Since the integral in the last term of (13) is bounded in $x$, putting $a_j = (-1)^{j-1}E^{(j-1)}(t)|_{t=0}$, (13) becomes

$$\int_{-3/4}^{0} E(t)x^{t+1}dt = x\sum_{j=1}^{a} \frac{a_j}{(\log x)^j} + O\left(\frac{x}{(\log x)^{a+1}}\right). \tag{14}$$

The integral of the error term in (10) gives

$$x^{1/2}\log x \int_{-3/4}^{0} x^t dt = O(x^{1/2}), \quad \text{and} \quad \int_{-3/4}^{0} tE(t)dt = O(1),$$

which, combined by (10), (11), (12) and (14), completes the proof of the Theorem.

The constructive remarks of the referee are greatly appreciated.

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# On the simultaneous diophantine approximation of values of certain hypergeometric and algebraic functions

by

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**Introduction.** In this paper we shall prove a number of theorems concerning the arithmetic properties of functions which we shall denote as $B$-functions. This class of functions includes many well known functions of classical analysis.

DEFINITIONS. A function $g(z)$ is an *A-function* if there exists an effective algorithm for computing a positive constant $\gamma$ and a finite set of ordered pairs $(a_j, \beta_j)$, where each $a_j$ is an algebraic number [1] and each $\beta_j$ is a non-negative integer, such that $g(z)$ may be written as a finite sum of functions of the form

$$f_j(z) = z^{a_j}(\log(z))^{\beta_j} g_{a_j,\beta_j}(z) \tag{1}$$

where: (a) The function $g_{a_j,\beta_j}(z)$ is analytic at $z = \infty$. (b) Each derivative of $g_{a_j,\beta_j}(z)$ at $z = \infty$ is algebraic. (c) There exist $T_j(n)$, a non-vanishing Gaussian integral valued function defined on the positive integers, and a positive integer $M_j$ such that (i) $M_j < \gamma$, (ii) $|T_j(n)| < \gamma^n$ for all $n \geqslant 1$, (iii) $M_j g_{a_j,\beta_j}(\infty)$ and each $T_j(n)(n!)^{-1}g^{(n)}_{a_j,\beta_j}(\infty)$ are algebraic integers, and (iv) the absolute values of the conjugates of $M_j g_{a_j,\beta_j}(\infty)$, and each $T_j(n)(n!)^{-1}g^{(n)}_{a_j,\beta_j}(\infty)$ are less than $\gamma$ and $\gamma^n$, respectively.

We shall say that a function $g(z)$ is a *B-function* if there exists an effective algorithm for calculating not only $\gamma$ and a set of $(a_j, \beta_j)$ as above but, also, a positive constant $\gamma_1$ such that, for a set of $f_j$ as above, $g(z) = \sum_j c_j f_j$ where each $c_j \in C$ and each $|c_j| < \gamma_1$.

Our first result is:

THEOREM I. *If $y(z)$ is a solution of $q(z, y) = 0$, where $q(z, y) \not\equiv 0$ is a polynomial in $z$ and $y$ with coefficients in $Q(i)$, then $y(z)$ is an A-function.*

[1] By effectively computing an algebraic number $a_j$ we mean being able to approximate it effectively to within any preassigned error by an element of $Q(i)$ as well as being able to effectively compute a non-zero polynomial equation with coefficients in $Q(i)$ which is satisfied by $a_j$. (Given $a_1, a_2, a_3$, and $a_4$ which have been effectively computed we may effectively determine, for example, if $a_1+a_2 = a_3+a_4$.)

DEFINITION. Let $W(y_1, \ldots, y_m)$ denote the Wronskian of $y_1, \ldots, y_m$.

Our next result is about the simultaneous diophantine approximation of $B$-functions, for large Gaussian integral values of the variable $z$.

Suppose that $y$ is a non-polynomial $B$-function which satisfies a given linear homogeneous differential equation of order $m$ with coefficients in $Q(i, z)$ which has as a fundamental system of solutions around $z = \infty$ a collection of $A$-functions $y_1, \ldots, y_m$ such that $(W(y_1, \ldots, y_m))^{-1}$ is also an $A$-function. Let $a_1, \ldots, a_r, \ldots, a_n$ denote $n \geqslant 2$ distinct elements of $Z[i]$ such that the difference of no two distinct $a_r$'s equals a singularity of the above linear differential equation satisfied by $y$. Let $(p_{0,1}, \ldots, p_{N-1,n})$ denote a non-zero $(Nn)$-tuple of Gaussian integers. Let $E_1^{\theta} y$ denote any $\theta$-fold integral of $y$ for $\theta = 0, 1, \ldots$ Let $N_1$ denote a Gaussian integer.

THEOREM II. *There exist a non-negative integer $N$ and two effectively computable functions $\varphi = \varphi(\varepsilon, y, a_1, \ldots, a_n)$ and $\delta = \delta(\varepsilon, y, a_1, \ldots, a_n)$ such that if $|N_1| > \varphi$ and $|q| > |N_1|^{\delta}$ then*

$$(2) \qquad \max_{\substack{0 \leqslant j \leqslant N-1 \\ 1 \leqslant r \leqslant n}} \{|D^j E_1^{N-m} y(N_1 + a_r) - p_{j,r} q^{-1}|\} > |q|^{-\frac{1+\varepsilon}{n-1}}.$$

Since it is inconvenient to always have to consider the integrals of $y$ we have the following result not involving them: Suppose that in the above differential equation for $y$, $D^m y$ has the coefficient $a_m(z) \not\equiv 0$ and the indicial equation at $z = \infty$ has no integral roots, i.e. no formal series in descending integral powers of $z$ can be a solution. Set $\beta = \deg a_m(z)$ and $\psi = (\beta - m + 1)(n - (m+1)\beta)^{-1}$.

THEOREM III. *If $n > (m+1)\beta$ then there exist two effectively computable functions $\varphi_1 = \varphi_1(\varepsilon, y, a_1, \ldots, a_n)$ and $\delta_1 = \delta_1(\varepsilon, y, a_1, \ldots, a_n)$ such that if $|N_1| > \varphi_1$ and $|q| > |N_1|^{\delta_1}$ then*

$$(3) \qquad \max_{\substack{0 \leqslant j \leqslant m-1 \\ 1 \leqslant r \leqslant n}} \{|D^j y(N_1 + a_r) - p_{j,r} q^{-1}|\} > |q|^{-(1+\varepsilon)\psi}.$$

EXAMPLE. Suppose that $y$ is a zero of an element $p(z, y)$ of $Q[i, z, y]$ of degree $l \geqslant 2$ in $y$. Suppose, further, that $p(z, y)$ is irreducible over the field of functions meromorphic at $z = \infty$ and that the coefficient of $y^{l-1}$ is zero. Suppose that $Ly = 0$ is a *minimal order* linear homogeneous differential equation with coefficients in $Z[i, z]$ which one can obtain, for all zeros of $p(z, y)$, by the process of differentiating $p(z, y) = 0$ repeatedly and writing each derivative in the basis $1, y, \ldots, y^{l-1}$ over $Q(i, z)$. Suppose that the order of the operator $L$ is $m$ and the degree of the coefficient of $D^m$ in the operator $L$ is $\beta$. Then we shall show that inequality (3) holds for all $y$ which are zeros of $p(z, y)$.

We must show that the hypotheses of Theorem III are satisfied. Each zero of $p(z, y)$ is an analytic continuation about $z = \infty$ of one of the zeros of $p(z, y)$ and no zero of $p(z, y)$ has any integral powers of $z$ in its expansion about $z = \infty$. If the solution space of $Ly = 0$ is spanned by the zeros of $p(z, y)$ we are through, since each zero of $p(z, y)$ is an $A$-function (which would mean, also, that $z = \infty$ is at worst a regular singular point of $Ly = 0$, so all roots of the indical equation at $z = \infty$ correspond to actual solutions), the reciprocal of the Wronskian of a collection of zeros of $p(z, y)$ is (if defined) an $A$-function, and no zero of $p(z, y)$ has any integral powers of $z$ in its expansion about $z = \infty$ (which means that the indicial equation of $Ly = 0$ at $z = \infty$ has no integral roots). If $y_j$ is a zero of $p(z, y)$ then $p(z, y)$ must be a minimal polynomial for $y_j$ over $Q(i, z)$, since otherwise $p(z, y)$ would be factorable over $Q(i, z)$. But then the linear homogeneous differential equation $Ly = 0$ obtained above must be, for each zero of $p(z, y)$, a minimal order linear homogeneous differential equation with coefficients in $Z[i, z]$ satisfied by that zero. Suppose that $y_1, \ldots, y_l$ are the zeros of $p(z, y)$. Since each analytic continuation of a $y_j$ is another zero of $p(z, y)$, we recall from [4] that $Ly = 0$ must equal, up to a factor in $Q[i, z]$,

$$W(y_1, \ldots, y_m, y)(W(y_1, \ldots, y_m))^{-1} = 0,$$

where $y_1, \ldots, y_m$ are a maximal linearly independent subset of $y_1, \ldots, y_l$. Then, clearly, the $y_1, \ldots, y_l$ generate the solution space of $Ly = 0$, so we have proven our assertion. It is clear that our proof will go through if $p(z, y)$ is irreducible over $Q[i, z]$ and each zero of $p(z, y)$ has no integral powers of $z$ in its expansion about $z = \infty$.

We wish to see what one may say, using the methods of the present paper, under more general conditions than in the Example above. Suppose that $y(z)$ is any non-rational function which is a solution to $p(z, y) = 0$, where $p(z, y) \in Q[i, z, y]$ has degree $l \geqslant 2$ in $y$. Form $Ly = 0$, as in the above Example and let $m$ and $\beta$ be as in the Example also. We are able to obtain the following result:

THEOREM IV. *There exists an effectively computable function $\varphi_2 = \varphi_2(y, a_1, \ldots, a_n)$ such that if $|N_1| > \varphi_2$ then the dimension of the field $Q(i, y(N_1 + a_1), \ldots, y(N_1 + a_n))$ over $Q(i)$ is at least $(n+1)\beta^{-1}$.*

(As we shall show later, a lower bound of $n\beta^{-1}$ may be obtained by a fairly simple argument which does not involve the methods of this paper. One would hope to eventually obtain a stronger result in this case. On the other hand, somewhat more will be proven for most cases than is actually asserted in Theorem IV.)

FURTHER EXAMPLES. In Theorem II we may set $y = z^{-1}$. Then $(zD + 1)y = 0$ has $y$ as a solution, $y$ is an $A$-function, and $(W(y))^{-1} = z$ is an $A$-function. Thus we see that

$$\max_{1 \leqslant j \leqslant n} (|\log((N_1 + a_j) + c) - p_i q^{-1}|) > |q|^{-\frac{1+\varepsilon}{n-1}}$$

under the conditions of Theorem II for any complex constant $c$. Setting $c = -\log(N_1+a_1)$ and choosing $a_1 = 0$ we have that

$$\max_{2\leqslant j\leqslant n}\{|\log(1+a_jN_1^{-1})-p_jq^{-1}|\} > |q|^{-\frac{1+\varepsilon}{n-1}},$$

under the conditions of Theorem II. The reader may wish to compare this result with those obtained in [2] by Fel'dman.

