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On a problem of Davenport and Schinzel

by

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We set for every integer l: $\boldsymbol{l} = \{0, 1, \ldots, l-1\}$.

Definition 1. A function $a\colon \boldsymbol{N}\to\boldsymbol{n}$ is said to be an *admissible n-sequence of length N* if $a_i \neq a_{i+1}$ for $i+1 < N$ (a_i is the value of the function at the place i).

We say that *a contains an alternating l-sequence* if there are numbers $b \neq c$ and $0 < i_0 < \ldots < i_{l-1} < N$ such that

$$(1)\qquad \begin{cases} a_{i_{2s}} = c & \text{if} \quad 0 \leqslant 2s < l, \\ a_{i_{2s+1}} = b & \text{if} \quad 1 \leqslant 2s+1 < l. \end{cases}$$

Definition 2. $N_l(n) = \max\big(N\colon$ there is an admissible n-sequence of length N not containing an alternating $(l+1)$-sequence$\big)$.

Remark. One can extend the notion of an admissible n-sequence of length N replacing in Definition 1 the set $\boldsymbol{n}$ by an arbitrary set of n elements. Clearly such an extension does not affect the definition of $N_l(n)$. A finite sequence $a\colon \boldsymbol{N}\to X$ will often be denoted by $\langle a_0, \ldots, a_{N-1}\rangle$ and the set of its elements by $\{a_0, \ldots, a_{N-1}\}$.

It is known from [1] and [2] that $N_l(n)$ exists for every l and n, and we have

$$(2)\qquad N_3(n) = 2n-1,$$

$$(3)\qquad N_l(n) > (l^2-4l+3)n - C(l)$$

if l is odd and $l > 3$,

$$(4)\qquad N_l(n) > (l^2-5l+8)n - C(l)$$

if l is even and $l > 4$, where $C(l)$ is a constant depending on l only, and

$$(5)\qquad N_4(n) \geqslant 5n-8, \quad \lim \frac{N_4(n)}{n} \geqslant 8.$$

As to the upper bounds in [1] and [2] it is proved

$$(6)\qquad N_4(n) = O\left(\frac{n\log n}{\log\log n}\right)$$

and

$$(7_1)\qquad N_l(n) \leqslant ln(n-1)+1,$$

$$(7_2)\qquad N_l(n) < An\exp\{B\sqrt{\log n}\}$$

where A, B depend only on l.

DEFINITION 3. Put

$$k(n) = \min\{k\colon \exp_k(1) > n\},$$

where $\exp_1(x) = e^x$ and $\exp_k(x) = \exp\exp_{k-1}(x)$. We shall prove the following improvement of (6) and (7_2).

THEOREM. *$N_l(n) < Ank(n)$, where A depends only on l.*

Proof will be carried by induction with respect to n. Let n_0 satisfy the following inequalities

$$(8)\qquad \log_3(ln_0) > 2^{(10l)^3},$$

$$(9)\qquad (ln^2) \leqslant k(n)+1 \quad \text{for} \quad n > n_0$$

and let $A = ln_0$.

It follows from (7_1) that the theorem is true for $n \leqslant n_0$. Let $n > n_0$ and assume the theorem is true for every $n' < n$. Let a be an admissible n-sequence of length $N = Ak(n)n$. Assume now that a does not contain an alternating $(l+1)$-sequence and we shall arrive at a contradiction.

Put $K = Ak(n)$, $A_i = \{j < N,\ a_j = i \text{ for } i < n\}$.

LEMMA 1. *Let $|A_i|$ be the number of elements of A_i. Then for $i < n$*

$$(10)\qquad |A_i| > \tfrac{1}{2}K.$$

Proof. If we remove from a all terms equal a_i and eliminate all immediate repetitions we get an admissible sequence formed from $n-1$ distinct integers of length $t = N-|A_i|-r$ where r is the number of the immediate repetitions. Since a is admissible $r \leqslant |A_i|$ thus $t \geqslant N-2|A_i|$. By the inductive assumption $t < Ak(n-1)(n-1)$, hence $|A_i| > \frac{1}{2}K$.

