an upper bound for $A(D, A')$.

If $c+2\sigma$ $(0 \leqslant \sigma \leqslant s-1)$ occurs in E' then, as in the proof of Theorem 7, $v(c+2\sigma) \notin A'$. Therefore from a pair $c+2\sigma, v(c+2\sigma)$ $(0 \leqslant \sigma \leqslant s-1)$ at most one occurs in A'.

Furthermore, since $c+2s \in O$, the odd integers $3c+6s$ and $3c+6s+2$ do not occur in A'.

Finally, from two consecutive odd integers in $\{3c-2, \ldots, 3c+6s-2\}$ at most one occurs in A'.

Therefore we get

$$A(D, A') \leqslant \begin{cases} s+2+\frac{1}{2}(3s+1) & \text{if } 3s+1 \text{ is even}, \\ s+2+\frac{1}{2}(3s+1)+\frac{1}{2} & \text{if } 3s+1 \text{ is odd}. \end{cases}$$

For s odd we write $s+2+\frac{1}{2}(3s+1) = \frac{5}{2}(s+1)$ and for s even we have

$$s+2+\tfrac{1}{2}(3s+1)+\tfrac{1}{2} = \left\{\frac{5}{2}+\frac{1}{2(s+1)}\right\}(s+1) \leqslant \left(\frac{5}{2}+\frac{1}{6}\right)(s+1).$$

This implies

$$\limsup \frac{A(C_3, A'_n)}{n} \leqslant \left(\frac{5}{2}+\frac{1}{6}\right)\cdot\frac{1}{24}. \tag{9}$$

From (7), (8) and (9) we get Theorem 9.

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