The hypergeometric function of Gauss, $F(z, a, b, c)$ satisfies the differential equation

$$(zD+a)(zD+b)y = D(zD+c-1)y.$$

Noticing that $D(zD) = (zD+1)D$ we see that

$$(zD+a-N)(zD+b-N)E_1^Ny = (zD+c-N)E_1^{N-1}y,$$

up to a polynomial of degree $N-1$, for any definition of $E_1^Ny$. Since the coefficient of $E_1^Ny$ is $(a-N)(b-N)$ we see that if neither $a$ nor $b$ are non-negative integers we may define $E_1^Ny$ so that each $E_1^Ny$ may be expressed as a linear combination of $y$ and $Dy$. Suppose that $y_1$ and $y_2$ are the two linearly independent solutions of $(zD+a)(zD+b)y = D(zD+c-1)y$ given by the method of Frobenius at $z = \infty$ when $a$, $b$, and $c$ belong to $Q$ and neither $a$ nor $b$ are non-negative integers. By Lemma 3 of [5] it is easy to see that $y_1$ and $y_2$ are each $A$-functions if $a$ and $c$ do not differ by an integer. (If a solution involving logarithms should occur it would be an $A$-function also since, by [6], the least common multiple of the numbers $1, 2, \ldots, N$ is less than $2^{\frac{3}{2}N}$.) If $W \stackrel{\text{def}}{=} W(y_1, y_2)$ then

$$-\left(\frac{d}{dz}\log(W)\right) = \big(c-(a+b+1)z\big)z^{-1}(z-1)^{-1}.$$

By the same argument as for the hypergeometric function we see that each $(z-\gamma)^\delta$ is an $A$-function for all $\gamma$ and $\delta$ in $Q$. Thus $W^{-1}$ is an $A$-function and we may apply Theorem II to obtain

$$\max_{\substack{\delta=0,1\\ 1\leqslant j\leqslant n}}\{|C_1y_1^{(\delta)}(N_1+a_j)+C_2y_2^{(\delta)}(N+a_j)-p_{\delta,j}q^{-1}|\} > |q|^{-\frac{1+\varepsilon}{n-1}}$$

for all fixed non-zero $(C_1, C_2)$ in $C^2$, if $|N_1| > \varphi$ and $|q| > |N_1|^\delta$.

Notice that if we consider the equation

$$\Big(\prod_{j=1}^{m}(zD+a_j)\Big)y = D\Big(\prod_{j=1}^{m-1}(zD+b_j+1)\Big)y$$

where each $a_j$ and each $b_j \in Q$ then the same sort of argument goes through.



If we consider the equation

$$z\Big(\prod_{j=1}^{m}(zD+a_j)\Big)y = \Big(\prod_{j=1}^{m}(zD+b_j)\Big)y$$

where each $a_j$ and each $b_j \in Q$ and no $a_j$ equals an integer, we can apply Theorem III. Here we see $\psi = 2\big(n-(m+1)^2\big)^{-1}$.

Suppose that $\frac{d}{dz}(\log y) = -\frac{1}{3}\big((z-a)^{-1}+(z-b)^{-1}+(z-c)^{-1}\big)$, where $a$, $b$, and $c$ each belong to $Q(i)$. Then in general any $E_1y$ will be related to elliptic functions and will be non-algebraic. Also in Theorem III we have $\psi = \frac{3}{n-6}$ and in Theorem IV the lower bound is $\frac{1}{3}(n+1)$. These hold if we do not have $a = b = c$.

If we choose $y = z^{hk^{-1}}$, where $h$ and $k$ are relatively prime positive integers with $k > 1$, then we would obtain results similar to those in [3].

Some corrections for [4] are included at the end of the paper.

## Section I

Proof of Theorem I. Suppose that $y(z)$ denotes any solution to $q_0(z, y) = 0$ where $q_0(z, y)$ belongs to $Z[i, z, y]$, has degree $m \geqslant 1$ in the variable $y$, and has no repeated zeros. One may easily obtain effective upper bounds on $|y(z)|$ near $z = \infty$ from the equation $q_0(z, y) = 0$. Thus we may effectively compute an integer $k \geqslant 0$ such that $y_1(z) = z^ky(z^{-m!})$ is analytic at $z = 0$. Using the integral representation of $y_1^{(k)}(z)$, for each $k \geqslant 0$, we may obtain effectively computable bounds on the absolute values of the coefficients of the expansion of $y(z)$, about $z = \infty$. Let $q_1(z, y) = 0$ be a polynomial equation with $q_1(z, y) \in Z[i, z, y]$ of degree $m$ in $y$ which is satisfied by $y_1(z)$. Set $y_2(z) = z^{-1}\big(y_1(z)-y_1(0)\big)$. One may effectively bound from above the absolute values of the coefficients (and also their conjugates over $Q(i)$) of an element $q_2(z, y)$ in $O_2[z, y]$ of degree at most $m$ in $y$ with $q_2(z, y_2) \equiv 0 \not\equiv q_2(0, y)$, where $O_2$ is the ring of algebraic integers of $Q\big(i, y_1(0)\big)$. We may continue in this fashion for any finite number of steps. (Note that these bounds hold for all solutions $y(z)$ of $q_1(z, y) = 0$.)

It is possible to effectively bound the number of initial power series coefficients which may be identical for two distinct roots of $q_1(z, y) = 0$ in terms of our bounds on the absolute values of the coefficients of $q_0(z, y)$, as we shall show. If $q_0(z, y)$ is monic then letting $v_1, v_2, \ldots, v_m$ be the solutions of $q_0(z, y) = 0$ we have $\deg\big(\prod_{j<k}(v_j-v_k)^2\big) \geqslant 1$ and for some effectively computable constants $\beta_1$ and $\beta_2$ if $|z| > \beta_1$ then $|v_1-v_k| \leqslant |z|^{\beta_2}$. Thus one may bound $|v_j-v_k|$ from below by $|z|^{-\beta_3}$, where $\beta_3 \geqslant 0$ is effectively computable, if $|z| > \beta_1$. (The general case follows easily now from this particular case.) Below let $N$ be fixed.

If $N$ is sufficiently large we shall see that $q_N(0, y) = ay+b$ for constants $a$ and $b$ in $O_N$ with $a \neq 0$. Further the above lower bound on $N$ is effective. If we choose $N$ sufficiently large that each of the other $m-1$ functions which are solutions of $q_N(z, y) = 0$ must have a singularity at $z = 0$, then $q_N(z, y)$ looks like

$$a_N(z)\Big(\prod_{j=1}^{m-1}\big(y - z^{-a_j}\varphi_j(z)\big)\Big)\big(y-y_N(z)\big)$$

where each $\varphi_j(z)$ is analytic at $z = 0$, each $\varphi_j(0) \neq 0$, each $a_j$ is a positive integer, and $a_N(z) \in O_N[z]$. Thus we see that $q_N(z, y)$ looks like

$$b_N(z)\Big(\prod_{j=1}^{m-1}\big(z^{a_j}y - \varphi_j(z)\big)\Big)\big(y-y_N(z)\big)$$

where $b_N(z) \in O_N[z]$ and $b_N(0) \neq 0$. Then $q_N(0, y) = ay+b$, with $a \neq 0$, where $a$ and $b$ are in $O_N$ since $q_N(0, y) \in O_N[y]$.

Choose a positive integer $M_1$ such that $M_1 a^{-1}$ is an algebraic integer and each $M_1 y_j(0)$, $1 \leqslant j \leqslant N$, is an algebraic integer. We may place an effective upper bound on $M_1$ since we can effectively bound from above the absolute values of the coefficients of each $q_j(z, y)$, in $O_j$, and we see that each $O_j$ has a quotient field $F_j$ with $[F_j\colon Q(i)] \leqslant m^j$.

Place the series $\sum_{n=0}^{\infty} a_n z^n$ for $y(z)$ in

$$q_N(z, y) = b_N(z)\Big(\prod_{j=1}^{m-1}\big(z^{a_j} - \varphi_j(z)\big)\Big)\big(y-y_N(z)\big)$$

and collect the coefficient of $z^n$ for each $n \geqslant 0$, in the resultant. We see that we have exactly one term involving $a_n$, i.e.

$$b_N(0)\Big(\prod_{j=1}^{m-1} -\varphi_j(0)\Big)a_n = aa_n.$$

The other terms in the coefficient of $z^n$ are each monomials, with coefficients in $O_N$, of the form $\prod_{k=1}^{m'} a_{j_k}$ where (i) $m' \leqslant m$, (ii) $\sum_{k=1}^{m'} j_k \leqslant n-1$, and (iii) each $0 \leqslant j_k \leqslant n-1$. Thus each $M_1^{mn} a_n \in O_N$ for every $n \geqslant 1$. The remainder of the proof of Theorem I is trivial, since one may bound the absolute values of the $a_n$'s (and of their conjugates) by $\delta^n$ for some $\delta > 0$ independent of $n$ by using the upper bounds on $|y_1(z)|$ and the Cauchy integral formula. One can then use what we have just proven above to see that our original (arbitrary) solution $y(z)$ of $q_0(z, y) = 0$ is an $A$-function. This proves Theorem I.



It is relatively easy to see that the class of $A$-functions ($B$-functions) is closed under addition of functional values, multiplication of functional values, and differentiation. If we define an integration operator $E$ by

$$E\Big(\sum_{c,d} a_{c,d} z^c\big(\log(z)\big)^d\Big) = \sum_{c\neq -1} a_{c,d} z^{c+1}(c+1)^{-1}\big(\log(z)\big)^d +$$
$$+\sum_{d} a_{-1,d}\big(\log(z)\big)^{d+1}(d+1)^{-1} - E\Big(\sum_{c\neq -1} a_{c,d}(c+1)^{-1} d z^c \big(\log(z)\big)^{d-1}\Big),$$

then it is not hard to show that the set of $A$-functions ($B$-functions) is closed under the operator $E$.

Probably the most difficult part of the proof of Theorem II is in showing that one may express, for each positive integer $N$, the different

$$D^j E^{N-m} y(z+\alpha_r)\text{'s}, \quad \text{for } 0 \leqslant j \leqslant N-1 \text{ and } 1 \leqslant r \leqslant n,$$

in terms of a basis (chosen from among themselves) of the vector space which they generate over $Q(i, z)$, while having effective upper bounds on the degrees and absolute values of the coefficients of the numerators and denominators (which we take to be in $Z[i, z]$) of the coefficients. Therefore we shall now begin to build up to a proof of this latter statement, in Theorem VII.

Definitions. Let $Q(i, z)[D]$ ($Z[i, z][D]$) denote the set of all operators of the form $G = \sum_{j=0}^{m} p_j(z) D^j$ where each $p_j(z) \in Q(i, z)$ ($Z[i, z]$). If $p_m(z) \not\equiv 0$ we say that $G$ has order $m$ or $\operatorname{ord} G = m$. If $m = 0$ and $p_0(z) \equiv 0$ we set $\operatorname{ord} G = -\infty$. For a set of parameters $c_1, \ldots, c_m$ set

$$M_K(c_1, \ldots, c_m) = K[z]c_1 + \ldots + K[z]c_m$$

for any field $K$ with $[K\colon Q(i)] < \infty$ and $K$ a subfield of $C$.

