6. Conclusions. We have shown, in this paper, that one of the best known theorems involving the sum of divisors function, as defined in $\mathscr{Z}$ has an analog, based on Spira's definition of σ, in $\mathfrak{S}$. While it is not absolutely certain that an alternate definition of σ would not yield comparable or even better results, the case now seems to be quite strong for the validity of Spira's definition of the sum of divisors function in $\mathfrak{S}$.

Our examination of the properties of $\sigma(\eta)/\eta$ suggests that the answers (or lack of them) to nearly all the questions which could be asked concerning the existence or the structure of odd perfect numbers in $\mathfrak{S}$, and concerning whether there exists a finite number of even and odd perfect numbers in $\mathfrak{S}$, may be similar to the answers to the same questions when posed about perfect numbers in $\mathscr{Z}$. The fact that if π is a prime and $a < b$, then $\|\sigma(\pi^a)/\pi^a\|$ is not necessarily less than $\|\sigma(\pi^b)/\pi^b\|$, and the resultant implication that imprimitive perfect numbers may exist, certainly suggests additional questions. A characterization of the even imprimitive perfect numbers or a proof that all perfect numbers in $\mathfrak{S}$ are primitive would be of considerable interest.

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Received on 5. 7. 1972 (302)

ACTA ARITHMETICA
XXV (1974)

$\Gamma^*(8)$

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1. Introduction. For each positive integer k, $\Gamma^*(k)$ is defined as the least s such that the congruence

$$(*)\qquad a_1x_1^k+\ldots+a_sx_s^k \equiv 0 \pmod{p^n}$$

has a primitive solution for all non zero integers $a_1, \ldots, a_s$ and all prime powers p^n.

Except for $k = 8$, $\Gamma^*(k)$ is known for $1 \leqslant k \leqslant 12$ and also when k is of the form $p-1$ or $p(p-1)$ where p is a prime ([3] and [5]). These results are given in Table 2. In all cases for which $\Gamma^*(k)$ is known it is true that $\Gamma^*(k) \equiv 1 \pmod{k}$ and Norton [5] has conjectured that $\Gamma^*(k) \equiv 1 \pmod{k}$ for all k. In this paper we show that $\Gamma^*(8) = 39$, disproving this conjecture.

Throughout this paper we use the same notation as Dodson in [3].

2. LEMMA 1. *Let n be a positive integer and suppose that for $i = 0, \ldots, n$, $F_i = \sum_{j=1}^{v_i} a_{ij}x_{ij}$ with all the a_{ij} odd and with $\sum_{i=0}^{k-1} v_i \geqslant 2^k$ for each $k = 1, \ldots, n$. Then for any positive integer $N > n$, $\sum_{i=0}^{n} 2^iF_i$ represents at least $\min(\sum_{i=0}^{n} v_i, 2^N)$ different residue classes* $\pmod{2^N}$ *where the $x_{ij} = 0$ or 1 and with at least one of the $x_{0j} = 1$.*

Proof. The proof is by induction on n. For $n = 0$ the result follows from Chowla's theorem on the addition of residue classes ([1] or [4], p. 49, Theorem 15).

Assume that the result is true for $n-1$. It is given that $\sum_{i=0}^{k-1} v_i \geqslant 2^k$, $k = 1, \ldots, n$ and so $\sum_{i=0}^{n-1} 2^iF_i$ represents at least 2^n different residue classes $\pmod{2^n}$ by the induction hypothesis, and at least $\sum_{i=0}^{n-1} v_i = N_0$ say, different residue classes $\pmod{2^N}$.

Let us represent these residue classes by numbers of the form $s+2^nY_{st}$, $t = 1, \ldots, u_s$; $s = 0, \ldots, 2^n-1$, where $1 \leqslant u_s \leqslant 2^{N-n}$ for each s.

We note that

$$\sum_{s=0}^{2^n-1} u_s = N_0.$$

Now $\sum_{i=0}^{n} 2^i F_i$ represents all the residue classes of the form

$$s+2^n\Big\{Y_{st}+\sum_{j=1}^{v_n} a_{nj}x_{nj}\Big\}$$

where $t = 1, \ldots, u_s$; $s = 0, 1, \ldots, 2^n-1$.