LEMMA 2. *Let T be the class of all triples of elements of $\mathbf{N}$. Divide T into two disjoint classes T_1, T_2, $T_1\cap T_2 \neq \emptyset$, $T_1\cup T_2 = T$. Then there exists a set $D \subset \mathbf{N}$ with $|D| > \frac{1}{2}\log_3(N)$ such that either all triples of elements of D belong to T_1 or all triples of elements of D belong to T_2.*

Proof. This quantitative form of a special case of Ramsey's theorem follows from the estimation of Ramsey numbers given in [3].

LEMMA 3. *Let C be a set of integers and $|C| > \frac{1}{2}K$. Then there exists a subset $C' \subset C$ such that $|C'| \leqslant (K^{1/2}+1)\log K$ and $C\setminus C'$ is a disjoint union of sets of the form $\{t_0, \ldots, t_{q-1}\}$, where*

$$(11)\qquad t_0 < \ldots < t_{q-1} \quad \text{or} \quad t_0 > \ldots > t_{q-1},$$

$$(12)\qquad |t_1-t_0| > K^{1/2}, \quad |t_{i+2}-t_{i+1}| > 2\,|t_{i+1}-t_i| \quad \text{for} \quad i+2 < q$$

and

$$q = \left[\frac{1}{2}\,\frac{\log_4 K}{\log 2}\right].$$

Proof. C is the disjoint union of the sets

$$C_t = \{j\in C;\ j \equiv t \bmod [\sqrt{K}+1]\} \qquad (t = 0, 1, \ldots, [\sqrt{K}]).$$

Let $x, y, z \in C_t$, $x < y < z$. Then either

$$\text{(i)}\qquad |x-y| < |z-y|$$

or

$$\text{(ii)}\qquad |x-y| \geqslant |z-y|.$$

Let D be any subset of C_t with $|D| > \log K$. By Lemma 1 there is a subset $D' \subset D$ with $|D'| > \frac{1}{2}\log_3 K$ such that either all the triples of elements of D' satisfy (i) or all the triples of elements of D' satisfy (ii). We shall show that D' contains a subset of q elements satisfying (11) and (12). Let $D' = \{d_1, \ldots, d_{|D'|}\}$, $d_1 < \ldots < d_{|D'|}$. Notice that

$$2^q < \tfrac{1}{2}\log_3 K < |D'|$$

and put $t_i = d_{2^i}$ for $i = 0, 1, \ldots, q-1$. Suppose that all triples of elements of D' satisfy (i). Then for $i+2 < q$ we have

$$\begin{aligned} t_{i+2}-t_{i+1} &= d_{2^{i+2}}-d_{2^{i+1}} = \sum_{s=0}^{2^{i+1}-1} d_{2^{i+2}-s}-d_{2^{i+2}-s-1} \\ &> 2^{i+1}(d_{2^{i+1}+1}-d_{2^{i+1}}) > 2^{i+1}(d_{2^{i+1}}-d_{2^{i+1}-1}) \\ &> 2\sum_{s=0}^{2^i-1}(d_{2^{i+1}-s}-d_{2^{i+1}-s-1}) = 2(d_{2^{i+1}}-d_{2^i}) = 2(t_{i+1}-t_i). \end{aligned}$$

Since $t_0 \neq t_1$, and $t_0 \equiv t \equiv t_1 \pmod{[\sqrt{K}+1]}$ we have $t_1-t_0 > \sqrt{K}$. Thus the set $\{t_0, \ldots, t_{q-1}\}$ satisfies (11) and (12). If all triples of elements of D' satisfy (ii) the proof that D' contains a subset of q elements satisfying (11) and (12) is analogous. It follows that for each C_t there is $C'_t \subset C_t$ such that $|C'_t| \leqslant \log K$ and $C_t\setminus C'_t$ is the disjoint union of sets satisfying (11) and (12). Putting $C' = \bigcup_{t<[\sqrt{K}]+1} C'_t$ we see that C' has all the properties asserted in Lemma 3.