Lemma I. *Suppose that $G$ and $G_1 \not\equiv 0$ each belong to $Q(i, z)[D]$ (and that $v$ and $v_1$ each belong to $M_K(c_1, \ldots, c_m)$). Then if both $Gy = 0$ and $G_1 y = 0$ (or $Gy = v$ and $G_1 y = v_1$) hold for $y = y_1, \ldots, y = y_p$ but no relation of this type of order less than $\operatorname{ord} G_1$ holds, we have that $G = G_2 G_1$ for some $G_2 \in Q(i, z)[D]$. (Additionally, $v = G_2 v_1$.)*

Proof. One sees that $Q(i, z)[D]$ has the property that if $\alpha$ and $\beta$ belong to $Q(i, z)[D]$ and $\operatorname{ord}\alpha \geqslant \operatorname{ord}\beta > -\infty$ then there exists $\gamma \in Q(i, z)[D]$ such that $\operatorname{ord}(\alpha - \gamma\beta) < \operatorname{ord}\alpha$. Thus one may construct $G_2 \in Q(i, z)[D]$ with $\operatorname{ord}(G - G_2 G_1) < \operatorname{ord} G_1$. But then $G = G_2 G_1$. Further we have that

$$G_2 v_1 = G_2(G_1 y) = (G_2 G_1) y = Gy = v.$$

This proves Lemma I.

Definition. If $L_1$ and $L_2$ belong to $Q(i, z)[D]$ then by $(L_1, L_2)$ we shall denote *any* $L_3 \in Q(i, z)[D]$ such that $L_3$ is a right divisor of both $L_1$

and $L_2$ (in the ring $Q(i,z)[D]$) of maximal order. We call $(L_1, L_2)$ *a greatest common divisor of* $L_1$ *and* $L_2$.

LEMMA II. *The right Euclidean Algorithm for determining a greatest common right divisor holds in* $Q(i,z)[D]$. *The greatest common right divisor obtained by this algorithm is right divisible by any other common right divisor of* $L_1$ *and* $L_2$; *hence, a greatest common divisor is uniquely determined up to multiplication on the left by a factor from* $Q(i,z)$.

Proof. Trivial.

LEMMA III. *The kernel of* $(L_1, L_2) \overset{\text{def}}{=} \operatorname{Ker}(L_1, L_2) = (\operatorname{Ker} L_1) \cap (\operatorname{Ker} L_2)$. *Further,* $(L_1, L_2)$ *may be defined as denoting any operator in* $Q(i,z)[D]$ *with kernel equal to* $V = (\operatorname{Ker} L_1) \cap (\operatorname{Ker} L_2)$.

Proof. Let $G \not\equiv 0$ be an element of $Q(i,z)[D]$ of minimal order which satisfies $GV = 0$. Then, by Lemma I, $L_1 = L_3 G$ and $L_2 = L_4 G$. Hence $\operatorname{Ker} G \subseteq (\operatorname{Ker} L_1) \cap (\operatorname{Ker} L_2)$. We know, by definition, that $\operatorname{Ker} G \supseteq (\operatorname{Ker} L_1) \cap (\operatorname{Ker} L_2)$. Since $G$ is a right divisor of $(L_1, L_2)$, by Lemma II, we have that $\operatorname{Ker}(L_1, L_2) \supseteq \operatorname{Ker} G = (\operatorname{Ker} L_1) \cap (\operatorname{Ker} L_2)$. On the other hand since $(L_1, L_2)$ is a right divisor of $L_1$ and $L_2$, $\operatorname{Ker}(L_1, L_2) \subseteq (\operatorname{Ker} L_1) \cap (\operatorname{Ker} L_2)$. Thus $\operatorname{Ker}(L_1, L_2) = (\operatorname{Ker} L_1) \cap (\operatorname{Ker} L_2)$. Any two elements of $Q(i,z)[D]$ with the same kernel $V$ must have the same order, i.e. the dimension of $V$ over $C$. Further, their greatest common divisor must also have the same order, hence; the two elements differ by a left factor in $Q(i,z)$. This proves Lemma III.

LEMMA IV. *If we are given a system of any number of non-zero linear homogeneous equations in* $n$ *variables with coefficients in* $Z[i]$ *which are each less in absolute value than some constant* $c \geqslant 1$, *then there exists an effectively computable number* $B(n, c)$ *such that there will always exist a basis* $v_1, \ldots, v_j, \ldots$ *of the solution space of the above system of equations, with each entry of each* $v_j$ *belonging to* $Z[i]$ *and having absolute value less than* $B(n, c)$.

Proof. If the zero vector is the only solution we are through. If the solution space has dimension exactly one and $n > 1$ then the statement of the Lemma is well known and the proof involves use of the "pigeon-hole principle". (If the solution space has dimension one and $n = 1$ then the number of equations was zero. In this case $X_1 = 1$ is a basis of the solution space.) Thus we may assume that the solution space has dimension $d \geqslant 2$. We may complete a maximal linearly independent collection of linear equations belonging to our system to a collection of $n$ linearly independent homogeneous equations by adjoining as many of the equations $X_j = 0$, $1 \leqslant j \leqslant n$, as may be necessary. The number of equations adjoined is $d \geqslant 2$. Without loss of generality we may assume that $X_1 = 0, \ldots, X_d = 0$ are adjoined. Then deleting in turn $X_1 = 0, \ldots, X_d = 0$ from our set of $n$ equations we obtain $d$ different systems each with a one dimensional solution space spanned by a non-zero vector $v_j \in (Z[i])^n$, for $1 \leqslant j \leqslant d$. The $j$th component of $v_j$ can not be zero or else $v_j \neq 0$ would be a solution of the homogeneous system of rank $n$. If $l \neq j$, however, the $l$th component of $v_j$ is zero for $1 \leqslant l \leqslant d$. Thus the $v_j$'s, $1 \leqslant j \leqslant d$, are linearly independent vectors in the solution space of our original system of equations. Then by the case $d = 1$, mentioned above, we may effectively compute $B(n, c)$. This proves Lemma IV.



DEFINITIONS. Let us denote by $(G_1, \ldots, G_t)$ any operator in $Q(i,z)[D]$ with kernel equal to $\bigcap_{j=1}^{t} (\operatorname{Ker} G_j)$. Note, using Lemma III, that

$$(G_1, \ldots, G_t) = \big((G_1, G_2), \ldots, G_t\big) = \big((G_2, G_1), \ldots, (G_t, G_1)\big)$$

so

$$(G_1, \ldots, G_t)$$

may be determined by using the right Euclidean algorithm $t-1$ times. Let $\lambda_1, \ldots, \lambda_n$ denote parameters. If we say that $p(z)$ in $Q[i,z]$ is effectively bounded from above we mean that for effectively computable constants $a_1$ and $a_2$ each coefficient of $p(z)$ has absolute value less than $a_1$ and $p(z)$ has degree less than $a_2$. If we say that $p(z)$ in $Q(i,z)$ is effectively bounded from above we mean that there are effectively computable constants $a_1$ and $a_2$ such that we may write $p(z) = p_1(z)\big(p_2(z)\big)^{-1}$, where both $p_1(z)$ and $p_2(z)$ belong to $Z[i,z]$, the coefficients of both $p_1(z)$ and $p_2(z)$ have absolute values bounded by $a_1$, and the degrees of both $p_1(z)$ and $p_2(z)$ are bounded by $a_2$. If we say that $v \in Q[i,z]\lambda_1 + \ldots + Q[i,z]\lambda_n$ (or $L \in Z[i,z][D]$) is effectively bounded from above we mean that the coefficients are effectively bounded from above (and in the case of $L$, additionally, $\operatorname{ord} L$ is effectively bounded from above).

Suppose that $\bar{y} = \sum_{j=1}^{m} c_j f_j$ where each $f_j$ is an $A$-function with coefficients in $K$, where $[K: Q(i)] < \infty$, and where each $c_j$ is a complex valued parameter. Suppose that $\bar{y}$ is the general solution of $Gy = 0$ where $G$ is any element of $Z[i,z][D]$ of order $m > 0$ and the coefficients of $G$ are effectively bounded from above.

THEOREM V. *Under the above conditions there exist, for some* $m_1 \leqslant m$, $\lambda_1, \ldots, \lambda_{m_1} \in Kc_1 + \ldots + Kc_m$ *and* $L \in Z[i,z][D]$ *such that:*

(i) *the* $\lambda_1, \ldots, \lambda_{m_1}$ *are linearly independent over* $K$;

(ii) $\operatorname{ord} L = m - m_1$;

(iii) $L\bar{y} = \sum_{j=1}^{m_1} p_j(z)\lambda_j$ *where the* $p_j(z)$*'s belong to* $Z[i,z]$;

(iv) $\bar{y}$ *satisfies no equation of the form* $L_1 y = v_1$ *with* $L_1 \in Z[i,z][D]$ *and* $v_1 \in M_K(c_1, \ldots, c_m)$ *(or even equal to any polynomial), of order less than* $m - m_1$;

(v) $L$ *and each* $p_j(z)$ *may be effectively bounded from above; and*

(vi) *each $\lambda_j$ may be written as a linear combination over $Q(i, z)$ of the $D^t\bar{y}$ (for $t = 0, 1, \ldots$) with coefficients which are effectively bounded from above.*

Proof. Let $L$ denote an element of $Z[i, z][D]$ of minimal order such that $L\bar{y}$ is a polynomial. By looking at the expansion of $\bar{y}$ about $z = \infty$ we see that $L\bar{y} \in M_K(c_1, \ldots, c_N)$. Suppose that $L_1 y = \sum_{j=1}^{m_1} p_j(z)\lambda_j$ where the $p_j(z)$'s and the $\lambda_j$'s are, respectively, linearly independent elements of $Z[i, z]$ over $Q(i)$ and linearly independent elements of $Kc_1 + \ldots + Kc_m$ over $K$. By Lemma I, $L$ is a right divisor of $G$, so $\operatorname{Ker} L \subseteq \operatorname{Ker} G$. Now $(\operatorname{Ker} L) \cap (\operatorname{Ker} G)$ has dimension exactly $m - m_1$ since $L(\bar{y}) = 0$ iff the parameters $c_1, \ldots, c_m$ are given values so that $\lambda_1 = \lambda_2 = \ldots = \lambda_{m_1} = 0$. Thus $\operatorname{ord} L = m - m_1$. If $m_1 = 0$ we are through. In what follows we shall assume that $m_1 \geqslant 1$. One may find $m_1$ elements $\mu_j(z) \in Q(i, z)$ beginning with $\mu_{m_1} = (p_{m_1}(z))^{-1}$ such that for some $G_1 \in Z[i, z][D]$, $G_1 = \mu_1(D\mu_2) \ldots (D\mu_{m_1})L$ and $G_1\bar{y}$ equals zero or $\lambda_1$. Since $DG_1\bar{y} = 0$ and $\operatorname{ord}(DG_1) = m$, we see that $G_1\bar{y} = \lambda_1$.

Now there must exist $\varphi_1 \neq 0$ in $Q(i, z)$ such that $DG_1 = \varphi_1 G$, or equivalently, $G^*\varphi_1 = 0$ (where $G^*$ is the "Lagrange adjoint" (²) of $G$, i.e. if $G = \sum_j a_j(z) D^j$ then $G^* = \sum_j (-D)^j a_j(z)$). If one repeats the above argument for each $1 \leqslant j \leqslant m_1$ one obtains $\varphi_1, \ldots, \varphi_{m_1}$, non-zero elements of $Q(i, z)$, and $G_1, \ldots, G_{m_1}$ in $Z[i, z][D]$ such that, for each $1 \leqslant j \leqslant m_1$, $\varphi_j G = DG_j$ and $G_j\bar{y} = \lambda_j$. Suppose that the $\varphi_j$'s are linearly dependent over $K$. Then for a collection of $\beta_j$ in $K$ which are not all zero we have $0 = \sum_{j=1}^{m_1} \beta_j\varphi_j$. It follows that

$$0 = \Big(\sum_{j=1}^{m_1} \beta_j\varphi_j\Big) G = D\Big(\sum_{j=1}^{m_1} \beta_j G_j\Big).$$

Then $\sum_{j=1}^{m_1} \beta_j G_j = 0$; however,

$$\sum_{j=1}^{m_1} \beta_j G_j \bar{y} = \sum_{j=1}^{m_1} \beta_j\lambda_j \neq 0.$$

This shows that the $\varphi_j$ are linearly independent over $K$.