Then by Chowla's theorem the part in brackets represents at least $\min(u_s+v_n, 2^{N-n})$ different residue classes $(\mathrm{mod}\ 2^{N-n})$. Thus $\sum_{i=0}^{n}{}' 2^t F_i$ represents at least $\sum_{s=0}^{2^n-1}\min(u_s+v_n, 2^{N-n})$ different residue classes $(\mathrm{mod}\ 2^N)$. By considering the two possibilities $u_s+v_n \geqslant 2^{N-n}$ for all s and $u_s+v_n \leqslant 2^{N-n}$ for some s, it can easily be seen that

$$\sum_{s=0}^{2^n-1}\min(u_s+v_n, 2^{N-n}) \geqslant \min\Big(\Big(\sum_{s=0}^{2^n-1} u_s\Big)+v_n, 2^N\Big)$$

which gives the required result.

Lemma 2. *Let k be an integer power of 2, then the numbers $1+32n$, $n = 0, \ldots, k-1$ are all the odd 8th powers* $(\mathrm{mod}\ 32k)$.

Proof. If x is odd, then $x^8 \equiv 1 \ (\mathrm{mod}\ 32)$. Since there are just k odd 8th powers $(\mathrm{mod}\ 32k)$ and just k numbers of the form $1+32n$ $(\mathrm{mod}\ 32k)$ the result follows.

Lemma 3. *Let $F = F_0+2F_1+4F_2$ where*

$$F_i = \sum_{j=1}^{v_i} a_{ij}x_{ij}^8, \quad i = 0, 1, 2,$$

and a_{ij} is odd for each i, j.

Suppose $v_0+v_1+v_2 = v$ with $8 \leqslant v \leqslant 14$, then F represents at least $v-6$ multiples of $8\ (\mathrm{mod}\ 2^8)$.

Proof. We deal with 3 different cases

(1) $v_0 \geqslant 2$, $v_0+v_1 \geqslant 4$. In this case it follows from Lemma 1 that we can solve $F \equiv 0 \ (\mathrm{mod}\ 8)$ with at least one of the $x_{0j} = 1$, and so we have that $F \equiv 8A \ (\mathrm{mod}\ 2^8)$ for some integer A. Now if x is an integer and we set $x_{0j} = x$ we get $F \equiv 8A+a_{0j}(x^8-1) \ (\mathrm{mod}\ 2^8)$ but by Lemma 2 with $k = 8$, we can solve $x^8-1 \equiv 32n \ (\mathrm{mod}\ 2^8)$ for $n = 0, \ldots, 7$ and so for $n = 0, \ldots, 7$ we can solve

$$F \equiv 8A+32na_{0j} \ (\mathrm{mod}\ 2^8).$$

The numbers $8A+32na_{0j}$, $n = 0, \ldots, 7$ are multiples of 8 and are incongruent $(\mathrm{mod}\ 2^8)$ and so, as $v \leqslant 14$, the result follows.

(2) $v_0 = 0$ or 1 and $v_1 \geqslant 2$. In this case we ignore F_0. First suppose $v_1+v_2 = 7$, then $v = 8$. By Lemma 1 we can solve $F_1+2F_2 \equiv 0 \ (\mathrm{mod}\ 4)$ with at least one of the $x_{1j} = 1$. Thus again by using Lemma 2 we can represent 4 multiples of $4 \ (\mathrm{mod}\ 2^7)$, say $4B+32na_{1j}$, $n = 0, 1, 2$ and 3, by F_1+2F_2. Hence we can represent $4 > v-6$ multiples of $8\ (\mathrm{mod}\ 2^8)$, as required.

Second suppose $8 \leqslant v_1+v_2$. By Lemma 1 we can solve $F_1+2F_2 \equiv 4 \ (\mathrm{mod}\ 8)$ and $F_1+2F_2 \equiv 0 \ (\mathrm{mod}\ 8)$, in each case with one of the $x_{1j} = 1$. By applying Lemma 2 as before, in each case we can represent at least 4 different multiples of $4\ (\mathrm{mod}\ 2^7)$ by F_1+2F_2 and so altogether we can represent at least $8 \geqslant v-6$ different multiples of $8\ (\mathrm{mod}\ 2^8)$.

(3) $v_0+v_1 \leqslant 3$. In this case we ignore F_0 and F_1. First suppose $5 \leqslant v_2 \leqslant 7$ so that $v \leqslant 10$. Then by Lemma 1, F_2 represents both 2 and $4\ (\mathrm{mod}\ 4)$ and as before, by Lemma 2, each of these gives rise to 2 multiples of $2\ (\mathrm{mod}\ 2^6)$. So altogether F_2 represents 4 multiples of $2\ (\mathrm{mod}\ 2^6)$ as required.