A_i satisfies the assumptions of Lemma 3, hence there exist disjoint sets D^i_ν, $\nu<\mu(i)$ of the form
$$D^i_\nu = \{t^{\nu i}_0, \ldots, t^{\nu i}_{q-1}\}$$
such that $|A_i - \bigcup_{\nu<\mu(i)} D^i_\nu| \leqslant (\sqrt{K}+1)\log K$ and D^i_ν satisfies (11) and (12). Put
$$E^i_\nu = \{t^{\nu i}_j;\ 10^{-3}q \leqslant j \leqslant (1-10^{-3})q\}.$$
It follows from (8) that E^i_ν is the inner part of D^i_ν.

For $\varrho \leqslant nK^{3/4}-1$ the intervals $I_\varrho = \{j;\ \varrho[K^{1/4}] \leqslant j < (\varrho+1)\times[K^{1/4}] \cap N\}$ have length $[K^{1/4}]$. Put
$$T=\{i\leqslant n;\ |A_i|\geqslant K^{3/2}\}.$$
Since the sets A_i are disjoint,
$$|T|K^{3/2} \leqslant \sum_{i\in T}|A_i| \leqslant \sum_{i<n}|A_i| = \Big|\bigcup_{i<n}A_i\Big| = N = Kn$$
hence

(13) $$|T| \leqslant K^{-1/3}\cdot n.$$

Lemma 4. *Let* $R = \{\varrho \leqslant n\cdot K^{3/4}-1;$ *there are* $j, s\in I_\varrho$ *such that* $a_j\neq a_s$; $a_j, a_s \in T\}$. *Then*

(14) $$|R| < 2n\cdot K^{1/2}.$$

Proof. Suppose that $|R|\geqslant 2n\cdot K^{1/2}$ and choose for each $\varrho\in R$ two elements $a_{j_\varrho}\neq a_{s_\varrho}$ where $j_\varrho, s_\varrho \in I_\varrho$, $j_\varrho<s_\varrho$; $a_{j_\varrho}, a_{s_\varrho}\in T$. In this way we get a subsequence $\langle a_{j_0}, a_{s_0}, a_{j_1}, a_{s_1}, \ldots, a_{j_\varrho}, a_{s_\varrho}, \ldots\rangle$ of the sequence a. Removing from this subsequence at most one element from each pair $a_{j_\varrho}, a_{s_\varrho}$ we get an admissible sequence $\tilde a$ of length $\tilde N \geqslant |R|/2 \geqslant n\cdot K^{1/2}$ which by (13) contains $\tilde n \leqslant n\cdot K^{-1/2}$ distinct integers. Since $\tilde N\geqslant n\cdot K^{1/2} \geqslant A\cdot k(K^{-1/2}\cdot n)K^{-1/2}\cdot\tilde n \geqslant A\cdot k(\tilde n)\tilde n$, by the inductive assumption $\tilde a$ contains an alternating $(l+1)$ sequence, contrary to the assumption that a does not.

Lemma 5. *Let*
$$V=\left\{\varrho\leqslant n\cdot K^{3/4}-1;\ \Big|\{i\in T;\ \bigcup_{\nu<\mu(i)} E^i_\nu\cap I_\varrho \neq \emptyset\}\Big| \geqslant \frac{K^{1/8}}{2\sqrt{l}}\right\}.$$
Then

(15) $$|V|\geqslant\tfrac13 n\cdot K^{3/4}.$$

Proof. Put
$$U=\Big\{\varrho\leqslant n\cdot K^{3/4}-1;\ \Big|I_\varrho\setminus\bigcup_{i<n}\bigcup_{\nu<\mu(i)}E^i_\nu\Big| < \tfrac14 K^{1/4}\Big\}.$$