Suppose that $\theta_1, \ldots, \theta_t$ are $t \geqslant m_1$ linearly independent solutions (over $K$) of $G^*y = 0$ which are in $Q(i, z)$. Let $DG_j \overset{\text{def}}{=} \theta_j G$ and set $\lambda_j \overset{\text{def}}{=} G_j\bar{y}$. By the choice of $\bar{y}$ each $\lambda_j \in Kc_1 + \ldots + Kc_m$. If $\sum_{j=1}^{t} \beta_j\lambda_j = 0$ where the $\beta_j$ belong to $K$ and are not all zero then $\sum_{j=1}^{t} \beta_j G_j\bar{y} = 0$ which implies that $\sum_{j=1}^{t} \beta_j G_j = 0$ since $\sum_{j=1}^{t} \beta_j G_j$ has order less than $m$. It follows that $\sum_{j=1}^{t} \beta_j DG_j$

(²) See [1]. One may verify that $G^{**} = G$ and $(G_1G_2)^* = G_2^*G_1^*$.



$= \big(\sum_{j=1}^{t} \beta_j\theta_j\big)G = 0$. Then $\sum_{j=1}^{t} \beta_j\theta_j = 0$, which is a contradiction. It follows that the $\lambda_j$ are linearly independent over $K$.

For each $1 \leqslant j \leqslant t$, $\operatorname{Ker} G_j = \{\sum_{l=1}^{m} c_l f_l \mid \lambda_l = 0\}$. We recall that $H = (G_1, \ldots, G_t)$ exists in $Z[i, z][D]$, has kernel exactly equal to

$$\Big\{\sum_{l=1}^{m} c_l f_l \mid \lambda_1 = \lambda_2 = \ldots = \lambda_t = 0\Big\},$$

and may be effectively determined from $G_1, \ldots, G_t$ using the right Euclidean algorithm. Carrying through the non-homogeneous terms in the above algorithm we see that we have an equation of the same type as

$$L\bar{y} = \sum_{j=1}^{m_1} p_j(z)\lambda_j$$

but of possibly lower order, i.e. of order $m - t$. This is not possible, by definition, so $t = m_1$ and, except for a factor on the left from $Q(i, z)$, $L = H$. Further the non-homogeneous terms are equal, up to multiplication by this factor.

We shall show that we can effectively bound from above a maximal linearly independent set of solutions of $G^*y = 0$ in $Q(i, z)$. (Obviously the number of functions in this collection must be $m_1$.) If we could accomplish this then we could effectively bound from above the numerators and denominators of a collection of corresponding $G_j$'s. (Then, using $G_j\bar{y} = \lambda_j$ to define a "new" set of $\lambda_j$'s, we have satisfied part (vi) of the theorem.) It would follow that we could effectively bound from above the coefficients of an operator $H$ in $Z[i, z][D]$ of order $m_1$ such that $H\bar{y} = \sum_{j=1}^{m_1} q_j(z)\lambda_j$ for a set of effectively bounded $q_j(z)$'s in $Z[i, z]$ and the "new" $\lambda_j$'s. All of Theorem V would then follow.

One may obtain, from the effective upper bounds on $G$, effective upper bounds on $G^*$. Then one may bound from above the absolute values of the finite singularities of $G^*$, the degree of the minimal field extension of $Q(i)$ which contains each zero, and the magnitude of the smallest positive integer such that multiplication by it takes each singularity into an algebraic integer. From this one may effectively bound from above a collection of elements of $Z[i, z]$ whose zeros include all of the roots of all of the indicial equations about all singularities (finite and infinite) of $G^*y = 0$. We may then effectively bound from above the absolute values of these roots. It follows that the denominator of each $\varphi_j$ may be taken to be the coefficient of the highest power of $D$ in $G$ raised to an effectively computable positive integral power while the numerator is effectively bounded in degree and is known to satisfy the system of homogeneous linear equations implied by $G^*\varphi_j = 0$. Thus by Lemma IV we may effectively bound each $\varphi_j$ from above. This proves Theorem V.

Let $G$ be as in Theorem V. For any $N \geqslant m$ set $E_1^{N-m}\bar{y} = \sum_{j=1}^{m} c_j E^{N-m} f_j + \sum_{j=m+1}^{N} c_j z^{j-1-m}$ where $c_{m+1}, \ldots, c_N$ are additional parameters. Obviously $GD^{N-m}y = 0$ is a minimal order linear homogeneous differential equation which is satisfied by $E_1^{N-m}\bar{y}$. By Theorem V applied to $E_1^{N-m}\bar{y}$ there exists an effectively bounded $L \in Z[i, z][D]$, of order $N-m_1$, and effectively bounded $p_1(z), \ldots, p_{m_1}(z) \in Z[i, z]$ such that, for a collection of linearly independent $\lambda$ in $Kc_1 + \ldots + Kc_N$, $L(E^{N-m}\bar{y}) = \sum_{j=1}^{m_1} p_j(z)\lambda_j$ and there exists no equation of this type of lower order satisfied by $E^{N-m}\bar{y}$. Recall that each $\lambda_j$ may be written as a linear combination over $Q(i, z)$ of the $D^s E_1^{N-m}\bar{y}(z)$, $0 \leqslant s \leqslant N-1$, with coefficients which are effectively bounded from above. Suppose that $a_1, \ldots, a_n$ are $n \geqslant 1$ distinct elements of $Q(i)$ such that no $a_k$ equals the difference of two singularities of $Gy = 0$.

THEOREM VI. *Under the above conditions*:

(i) *The* $D^s E_1^{N-m}\bar{y}(z+a_1)$, $0 \leqslant s \leqslant N-1$, *and the* $D^s E_1^{N-m}\bar{y}(z+a_r)$, $2 \leqslant r \leqslant n$ *and* $0 \leqslant s \leqslant N-m_1-1$, *are linearly independent over* $Q(i, z)$.

(ii) *The* $D^s E_1^{N-m}\bar{y}(z+a_r)$, $0 \leqslant s \leqslant N-1$, *may each be written as a linear combination of the functions listed in part* (i), *with coefficients in* $Q(i, z)$ *which are effectively bounded from above.*

(iii) *The* $D^s E_1^{N-m}\bar{y}(z)$, *for* $0 \leqslant s \leqslant N-m_1-1$ *are a basis of the vector space* $V$ *which is the vector space generated over* $Q(i, z)$ *by the* $D^s E_1^{N-m}\bar{y}(z)$, *for* $0 \leqslant s \leqslant N-1$, *taken modulo the subspace of all elements which are rational functions.*

*There exists* $R(z)$, *a non-zero element of* $Z[i, z]$, *which is effectively bounded from above, such that each* $R(z) D^s E_1^{N-m}\bar{y}(z)$, *for* $0 \leqslant s \leqslant N-1$, *may be written as a linear combination over* $Z[i, z]$ *of the* $D^s E_1^{N-m}\bar{y}(z)$, *for* $0 \leqslant s \leqslant N-m_1-1$, *with coefficients which are effectively bounded from above, plus a polynomial function having degree effectively bounded from above.*

Proof. First we shall show that the $D^s E_1^{N-m}\bar{y}(z+a_r)$, $0 \leqslant s \leqslant N-m_1-1$ and $1 \leqslant r \leqslant n$, and $\lambda_1, \ldots, \lambda_{m_1}$ are linearly independent over $Q(i, z)$. Since each $\lambda_j$ may be written as a linear combination over $Q(i, z)$ of the $D^s E_1^{N-m}\bar{y}(z+a_1)$, $0 \leqslant s \leqslant N-1$, this will prove part (i). Suppose that a non-trivial linear combination over $Q(i, z)$ of these elements is zero. Without loss of generality we may assume that the coefficients are in $Q[i, z]$. Then, for each $1 \leqslant r \leqslant m$, the sub-sum consisting of all terms involving derivatives of $E_1^{N-m}\bar{y}(z+a_r)$, for some one $a_r$, is entire and bounded by $|z|$ to some power if $|z|$ is sufficiently large (we use here our restriction on the differences of the $a_k$'s). Hence $E_1^{N-m}\bar{y}(z+a_r)$ satisfies an equation of the form $L_1 y \in K[z]c_1 + \ldots + K[z]c_N$ for some $L_1 \in Z[i, z][D]$



of order less than $N-m_1$. This contradicts Theorem V, so part (i) has been proven.

By Theorem V parts (i)–(iv) we see that the $D^s E_1^{N-m}\bar{y}(z)$, for $0 \leqslant s \leqslant N-m_1-1$, are a basis for $V$. The remainder of part (iii) follows from parts (v) and (vi) of Theorem V.

To see part (ii) differentiate

$$L(E_1^{N-m}\bar{y}) = \sum_{j=1}^{m_1} p_j(z)\lambda_j$$

repeatedly and express each $D^s E^{N-m}\bar{y}(z+a_r)$, $0 \leqslant s \leqslant +\infty$ and $1 \leqslant r \leqslant n$, in terms of $\lambda_1, \ldots, \lambda_{m_1}$ and the $D^s E^{N-m}\bar{y}(z+a_r)$ for $0 \leqslant s \leqslant N-m_1-1$. Then expressing the $\lambda_j$'s in terms of the $D^s E_1^{N-m}\bar{y}(z+a_r)$, for $0 \leqslant s \leqslant N-1$, by Theorem V we are through. This proves Theorem VI.

## Section II

In this section we shall prove Theorem II. Suppose that we are given, for any $m > 0$, an $m$th order linear differential equation

$$L(y) = 0 \tag{4}$$

which has coefficients in $Z[i, z]$. After equation (4) has been multiplied through by an appropriate power of $z$ and one has used $zD = Dz-1$ repeatedly it may be put in the form $\sum_{j=0}^{c} z^j p_j(zD) y = 0$, for some non-negative integer $c$, where each $p_j(zD) \in Z[i, zD]$ and $p_0(zD)p_c(zD) \not\equiv 0$. Without loss of generality we may assume that (4) is in this form already. We have then that $p_c(t) = 0$ is the indicial equation corresponding to expansions of solutions of (4) about $z = \infty$. Since $y_1, \ldots, y_m$ are a fundamental system of solutions of our equations of type (4) and they are each $A$-functions then $z = \infty$ is, at worst, a regular singular point. Thus $p_c(t)$ has degree exactly $m$ and roots $r_1, \ldots, r_m$ which are not necessarily all distinct. Let us now assume, without loss of generality, that $y_1, \ldots, y_m$ are the $m$ canonical expansions about $z = \infty$ given by the method of Frobenius and that each $y_j$ corresponds to the root $r_j$. Thus the order of vanishing, at $z = \infty$, of each $y_j(z)$ is at least $-\max_j\{\text{the real part of } r_j\} - \varepsilon$ for every $\varepsilon > 0$.