Second suppose $v_2 \geqslant 8$. Then by Lemma 1 F_2 represents 2, 4, 6 and $8\ (\mathrm{mod}\ 8)$ and again each of these gives rise to 2 multiples of $2\ (\mathrm{mod}\ 2^6)$, giving in all 8 multiples of $2\ (\mathrm{mod}\ 2^6)$ as required.

Theorem 1. $\Gamma^*(8, 2) = 39$.

Proof. Clearly the congruence

$$\sum_{i=1}^{7} x_i^8+8\sum_{i=1}^{31} y_i^8 \equiv 0 \ (\mathrm{mod}\ 2^8)$$

has no non trivial solution and so we know that $\Gamma^*(8, 2) \geqslant 39$.

Suppose that we have a form F with 39 variables. Then without loss of generality ([3], pp. 182–183 or [2]) we can write F in the form

$$F = F_5+2F_6+4F_7+8(F_0+2F_1+4F_2+8F_3+16F_4)$$

with $F_i = \sum_{j=1}^{v_i} a_{ij}x_{ij}^8$ for each $i = 0, \ldots, 7$; a_{ij} all odd and $\sum_{i=0}^{7} v_i = 39$ and such that $\sum_{i=0}^{k}{}' v_i \geqslant \dfrac{39(k+1)}{8}$, $k = 0, \ldots, 7$.

If we can show that we can always solve $F \equiv 0 \ (\mathrm{mod}\ 2^8)$ with at least one of the variables x_{0j} odd, then the problem is solved.

We have

$$\begin{aligned} v_0 &\geqslant 5 > 2^1, \\ v_0+v_1 &\geqslant 10 > 2^2, \\ v_0+v_1+v_2 &\geqslant 15 > 2^3, \\ v_0+v_1+v_2+v_3 &\geqslant 20 > 2^4 \end{aligned}$$

and so by Lemma 1 $\sum_{i=0}^{4} 2^i F_i$ represents, with at least one of the x_{0j} odd, at least $v' = \sum_{i=0}^{4} v_i \geqslant 25$ different residue classes (mod 32).

Now if $v' \geqslant 32$ the problem is solved and so we take $25 \leqslant v' \leqslant 31$ without loss of generality. This implies that $8 \leqslant v_5+v_6+v_7 \leqslant 14$ which is the hypothesis for Lemma 3.

Thus we have that $F_5+2F_6+4F_7$ represents at least $v_5+v_6+v_7-6 = 39-v'-6 = 33-v'$ different multiples of 8 (mod 2^8).

We have already shown that $8\left(\sum_{i=0}^{4} 2^i F_i\right)$ represents v' multiples of 8 (mod 2^8) and so, as there exists only 32 multiples of 8 (mod 2^8) the problem is solved.

All that remains is to show that $\Gamma^*(8, p) \leqslant 39$ for all odd primes p. Now if p does not divide k we note that

$$\Gamma^*(k, p) \leqslant k\left(\gamma^*(k, p)-1\right)+1 \qquad ([3], \text{ Lemma } 4.2.1)$$

where $\gamma^*(k, p)$ is the least s such that (*) has a primitive solution for all integers $a_1, \ldots, a_s$ prime to p and $n = 1$, and so we have to show that $\gamma^*(8, p) \leqslant 5$ for all odd primes p. If $d = (k, p-1)$ then we know that ([3], p. 166–167) $\gamma^*(k, p) = \gamma^*(d, p) \leqslant d+1$, and so we are only interested in primes $\equiv 1 \pmod 8$, the first few of which are

$$17,\ 41,\ 73,\ 89,\ 97,\ 113,\ 137,\ 193.$$

It is fairly easy to show using exponential sums that if $(d-1)^{s-1} < p^{\frac{1}{2}s-1}$ then $\gamma^*(d, p) \leqslant s$ ([3], p. 168). For $s = 5$ and $d = 8$ this is certainly true for all $p \geqslant 193$.

Another well known result (see [3], p. 166) is that if $d = \frac{1}{2}(p-1)$ then

$$\gamma^*(d, p) = \left[\frac{\log p}{\log 2}\right]+1.$$

Applied to $p = 17$, $d = 8$ this gives $\gamma^*(8, 17) = 5$.

Thus we are left with the primes

$$41,\ 73,\ 89,\ 97,\ 113,\ 137.$$

LEMMA 4. *Let N be a positive integer. Let f be a real positive function defined on the integers* (mod N) *and let $c_1, \ldots, c_s$ be integers prime to N. Then*

$$\sum_{n=1}^{N} f(nc_1) \ldots f(nc_s) \leqslant \sum_{n=1}^{N} f(n)^s.$$

Proof. This is straightforward using Hölder's inequality and induction on s.