Let $L^*$ denote as before the Lagrange adjoint of $L$, i.e. $\sum_{j=0}^{c} p_j(-Dz)z^j$. Choose $K_0 \geqslant \max_{1 \leqslant l \leqslant m}\{|r_l|\}$. Set

$$L_1 = (-1)^{c+K_0+1} z^{K_0+1} L D^{c+K_0+1}. \tag{5}$$

Note that for each $n \in C$,

$$L_1^*(z^n) = \sum_{j=0}^{c} p_j(-n-j-K_0-1)\,\Gamma(n+j+K_0+2)\big(\Gamma(n+j-c+1)\big)^{-1} z^{n+j-c}.$$

Since the indicial polynomial of $L_1^*$ has degree $m+c+K_0+1$ we see that $L_1^*$ has a regular singular point at $z=\infty$. Also each solution of $L_1^*$ vanishes at $z=\infty$ to the order $1-\varepsilon$ for each $\varepsilon>0$. We may rewrite (5) as

$$L_1^*(z^n) = D^c\Big(\sum_{j=0}^{c} p_j(-n-j-K_0-1)\,\Gamma(n+j+K_0+2)\big(\Gamma(n+j+1)\big)^{-1}z^{n+j}\Big).$$

Since $p_c(-n-c-K_0-1)\neq 0$ if $n=0,1,\ldots$, we see that given $n=0,1,\ldots$ we may produce a polynomial $\mu_n$ of degree exactly $n$ such that

$$L_1^*(\mu_n) = D^c\big((n+1)^{-1}\ldots(n+c)^{-1}z^{n+c}+s(z)\big),$$

where $s(z)$ is a polynomial of degree less than or equal to $c-1$. That is, there exists a polynomial of degree exactly $n$ which solves $L_1^*(\mu_n)=z^n$, $n=0,1,\ldots$ Set $L_1(y)=a_N(z)D^Ny+\ldots$, where $N=m+c+K_0+1$. Set $\{U_1,\ldots,U_m\}$ equal to any collection of $(c+K_0+1)$-fold integrals of the different $y_j(z)$, $1\leqslant j\leqslant m_1$, which were given as solutions of (4). Extend $\{U_1,\ldots,U_m\}$ to a fundamental system of solutions of $L_1(y)=0$, i.e. $\{U_1,\ldots,U_N\}$, by adjoining the solutions $1,z,\ldots,z^{N-m-1}$.

According to [1] (see page 70) if $x'=A(t)x$ is a vector differential equation and $\Phi$ is a fundamental matrix of solutions of this equation then $(\Phi^*)^{-1}$ is a fundamental matrix for the adjoint system, $x'=-A^*(t)x$ where by $A^*$ is meant the conjugate transpose of $A$ (but clearly the result is also true if we interpret $M^*$ to be the transpose of $M$ for each matrix $M$). If one uses the canonical representation of an $m$th order linear homogeneous differential equation $Hy=0$, with $H(1)=1$, as a first order system with a fundamental matrix of $(y_j^{(k)})$, $1\leqslant j$, $k+1\leqslant m$ (where $y_1,\ldots,y_m$ are a fundamental system of solutions of $Hy=0$) and $\psi$ is any fundamental matrix of the adjoint of this first order system then according to [1] (page 85) the functions in the bottom row of $\psi$ are a fundamental system of solutions of $H^*y=0$. (Again the result follows for the Lagrange adjoint, also, with $M^*$ denoting $M$ transpose.)

Using these two results above we see that the

$$U_j^*(z) \overset{\text{def}}{=} \big(a_N(z)\big)^{-1}w_j(z) = \big(a_N(z)\big)^{-1}W^{-1}\frac{\partial W}{\partial U_j^{(N-1)}},\qquad 1\leqslant j\leqslant N,$$

are a fundamental system of solutions of $L_1^*y=0$, where $W$ is the Wronskian of $U_1,\ldots,U_N$. By our assumption that $W^{-1}$ is an $A$-function we see that each $w_j(z)$ is an $A$-function. By Theorem I $\big(a_N(z)\big)^{-1}$ is an $A$-function. It follows that the $U_j^*(z)$ are $A$-functions.

Now let us reverse the above procedure, going from $L_1^*$ to $(L_1^*)^*=L_1$. We see that the

$$v_j(z) = (-1)^N W_1^{-1}\frac{\partial W_1}{\partial w_j^{(N-1)}},\qquad 1\leqslant j\leqslant N,$$



are a fundamental system of solutions of $L_1y=0$, where $W_1$ is the Wronskian of $w_1,\ldots,w_N$. The logarithmic derivative of $W_1$ equals $-a_{N-1}(a_N)^{-1} = -W'(W)^{-1}$. Thus $W_1=KW^{-1}$. We may multiply $W_1$ and $W$ together and calculate, in terms of the expansions of the $U_j$ and $w_j$ about $z=\infty$, the constant $K$. Hence we may bound $|K|$ effectively from above and may also effectively bound from above a positive integer $M$ such that $MK$ is an algebraic integer. Therefore $W_1^{-1}$ is an $A$-function and each $v_j(z)$ is an $A$-function. (We have just used the remark in footnote 1 on the first page of this paper.)

We wish to use "variation of parameters" in order to write the general solution of $L_1^*(\mu_n)=z^n$ for $n=0,1,\ldots$ Recall the definition of the integral operator $E$, given before Lemma I. Recall, from differential equations, that

$$W\Big(\big(a_N(z)\big)^{-1}w_1,\ldots,\big(a_N(z)\big)^{-1}w_n\Big) = \big(a_N(z)\big)^{-m}W(w_1,\ldots,w_m).$$

We see that the general solution looks like

$$\sum_{j=1}^{N}\Big(E\big(v_j(z)z^n\big)+b_j\Big)U_j^*(z)$$

for a collection of arbitrary constants $b_j$. We wish to find a polynomial solution by choosing the $b_j$'s appropriately. The different $U_j^*(z)$ all vanish at $z=\infty$ to the order $1-\varepsilon$ for each $\varepsilon>0$, since they are solutions of $L_1^*$. Thus we may take $\mu_n$ to be exactly those terms in the expansion of $\sum_{j=1}^{N}\big(Ev_j(z)z^n\big)U_j^*(z)$ about $z=\infty$ which do not vanish at $z=\infty$. Each $v_j(z)$ and each $U_j^*(z)$ is an $A$-function. In [6] it was shown that the least common multiple of $\{1,2,\ldots,n\}<2^{\frac{3}{2}n}$. Using all of these facts we see that:

Lemma V. *Suppose that $L_1$ and $U_1,\ldots,U_N$ are as above. Then for each positive integer $n$, $L_1^*\mu_n=z^n$ has a solution $\mu_n$ in $Q[i,z]$ such that there exists a non-zero Gaussian integer $d_n$ of absolute value less than $K_1^n$ which when multiplied times $\mu_0$, for each $1\leqslant\theta\leqslant n$, gives an element of $Z[i,z]$ with coefficients having absolute value less than $K_2^{n+1}$ for some pair of effectively computable constants $K_1$ and $K_2$ independent of $n$.*

Let

$$E_1^{N-m}\bar{y}(t) \overset{\text{def}}{=} \sum_{j=1}^{m}c_jE^{N-m}y_j(t)+\sum_{j=m+1}^{N}c_jt^{j-(m+1)}$$

where $\bar{y}=\sum_{j=1}^{m}c_jy_j$ is as in this section and $N=m+c+K_0+1$. Then $L_1\bar{y}=0$ has $E_1^{N-m}\bar{y}$ as its general solution. We would like to be able to define $E_1^{N-m+\theta}\bar{y}(t)$, for $\theta=1,2,\ldots$, in such a way that we may express it as a linear combination over $Q[i, z]$ of the $D^s E_1^{N-m+\theta}\bar{y}(t)$, for $0 \leqslant s \leqslant N-1$, with the coefficients having a common denominator in $Z[i]$ which does not have "too large" an absolute value. Since we are regarding $t$ as the variable now $z+a_1$ is regarded as a constant and we may define $E_1^{N-m+\theta}\bar{y}(t)$ to be

$$\big((\theta-1)!\big)^{-1}\int_{z+a_1}^{t}(t-u)^{\theta-1}\big(E_1^{N-m}\bar{y}(u)\big)\,du$$

$$= \big((\theta-1)!\big)^{-1}\sum_{k=0}^{\theta-1}\binom{\theta-1}{k}t^{\theta-1-k}\int_{z+a_1}^{t}(-u)^k\big(E_1^{N-m}\bar{y}(u)\big)\,du.$$

Now consider, for each $1 \leqslant k \leqslant \theta$, the identity

$$0 = \int_{z+a_1}^{t}\mu_k(u)\big(L_1E_1^{N-m}\bar{y}(u)\big)\,du$$

$$= \int_{z+a_1}^{t}\big(E_1^{N-m}\bar{y}(u)\big)\big(L_1^*\mu_k(u)\big)\,du - \int_{z+a_1}^{t}D\big(H_kE_1^{N-m}\bar{y}(u)\big)\,du$$

$$= \int_{z+a_1}^{t}u^k\big(E_1^{N-m}\bar{y}(u)\big)\,du - (H_kE_1^{N-m}\bar{y})(t) + (H_kE_1^{N-m}\bar{y})(z+a_1),$$

where the $H_k$'s are linear differential operators of order at most $N$ with coefficients in $Q[i, t]$ which have degree at most $K_4+\theta$ and which have a common denominator $d_\theta$ in $Z[i]$ where $|d_0| < K_5^k$, for effectively computable constants $K_4$ and $K_5$ which are independent of $k$. Thus we see:

Lemma VI. *For each $1 \leqslant r \leqslant n$, each $\theta = 1, 2, \ldots$, and each $1 \leqslant \varphi \leqslant \theta$,*

$$(\varphi-1)!\,d_0E_1^{N-m+\varphi}\bar{y}(z+a_r)$$

*equals a linear combination over $Z[i, z]$ of the $D^sE_1^{N-m}\bar{y}(z+a_r)$ and the $D^sE_1^{N-m}\bar{y}(z+a_1)$, $0 \leqslant s \leqslant N-1$, with coefficient polynomials which are bounded from above in degree by $\theta+K_8$ and whose coefficients are smaller in absolute value than $K_7^\theta$, for effectively computable positive constants $K_6$ and $K_7$ independent of $r$, $\theta$, and $\varphi$. Finally, each $\frac{d}{dt}E^{N-m+\varphi}\bar{y}(t) = E^{N-m+\varphi-1}\bar{y}(t)$.*

Suppose now that we define $E_2^\gamma\bar{y}(t)$ for $\gamma = 0, 1, \ldots$ to equal $E_1^\gamma\bar{y}(t)$, if $\gamma \leqslant N-m$, and to equal $\big((\gamma-1)!\big)\int_a^t(t-u)^{\gamma-N+m-1}\big(E_1^{N-m}\bar{y}(u)\big)\,du$ if $\gamma > N-m$, where $a$ is any point of analyticity of $\bar{y}$.

Lemma VII. *For each $1 \leqslant r \leqslant m$, each $\theta = 1, 2, \ldots$, and each $1 \leqslant \varphi \leqslant \theta$,*

$$(\varphi-1)!\,d_\theta E_2^{N-m+\varphi}\bar{y}(z+a_r)$$



*equals a linear combination over $Z[i, z]$ of the $D^sE_1^{N-m}y(z+a_r)$ with coefficient polynomials which are bounded from above in degree by $\theta+K_6$ and whose coefficients are smaller in absolute value than $K_7^\theta$, for effectively computable positive constants $K_6$ and $K_7$, plus a polynomial of degree at most $\varphi-1$ in $z$. Each $d_0$ is an element of $Z[i]$ with $|d_0| < K_5^\theta$, for an effectively computable positive constant $K_5$ independent of $\theta$. Finally each*

$$\frac{d}{dz}E_2^{N-m-\varphi}\bar{y}(z) = E_2^{N-m+\varphi-1}\bar{y}(z).$$

(Lemma VII is for use in Section III.)

We next wish to apply Theorem V of [4]. We shall first verify that Condition A is satisfied by the class $\Pi$ of "functions" consisting of the one function $D^\lambda E_1^{N-m}\bar{y}$ where $\lambda$ is a non-negative integer chosen so that each $D^\lambda E_1^{N-m}\bar{y}$ satisfies a linear differential equation of type (15) in [4] in which $q_0(s)$ has no non-negative integral zeros, and where each parameter $c_j$, $1 \leqslant j \leqslant m$, is bounded effectively from above. The argument after equation (10) of [4] (see page 32) shows how to place our given differential equation for $E_1^{N-m}\bar{y}$ in the form of (10) and also how to obtain an analogous equation of type (10) for each $D^\lambda E_1^{N-m}\bar{y}$. One may use this argument to effectively bound from above both $\lambda$ and the differential equation of type (10) for $D^\lambda E_1^{N-m}\bar{y}$. We also choose $\lambda \geqslant N-m$. Since $\lambda \geqslant N-m$ and we assume given bounds on the absolute values of the exponents and the coefficients $c_1, \ldots, c_m$ in the expansion of $\bar{y}$ about $z = \infty$ we may show that

$$|D^\lambda E_1^{N-m}\bar{y}(t)| < K_8|t|^{K_9}$$

if $|z| > K_{10}$ and $|z-t| < \frac{1}{2}|z|$, for effectively computable constants $K_8$, $K_9$, and $K_{10}$ independent of $z$ regardless of $c_{m+1}, \ldots, c_N$. In [4] set $K_1(y, a_1, \ldots, a_n) \geqslant K_{10}+1+\max_r\{|a_r|\}$. Set $K_2(y) = 0$, $K_3(y) = +\infty$, and $\eta = \frac{1}{2}$. Then Condition A holds. Without loss of generality we may assume that $\lambda = N-m$, since we may substitute a derivative of $\bar{y}$ for $\bar{y}$ in the previous arguments.

We next wish to see that condition B is satisfied (see the end of this paper where corrections for [4] are listed). By Lemma VI and Theorem VI we may write, for some non-zero $g(z) \in Z[i, z]$ which is effectively bounded from above, each $g(z)d_{m+K-1}(\varphi-1)!\,E_1^{N-m+\varphi}\bar{y}(z+a_r)$, for $0 \leqslant \varphi \leqslant m+K-1$, as a linear combination over $Z[i, z]$ of the elements of a basis of $p_1(y)$ elements from among the different $D^sE_1^{N-m}y(z+a)_t$, $0 \leqslant s \leqslant N-1$ and $1 \leqslant t \leqslant n$, with coefficients bounded as required in condition B for $\gamma = 1$. One may effectively bound from above the coefficients, in $Q(i, z)$, obtained when the generators of $P_1(y)$ are expressed in terms of our basis. We may choose $p \leqslant p_1$ of these generators of $P_1(y)$

to replace $p$ elements of our basis above and form a new basis, i.e. the $U_{j,y}(z)$. Now $T_{1,y}(z) = 1$ and $S_y(m) = d_{m+k+1}$. Also $T_y(z)$ may be effectively bounded from above from what we know. We see that $T_y(z)$ is $g(z)$ times a least common denominator in $Z[i, z]$ of the coefficients giving the elements of the old basis in terms of the different $U_{j,y}(z)$'s times a polynomial which enables us to satisfy the finite number of conditions involving derivatives of $\bar{y}$. The remaining conditions are easily verified. Then Theorem V of [4] applies. The $U_{j,y}(z)$ are linear combinations over $Q(i, z)$ of the $D^s E_1^{N-m}\bar{y}(z+a_r)$, $0 \leqslant s \leqslant N-1$, with coefficients which are effectively bounded from above. Thus we may conclude the statement of Theorem II, but for the $E_1^{N-m}\bar{y}$ as above. We may now set $E_1^{N-m}\bar{y} = E_1^{N-m}y$ and we have proven Theorem II.

## Section III

In this section we shall prove Theorems III and IV. Suppose that $\bar{y} = \sum_{j=1}^{m} c_j y_j$, where the $c_j$'s are parameters which assume values in $C$ with absolute values bounded from above by some known constant, and each $y_j$ is an $A$-function. Suppose that $\bar{y}$ is the general solution of

$$Hy = \sum_{j=0}^{m} a_j(z) D^j y = 0 \tag{6}$$

where each $a_j(z) \in Z[i, z]$, the $a_j(z)$ are relatively prime, and $a_0(z) \not\equiv 0$. Set $\beta = \deg a_m(z)$. Since $\infty$ is a regular singular point of (6) the indicial polynomial there must have $m$ zeros. Therefore we have $\deg a_m(z) - m = \max_j \{\deg a_j(z) - j\} \geqslant 0$, and $\beta = m + \max_j \{\deg a_j(z) - j\}$. Let $\operatorname{ord} a_j(z)$ denote the order of vanishing of $a_j(z)$ at $z = 0$. Let $f = \max_{1 \leqslant j \leqslant m} \{j - \operatorname{ord} a_j(z)\} \geqslant 0$. If we multiply (6) through by $z^f$ and use $zD = Dz - 1$ repeatedly we may write (6) as

$$H_1 y = \sum_{j=0}^{c} \varphi_j(-Dz) z^j y = 0 \tag{7}$$

for some $0 \leqslant c \leqslant m$ and some collection of $\varphi_j(-Dz) \in Z[i, -Dz]$ with $\varphi_0(z)\varphi_c(z) \not\equiv 0$. (We could rewrite $H_1 y$ as $\sum_{j=0}^{c} z^j \varphi_j(-zD + j - 1) y$ so that it is put in exactly the same form as was (4).)

Letting $N = m + c + K_0 + 1$, as in Section II, we may assume without loss of generality that $\beta < N$, since $K_0$ may be taken larger if necessary. Define $E_2^{N-m}\bar{y}(t)$ to be as defined before Lemmas VI and VII. Integrating equation (6), $N - \beta + m + \gamma$ times for each $\gamma \geqslant \beta - N - m$ using integration by parts, repeatedly, differentiating each power of $z$ and integrating each $D^\delta \bar{y}$ into $D^{\delta-1}\bar{y}, \ldots, \bar{y}, E_2\bar{y}, \ldots$, we have a linear differential equation of order exactly $\beta$ with coefficients in $Q[i, z, \gamma]$ which is satisfied by $E_2^{N-m+\gamma}\bar{y}$ and which has a polynomial non-homogeneous term. The coefficient of $E_2^{N-m+\gamma}\bar{y}$ is a not identically zero polynomial in $\gamma$ of degree $\beta$. Thus the dimension of the vector space over $Q[i, z]$ spanned by the $\ldots D^\delta\bar{y}, D^{\delta-1}\bar{y}, \ldots, \bar{y}, E_2\bar{y}, \ldots$, modulo all polynomial functions is at most $m$ plus the number of zeros of the coefficient of $E_2^{N-m+\gamma}\bar{y}$, i.e. $m + \beta$.



Choose $N$ sufficiently large that the coefficient of $E_2^{N-m+\gamma}\bar{y}$ does not vanish if $\gamma \geqslant 0$. We see by considering our above equation with $\gamma = \beta - N$, $\beta - N + 1, \ldots$, that for each positive integer $\gamma$ we may write $(a_m(z))^{N-m+\gamma}\bar{y}$, hence each $(a_m(z))^{N-m+\gamma+j}\bar{y}^{(j)}$, as equal to a linear combination over $Q[i, z]$ of $E_2^{N-m+\gamma}\bar{y}, \ldots, E_2^{N-m+\gamma+\beta-1}\bar{y}$, modulo all polynomials. We may replace our basis for $V$, given in Theorem VI by a new basis $B$ which is formed by completing to a basis a maximal linearly independent subset in $V$ of the $\bar{y}, \ldots, \bar{y}^{(m-1)}$ and have all of the assertions in Theorem VI (iii) about the old basis still hold for $B$. Since $\bar{y}$ is not a rational function $\bar{y}$ does appear in $B$.

In the remainder of this paper $K$ with a subscript will always denote an effectively computable constant. Using Lemma VI as well as the statements above we see that there exists $\varrho(z)$, a non-zero element of $Z[i, z]$ which is effectively bounded from above such that, for every $\gamma \geqslant 1$, each

$$\varrho(z) d_\gamma (\gamma - 1)!\, E_2^{N-m+\gamma}\bar{y}, \ \ldots, \ \varrho(z) d_{\gamma+\beta-1}(\gamma + \beta - 2)!\, E_2^{N-m+\gamma+\beta-1}\bar{y}$$

may be written as a linear combination over $Z[i, z]$ of the elements of $B$, with coefficients $\ll K_{11}^{\gamma}(z+1)^{\gamma+K_{11}}$, plus a polynomial of degree at most $\gamma + K_{11}$, where $K_{11}$ does not depend on $\gamma$. (For the definition of $\ll$ see [4] p. 359). Let $\eta$ denote the dimension in $V$ of $E_2^{N-m}\bar{y}, \ldots, E_2^{N-m+\beta-1}\bar{y}$ and, hence, of each $E_2^{N-m+\gamma}\bar{y}, \ldots, E_2^{N-m+\gamma+\beta-1}\bar{y}$. Let $\eta_1$ denote the dimension in $V$ of the derivatives of $\bar{y}$. We see that $\eta \leqslant \beta$ and $\eta_1 \leqslant m$. Let $z_1$ denote any point in $C$ which is not a zero of $a_m(z)\varrho(z)$.

LEMMA VIII. *For every $z_1$ and positive integer $\theta$ there exist elements of $Z[i, z]$*

$$s_\theta(z) \text{ having degree at most } \eta\theta + K_{12},$$

$$t_{\theta,p}(z) \text{ having degree at most } \eta_1\theta + K_{12},$$

*and*

$$U_{\theta,p,l}(z) \ll K_{12}^{\theta}(z+1)^{(\eta-\eta_1)\theta+K_{12}}$$

*such that:*

(i) *each*

$$L_{\theta,p} \overset{\text{def}}{=} \sum_{l=0}^{\beta-1} U_{\theta,p,l}(z)\varrho(z) d_{\theta+l}(\theta + l - 1)!\, E_2^{N-m+\theta+l}\bar{y}$$

*equals a linear combination over* $Z[i, z]$ *of derivatives of* $\bar{y}$ *plus a polynomial of degree at most* $(\eta-\eta_1)\theta+K_{12}$;

(ii) $\sum_p t_{\theta,p}(z) L_{\theta,p}(z) = s_\theta(z)\bar{y}$ *plus a polynomial of degree at most* $\eta\theta+K_{12}$;

(iii) $s_\theta(z_1) \neq 0$; *and* (iv) $K_{12}$ *is independent of* $\theta$ *and* $z_1$.

Proof. The various effective upper bounds will follow trivially once we have constructed our polynomials so as to satisfy the other conditions. Set each $\varrho(z) d_{\theta+l}(\theta+l-1)!\, E_2^{N-m+\theta+l}\bar{y} = v_l$, for $0 \leqslant l \leqslant \beta-1$. We may write each $v_l$, in $V$, as a linear combination of the elements of $B$. Set $z = z_1$ and choose out of the above coefficient matrix any maximal non-singular matrix. Since at $z = z_1$ we may write each $\bar{y}^{(j)}$ as a linear combination of the $v_l$ one may use elementary row operations to see that our submatrix must contain the $\eta_1$ columns corresponding to the $\eta_1$ elements of $B$ which are derivatives of $\bar{y}$.