Suppose that $c_1x_1^d + \ldots + c_sx_s^d \equiv 0 \pmod p$ has only the trivial solution. Then

$$\sum_{x_1, \ldots, x_s=0}^{p-1} \sum_{t=0}^{p-1} e_p(c_1x_1^d t) \ldots e_p(c_sx_s^d t) = p,$$

i.e.

$$\sum_{t=1}^{p-1} S(c_1t) \ldots S(c_st) = p-p_s,$$

where, as usual, $e_p(x) = \exp(2\pi ix/p)$ and

$$S(b) = \sum_{x=0}^{p-1} e_p(x^d b).$$

Taking moduli and applying Lemma 4 we get

$$\sum_{t=1}^{p-1} |S(t)|^s \geqslant p^s - p$$

and so if we now define, for $s > 1$,

$$Q(d, p, s) = \sum_{t=1}^{p-1} |S(t)|^s/(p^s-p)$$

we get:

LEMMA 5. *If $Q(d, p, s) < 1$ then*

$$\gamma^*(d, p) \leqslant s.$$

The values of $Q(8, p, 5)$ for the six relevant primes were calculated on York University's ICL 4100 computer and they are listed, accurate to three figures, in Table 1. It can be seen that they are all well less than 1.

We have now shown that $\Gamma^*(8, p) \leqslant 33$ for all odd primes p, and so, combining this with Theorem 1 we get our final result.

THEOREM 2.

$$\Gamma^*(8) = 39.$$

Table 1

p	41	73	89	97	113	137
$Q(8, p, 5)$	0.880	0.206	0.338	0.319	0.446	0.249

Table 2. Values of $\Gamma^*(k)$

k	1	2	3	4	5	6	7	8	9	10	11	12
$\Gamma^*(k)$	2	5	7	17	16	37	22	39	37	101	45	145

If $k+1$ is prime then $\Gamma^*(k) = k^2+1$ and if $k+1$ is composite and k can be written in the form $p(p-1)$ for prime p then $\Gamma^*(k) = \frac{1}{2}k(p^2-1)+1$.

Acknowledgements. I would like to thank the Science Research Council for my Maintenance Grant while this research was being done.

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Received on 22. 8. 1972 (314)

ACTA ARITHMETICA
XXV (1974)

On difference-sequences

by

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1. Introduction. Throughout this paper capitals will be used for denoting (strictly increasing) sequences of non-negative integers; if $A, B, \ldots, A', A^{(1)}, \ldots$ denotes some sequence, then $a_n, b_n, \ldots, a'_n, a_n^{(1)}, \ldots$ is its nth element, and $A(x), B(x), \ldots, A'(x), A^{(1)}(x), \ldots$ the number of its elements less than or equal to x.

Let $A+B$ denote the sequence of all numbers which can be written in the form a_i+b_j. P. Erdös ([1], page 30, problem 15) asked the following: if $A^{(2)} = A+A$, how much denser will $A^{(2)}$ be than A. In general, we can state nothing, as is seen from the example of the multiples of some fixed number d; Erdös conjectured that $A(x) = o(x)$ implies

$$\limsup A^{(2)}(x)/A(x) \geqslant 3.$$

(In all the limits, if not stated differently, we mean $x\to\infty$.) Freyman in his book ([2], page 120) has proven this (in fact, a little more).

Also Erdös asked the analogous question for the sequence of differences (that is, the numbers which can be represented in the form a_i-a_j, $i \geqslant j$; this sequence will be called the *difference-sequence of* A). If A, A', $A^{(1)}, \ldots$ is a sequence, its difference-sequence will be denoted by D, D', $D^{(1)}, \ldots$ Erdös conjectured that if $A(x) = o(x)$, then $D(x)/A(x)\to\infty$ (unlike the case of sums, where $\limsup A^{(2)}(x)/A(x) \geqslant 3$ is best possible), and with Sárközy he proved that if the upper density of A is positive, then $d_{n+1}-d_n = O(1)$. In Section 2 we prove Erdös's conjecture, and in Section 3 we determine exactly what can be asserted about the lower density of D, knowing the upper density of A. Finally, in Section 4 we shall mention some more problems and some results, only outlining the proofs.

2. The case $A(x) = o(x)$.

Theorem I. *If A is an infinite sequence for which $A(x) = o(x)$, then*

$$\lim D(x)/A(x) = \infty.$$