Now look at any maximal non-singular submatrix $S$, with $z$ indeterminate, which contains a submatrix which is a maximal non-singular submatrix when $z = z_1$ that in turn contains the $\eta_1$ columns of coefficients of derivatives of $\bar{y}$.

We may apply Cramer's rule to solve, in $V$, for

$$\Delta\bar{y} \overset{\text{def}}{=} (z-z_1)^\gamma s_\theta(z)\bar{y} \overset{\text{def}}{=} (z-z_1)^\gamma s_\theta\bar{y}$$

as a linear combination over $Z[i, z]$ of the $v_l$, where $\Delta$ is the determinant of $S$, and $s_\theta(z_1) \neq 0$. If the coefficients of the $v_l$ are denoted as $\Delta_l$, where $\Delta_l$ is either a determinant or zero, we shall see that each $\Delta_l$ must be divisible by $(z-z_1)^\gamma$. We wish to see that any minor of $S$ gotten by expanding $\Delta$ along any column of coefficients of a derivative of $\bar{y}$ must be divisible by $(z-z_1)^\gamma$. The column operations which one goes through in order to show that the matrix has a determinant divisible by $(z-z_1)^\gamma$ are not affected by the loss of such a column, since at $z = z_1$ such a column can not enter into any dependence relations, and an arbitrary row. Thus each $\Delta_l$ is divisible by $(z-z_1)^\gamma$. Let us assume for now that $\eta_1 > 1$. Expanding each $\Delta_l$ along a column of coefficients of a derivative of $\bar{y}$ we may write $\Delta_l = \sum_{j\neq l} c_j \Delta_{j,l}$ where each $\Delta_{j,l}$ is divisible by $(z-z_1)^\gamma$ and each $c_j \in Q[i, z]$. Then

$$(z-z_1)^\gamma s_\theta\bar{y} = \sum_j c_j\Big(\sum_{l\neq j}\Delta_{j,l}v_l\Big).$$

Each $\sum_{l\neq j}\Delta_{j,l}v_l$ is a linear form in the derivatives of $\bar{y}$ (in fact in $\bar{y}$ and one other derivative of $\bar{y}$) inside of $V$. We may continue the above process $\eta_1-2$ more times until we have an equation which we write in $V$ as

$$(z-z_1)^\gamma \sum_p t_{\theta,p}(z) L_{\theta,p}(z) = (z-z_1)^\gamma s_\theta(z)\bar{y}.$$



Note that the non-homogeneous term must be divisible by $(z-z_1)^\gamma$ also. As was remarked the upper bounds are trivial. This proves Lemma VIII.

Notice that Lemmas VII and VIII do not depend on the solutions of (6) being anything more than $A$-functions. In what follows we assume that $\varphi_c(x)$ has no integral zeros. Note that $\eta_1 = m$. Let $N_1$ denote a parameter which takes on Gaussian integral values. Let $H_1$ be as in equation (7). Consider for any positive integer $u$, any non-negative integer $0 \leqslant h \leqslant n-1$, any circular path $\Gamma$ which winds once about each $N_1+\alpha_r$ in the positive direction, and any $s_0(z)$ as in Lemma VIII,

$$\begin{aligned} 0 &= (2\pi i)^{-1}\int_\Gamma \big(s_0(z)\big)^{m+1}(H_1\bar{y})z^h\Big(\prod_{r=1}^n (z-N_1-\alpha_r)^{-u}\Big)dz \\ &= (2\pi i)^{-1}\int_\Gamma \Big[\sum_{j=0}^c \big(s_0(z)\big)^{-1}z^j\varphi_j(zD)z^h\big(s_0(z)\big)^{m+1}\Big(\prod_{j=1}^n (z-N_1-\alpha_j)^{-u}\Big)\Big]s_0(z)\bar{y}(z)\,dz. \end{aligned}$$

Let $R_{un-h}(s_0\bar{y})$ denote

$$(2\pi i)^{-1}\int_\Gamma s_0(z)\bar{y}(z)\Big(\prod_{j=1}^n (z-N_1-\alpha_j)\Big)^{-u}z^h\,dz$$

for all $u$ and $h$ as above. Note that the final integral in the above equation may be written as a linear combination of $R_l(s_\theta\bar{y}), \ldots, R_{l+\delta}(s_\theta\bar{y})$ over $Q[i, N_1]$ where $l = un-(h+c)+m\big(\deg s_\theta(z)\big)$ and $\delta \leqslant m\eta\theta+mn+K_{13} = \delta_1 = \delta_1(\theta)$, where $K_{13}$ is independent of $\theta$, $u$, and $N_1$.

The coefficient of $z^{-l}$ in the expansion about $z = \infty$ of

$$\sum_{j=0}^c \big(s_\theta(z)\big)^{-1}z^j\varphi_j(zD)\Big[z^h\big(s_\theta(z)\big)^{m+1}\prod_{j=1}^n (z-N_1-\alpha_j)^{-u}\Big]$$

is essentially $\varphi_c\big(-l-c+\deg s_\theta(z)\big)$, which is not zero since $\varphi_c(z)$ has no integral zeros. Thus $R_l(s_\theta\bar{y})$ actually appears in the above equation. For each pair of non-negative integers $k$ and $\theta$ set $M_{\theta,k}$ equal to the module generated over $Q[i, N_1]$ by the $R_k(s_\theta\bar{y}), \ldots, R_{k+\delta_1}(s_\theta\bar{y})$, for each $s_\theta(z)$. (Recall if $\varrho(z_1)a_m(z_1) \neq 0$ then by Lemma VIII there exists an $s_\theta(z)$ with $s_\theta(z_1) \neq 0$.) We see that:

LEMMA IX. *If* $\varphi_c(x)$ *has no integral zeros and* $k_1 \leqslant k_2$ *then* $M_{\theta,k_1} \subseteq M_{\theta,k_2}$, *for all* $\theta \geqslant 1$.

If $\varphi_c(x)$ has no integral zeros it follows that $H_1^*\mu = 0$ has no rational function solutions. Thus (see the proof of Theorem V) it follows that $\bar{y}$ satisfies no equation of type (6) of order less than $n$, even allowing a polynomial non-homogeneous term. Therefore (see the proofs of Theorems V and VI) if no $\alpha_j-\alpha_{j_1}$ is a singularity of $\bar{y}$ the functions $\bar{y}^{(t)}(N_1+\alpha_j)$, for $0 \leqslant t \leqslant m-1$ and $1 \leqslant j \leqslant n$, are linearly independent over $Q(i, N_1)$.

Each

$$(2\pi i)^{-1}\int_\Gamma \big(s_\theta(z)\big)^{t+1}\bar{y}(z)(z-N_1-a_j)^{-(t+1)}\,dz,$$

for $t = 0, 1, \ldots, m-1$, may be written as a linear combination over $Q[i, N_1]$ of the elements of $M_{\theta,1}$. Then each

$$\big(s_\theta(N_1+a_j)\big)^{t+1}\bar{y}^{(t)}(N_1+a_j)\in M_{0,1}$$

for each $0\leqslant t\leqslant m-1$, as we may show by induction. We see then:

LEMMA X. *If $\varphi_c(x)$ has no integral zeros and no two $a_j$'s have their difference equal to a singularity of $\bar{y}$, then the $\bar{y}^{(t)}(N_1+a_j)$, for $0\leqslant t\leqslant m-1$ and $1\leqslant j\leqslant n$, are linearly independent over $Q[i, N_1]$ and for all positive integers $\theta$ and $k$ each $\big(s_0(N_1+a_j)\big)^m y^{(t)}(N_1+a_j)$ above belongs to $M_{0,k}$.*

What we are actually interested in are not the $M_{0,k}$ but the module $\overline{M}_{0,k}$ generated over $Q[i, z]$ by all of the

$$R_{un-h}(L_{0,p})\overset{\mathrm{def}}{=}(2\pi i)^{-1}\int_\Gamma L_{0,p}(z)\Big(\prod_{j=1}^{n}(z-N_1-a_j)^{-u}\Big)z^h\,dz,$$

where each $L_{0,p}(z)$ is in Lemma VIII and

$$k\leqslant un-h\leqslant k+\delta_2\overset{\mathrm{def}}{=}k+\delta_1+\eta_1\theta+K_{12}.$$

We see, using (ii) of Lemma VIII, that we may write each $R_k(s_0\bar{y})$ as a linear combination of the different $R_k(L_{0,p})$. From this we have:

LEMMA XI. *If $\varphi_c(x)$ has no integral zeros then, for all positive integers $\theta$ and $k$, each $\big(s_\theta(N_1+a_j)\big)^m y^{(t)}(N_1+a_j)$ in Lemma X belongs to $\overline{M}_{0,k}$, for all choices of $s_\theta(z)$.*

We wish to evaluate the $R_{n\theta}(L_{0,p}), \ldots, R_{n\theta-\delta_2}(L_{0,p})$, where $n\theta-\delta_2-1 > \theta(\eta-\eta_1)+K_{12}$. (If $n>(m+1)\eta$ this condition will be satisfied for all $\theta$ larger than some effectively computable constant.) For $un-h$ in the above range we may write

$$(8)\qquad R_{un-h}(L_{0,p}) = \int_\Gamma L_{0,p}(z)\Big(\prod_{j=1}^{n}(z-N_1-a_r)^{-u}\Big)z^h\,dz$$

as a linear combination over $Q(i, N_1)$ of the $D^s y(N_1+a_j)$, $0\leqslant s\leqslant m-1$, using the residue theorem, our representation of $L_{0,p}$ as a linear form over $Z[i, z]$ in the derivatives of $\bar{y}$ with a non-homogeneous term of degree less than $un-h-1$, and the differential equation for $\bar{y}$. Then using our representation of the $L_{0,p}$ as a linear combination over $Z[i, z]$ of the $v_l$, with a non-homogeneous term of degree less than $un-h-1$, and the partial fraction decomposition of $z^h\Big(\prod_{j=1}^{n}(z-N_1-a_j)^{-u}\Big)$ and simplifying



we may express (8) as a linear combination over $Q[i, N_1]$ of terms of the form

$$(9)\qquad (2\pi i)^{-1}\int_C (z-N_1-a_j)^{-u_1}d_{\theta+l}(\theta+l-1)!\,E_2^{N-m+\theta+l}\bar{y}\,dz$$

for $0\leqslant l\leqslant\beta-1$ and $u_1\leqslant u$. As in the proof of Lemma VI of [4] we see that there exists a common denominator in $Z[i]$, for the coefficients of our elements of type (9), whose absolute value is bounded from above by $K_{13}^\theta$ for some $K_{13}$ independent of $\theta$. The coefficient polynomials in $Q[i, N_1]$ have degrees bounded from above by $(\eta-\eta_1)\theta+K_{12}$ and their coefficients have absolute values less than $K_{14}^\theta$, for some $K_{14}$ independent of $\theta$, $u_1$, $l$, and $N_1+a_j$. Each element of type (9) may be written as

$$(10)\qquad \binom{\theta+l-1}{u_1-1}(\theta+l-u_1)!\,d_{\theta+l}E_2^{N-m+l+\theta-u_1+1}\bar{y}(N_1+a_j),$$

since $\theta+l\geqslant u+l\geqslant u_1\geqslant 1$. Now

$$\binom{\theta+l-1}{u_1-1}<2^{\theta+l-1}\leqslant 2^{\theta+K_{15}},$$

where $K_{15}$ is independent of $\theta$, $u_1$, $l$, and $N_1+a_r$.

One may effectively bound $|\bar{y}(z)|$ from above by a power of $|z|$, if $|z|$ is larger than some effectively computable constant. Then we may estimate $|R_k(L_{0,p})|$ by integrating along the circle $|z-N_1| = \frac{1}{2}|N_1|$. Using Lemmas VII and X and all of the above remarks we see that if $N_1$ is a Gaussian integer and $|N_1|$ is larger than some effectively computable bound then for each positive integer $\theta$:

LEMMA XII. *There exists some non-zero element of $Z[i]$ of absolute value less than $K_{16}^\theta$ such that if it is multiplied times each form in (8) the products each equal a linear form $S_{un-h}(L_{\theta,p})$ in the $y^{(t)}(N_1+a_j)$, $0\leqslant t\leqslant m-1$ and $1\leqslant j\leqslant n$ with coefficients in $Z[i]$ having absolute value less than $K_{16}^\theta|N_1|^{(\eta-\eta_1+1)\theta+K_{16}}$ and such that the absolute value of each linear form is less than*

$$K_{16}^\theta|N_1|^{-\theta(n-(m+1)\eta)+K_{16}}$$

*for some $K_{16}$ independent of $\theta$ and $N_1$.*

We next wish to use Lemmas VIII to XII, along with the Lemma of [3] to conclude our proof of Theorem III. We note that for each $N_1$ with $|N_1|$ larger than the absolute values of the zeros of $a_m(z)\varrho(z)$ we may pick out of the $S_{un-h}(L_{\theta,p})$, for $n\theta-\delta_2\leqslant un-h\leqslant n\theta$, $nm$ linear forms in the $\bar{y}^{(t)}(N_1+a_j)$, $0\leqslant t\leqslant m-1$ and $1\leqslant j\leqslant n$, which have a non-singular coefficient matrix. Let us call this coefficient matrix $M_\theta$. Our system of forms may be written as $M_\theta V$, where $V$ is a column vector. Then, by Lemma XII,

(i) $$\|M_\theta\| \leqslant K_{16}^{\theta} |N_1|^{+(\eta-\eta_1+1)\theta+K_{16}}$$

and

(ii) $$\|M_\theta V\| \leqslant K_{16}^{\theta} |N_1|^{-\theta(n-(m+1)\eta)+K_{16}},$$

where $\|\text{matrix}\|$ equals the maximum of the absolute values of its entries.

Next set $f(\theta) = |N_1|^{(\eta-\eta_1+1)\theta}$. Then in the Lemma of [3] choose $r = 5'$ and $\varepsilon < 1$ so $(1+\varepsilon/5)^2(1-\varepsilon/5)^{-1} < 1+\varepsilon$. We see that $f(0) = 1$ and that $f(\theta)$ is monotone increasing and onto $[1, +\infty)$. If $\theta$ is larger than some effectively computable number $\theta_1$ then

(i) $$\|M_\theta\| \leqslant \big(f(\theta)\big)^{1+\varepsilon/5},$$

(ii) $$\|M_\theta V\| \leqslant \big(f(\theta)\big)^{-\Lambda(1-\varepsilon/5)}$$

and

(iii) $$f(\theta) \leqslant \big(f(\theta-1)\big)^{1+\varepsilon/5},$$

where

$$\Lambda = \big(n-(m+1)\eta\big)(\eta-\eta_1+1)^{-1} \geqslant \big(n-(m+1)\beta\big)(\beta-m+1)^{-1}.$$

Thus the hypotheses of Lemma [3] are satisfied with $r = 5$. By that Lemma we see that if $q$ is a (Gaussian) integer with $|q| > \frac{1}{2}\big(f(\theta_1)\big)^{\Lambda(1-\varepsilon/5)}$, then for all column matrices $P$ with (Gaussian) integral entries

$$\|V-Pq^{-1}\| \geqslant (mn)^{-1}|2q|^{-(1+(1+\varepsilon)\Lambda^{-1})}.$$

This proves Theorem III.

Let $\operatorname{ord} f$ denote the order of vanishing of $f$ at $z = \infty$. We shall need the following lemma in order to prove Theorem IV.

Lemma XIII. *Suppose that $\big(l_\theta(z)\big) \equiv \big(\sum_{j=1}^{t} a_{j,\theta}(z) w_j(z)\big)$ is a sequence of not identically zero linear forms over $Q[i, z]$ in $w_1(z), \ldots, w_t(z)$, where each $w_j(z)$ is algebraic over $Q(i, z)$ and its minimal polynomial over $Q(i, z)$ is known. Set each $d(\theta) = \max\{\deg a_{j,\theta}(z)\}$. Let $\gamma > 0$. Suppose that for each $\varepsilon > 0$ and each positive integer $N$ there exist, respectively, effectively computable positive integers $\theta_1(\varepsilon)$ and $\theta_2(N)$ such that if $\theta \geqslant \theta_1(\varepsilon)$ then*

$$\gamma - \big(\operatorname{ord}(l_\theta(z))\big)\big(d(\theta)\big)^{-1} < \varepsilon$$

*and if $\theta \geqslant \theta_2(N)$ then*

$$d(\theta) \geqslant N.$$

*Then there exists $K_{17} > 0$ such that if $N_1$ is a Gaussian integer and $|N_1| > K_{17}$ the dimension of $Q\big(i, w_1(N_1), \ldots, w_t(N_1)\big)$ over $Q(i)$ is at least $\gamma+1$.*



Proof. Without loss of generality we may take the $w_j(z)$ to be algebraic integers over $Q[i, z]$ and the $a_{j,n}(z)$ to belong to $Z[i, z]$. For each positive integer $\theta$ it is possible to find $K_{18}$, depending on $\theta$, such that if $|N_1| > K_{18}$ then the absolute value of each (algebraic) conjugate over $Q(i)$ of $l_\theta(N_1)$ is less than $|N_1|^{d(\theta)+K_{19}}$, where $K_{19}$ is independent of $N_1$ and $\theta$. The product of all of the conjugates of each $l_\theta(N_1)$ is a non-zero Gaussian integer. Hence if $\theta$ is sufficiently large and $|N_1| > K_{18}$ we see that $l_\theta(N_1)$ must have at least $\gamma+1$ conjugates (including itself) over $Q(i)$. Thus the $w_j(N_1)$ generate a field of dimension at least $\gamma+1$ over $Q(i)$. This proves Lemma XIII.

Proof of Theorem IV. We wish to apply Lemma XIII to the sequence $\big(R_{\theta_n}(L_{\theta,p})\big)$, where the $p$ is chosen arbitrarily except that $L_{\theta,p}(z) \not\equiv 0$ in $V$ and $\bar{y} = y$. First we wish to see that no such $R_{\theta n}(L_{\theta,p})$ is identically zero. Assume one is 0. As we have seen before any such identity would imply that the sum over all terms involving any one value of $a_j$ equals a polynomial in $N_1$. We would then have that $L_{\theta,p}(z)$ is the solution of a non-zero linear differential equation with constant coefficients and polynomial non-homogeneous term. Since $L_{\theta,p}(z)$ must also be algebraic we see that it must be a polynomial (polynomials are the only entire algebraic functions). This is a contradiction. Hence $R_{\theta n}(L_{\theta,p}) \not\equiv 0$. One may look back to see that here

$$\gamma = (n-\eta+\eta_1)(\eta-\eta_1+1)^{-1} \geqslant (n-\beta+1)\beta^{-1},$$

since $\eta \leqslant \beta$ and $\eta_1 \geqslant 1$. This proves Theorem IV.

As was remarked in the Introduction, one can do nearly as well as the statement in Theorem IV without using any deep results. Where $a_m(z)$ is the coefficient of $D^m y$ we could set

$$l_\theta(N_1) = (2\pi i)^{-1} \int_\Gamma \big(a_m(z)\big)^\theta y(z) \Big(\prod_{j=1}^{n} (z-N_1-a_j)\Big)^{-(\theta+1)} dz,$$

where $\Gamma$ is a circular path enclosing the points $N_1+a_j$. As in the proof of Theorem IV, no $l_\theta(z)$ is identically equal to zero. Since $z = \infty$ is at worst a regular singular point of our differential equation for $y$ we see that $\deg a_m(z) = \beta > \deg a_j(z)$ if $j \neq m$. Thus we have that here $\gamma = (n-\beta)\beta^{-1}$.

The following are corrections for [4]: On page 359 in 3rd line from the bottom $Dy(z+a_r)$ should be $D^\varphi y(z+a_r)$ and in the 4th line from the bottom the $E_y^\theta y(z+a_r)$ should be "$E^\theta y(z+a_r)$". On the same page in lines 9 and 8 from the bottom (i) should read "some sequence of repeated integrals of $y(t), E^1 y(t), \ldots, E^\theta y(t), \ldots$ with each $E^{\theta-1} y(t) = \frac{d}{dz} E^\theta y(t)$". In line 13 from the bottom on page 359 we should have "$1 \leqslant j \leqslant p$" not $1 \leqslant j \leqslant p_1$.

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# On the upper asymptotic density of $(0, r)$-primitive sequences

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**1.** In this paper $A$ will denote a subsequence of the sequence of positive integers. For a set $V$ we denote by $A(V) = A(V, A)$ the number of elements of $A \cap V$. Moreover we put

$$\underline{d}A = \liminf \frac{A([1, n])}{n} \quad \text{and} \quad \bar{d}A = \limsup \frac{A([1, n])}{n}$$

for the lower and upper asymptotic density of $A$; if $\underline{d}A = \bar{d}A$ we write $dA$ for the asymptotic density of $A$.

A sequence $A = (a_i)$ is called *primitive* if $a_i \not\equiv 0 \pmod{a_j}$ if $i \neq j$. For a survey of the theory of primitive sequences we refer to [5], chapter V and [4]. We only state here three well-known results, see [5], p. 244–245.

THEOREM 1. *If $A$ is a primitive sequence, then $\bar{d}A < \frac{1}{2}$.*

THEOREM 2. (Behrend [1].) *For every primitive sequence, $\underline{d}A = 0$.*

THEOREM 3. (Besicovitch [2].) *Corresponding to every $\varepsilon > 0$, there exists a primitive sequence $A$, depending on $\varepsilon$, such that $\bar{d}A > \frac{1}{2} - \varepsilon$.*

Let $r$ be a positive integer. We will call in this paper a sequence $A = (a_i)$ $(0, r)$-*primitive* if $a_i \not\equiv 0, r \pmod{a_j}$ if $i \neq j$. In the following sections we give estimations for $\bar{d}A$ of $(0, r)$-primitive sequences, similar to the Theorems 1 and 3.

**2.** In this section we study $(0, r)$-primitive sequences with $r$ odd.

THEOREM 4. *Let $r$ be an odd positive integer. If $A$ is a $(0, r)$-primitive sequence then $\bar{d}A \leqslant \frac{1}{4}$.*

Proof. Let $n$ be a positive integer and $a_1, \ldots, a_t$ the elements of $A$ not exceeding $n$. Let $a_i'$ $(1 \leqslant i \leqslant t)$ denote the greatest odd divisor of $a_i$ and $A' = (a_i')_{i=1}^{t}$. Since $a_i' = a_j'$ implies $a_i | a_j$ or $a_j | a_i$ all numbers $a_i'$ are distinct.

We construct a one-to-one correspondance between the odd integers in $[1, \frac{1}{2}n]$ and the odd integers in $(\frac{1}{2}n + r, n + r]$. To every odd integer $c$ in $[1, \frac{1}{2}n]$ there exists exactly one integer of the form $2^k c$ in $(\frac{1}{2}n, n]$ and