Notice that
$$|E^i_\nu| \geqslant \left(1-\frac{2}{10^3}\right)q = \left(1-\frac{2}{10^3}\right)|D^i_\nu|,$$
$$\Big|\bigcup_{\nu<\mu(i)}E^i_\nu\Big| \geqslant \left(1-\frac{2}{10^3}\right)\Big|\bigcup_{\nu<\mu(i)}D^i_\nu\Big| \geqslant \left(1-\frac{2}{10^3}\right)\big(|A_i|-(K^{1/2}+1)\log K\big),$$
$$\Big|\bigcup_{i<n}\bigcup_{\nu<\mu(i)}E^i_\nu\Big| = \sum_{i<n}\Big|\bigcup_{\nu<\mu(i)}E^i_\nu\Big| \geqslant \left(1-\frac{2}{10^3}\right)\Big(\sum_{i<n}|A_i| - n\cdot(K^{1/2}+1)\log K\Big)$$
$$= \left(1-\frac{2}{10^3}\right)\big(N-n(K^{1/2}+1)\log K\big) \geqslant \frac{9}{10}n\cdot K.$$
If $\varrho\leqslant n\cdot K^{3/4}-1$ and $\varrho\notin U$ then I_ϱ contains $\geqslant\frac14K^{1/4}$ integers not belonging to $\bigcup\bigcup E^i_\nu$ thus N contains at least $\Big(\sum_{\substack{\varrho\notin U\\ \varrho\leqslant n\cdot K^{3/4}-1}} 1\Big)\frac14K^{1/4}$ integers not belonging to $\bigcup\bigcup E^i_\nu$. Therefore,
$$\Big(\sum_{\substack{\varrho\leqslant n\cdot K^{3/4}-1\\ \varrho\notin U}} 1\Big)\tfrac14 K^{1/4} \leqslant \tfrac1{10}nK.$$
Hence
$$|U| = \sum_{\varrho\leqslant n\cdot K^{3/4}-1}1 - \sum_{\substack{\varrho\leqslant n\cdot K^{3/4}-1\\ \varrho\notin U}}1 \geqslant n\cdot K^{3/4}-1-\tfrac25 n\cdot K^{3/4} = \tfrac35 n\cdot K^{3/4}-1$$
and $|U\setminus R| > \frac13 n\cdot K^{3/4}$. To complete the proof of Lemma 5 it is enough to show that $U\setminus R\subset V$. Let $\varrho\in U\setminus R$ and consider the sequence b obtained from $a|_{I_\varrho\cap\bigcup\bigcup E^i_\nu}$ ($a|_x$ is the restriction of the function a to the set x) by elimination of all immediate repetitions. b is admissible and has length
$$N^*\geqslant |I_\varrho| - 2\Big|I_\varrho - \bigcup\bigcup E^i_\nu\Big| \geqslant [K^{1/4}] - 2[\tfrac14K^{1/4}] \geqslant \tfrac12[K^{1/4}]$$
(see the proof of Lemma 1). As a subsequence of a, b does not contain an alternating $(l+1)$-sequence, thus by (7_1) the number of distinct integers in b satisfies the estimation
$$N^*\leqslant l(n^*-1)n^*+1$$
hence by (8) $n^* > \frac{K^{1/8}}{2\sqrt l}+1$. Since $\varrho\notin R$ at most one integer occurring in b belongs to T. Therefore, $\bigcup_{\nu<\mu(i)}E^i_\nu\cap I_\varrho\neq\emptyset$ for at least $n^*-1 > \frac{K^{1/8}}{2\sqrt l}$ different $i\notin T$. Thus $U-R\subset V$, q.e.d.

Put $\gamma=[10^{-3}q]$. Let $\varrho\in V$. We choose one element from each nonvoid set $\bigcup_{\nu<\mu(i)}E^i_\nu\cap I_\varrho$ where $i\notin T$ and obtain in this way a set $Z_\varrho\subset I_\varrho$ with

$|Z_\varrho| \geqslant \frac{K^{1/8}}{2\sqrt{l}}$ such that $a(Z_\varrho) \cap T = \emptyset$ and the function a is one-to-one on Z_ϱ.

Let $x \in Z_\varrho$. There exists one and only one ν such that $x \in E_\nu^{a_x}$. We have

$$D_\nu^{a_x} - E_\nu^{a_x} = \{t_0^{\nu, a_x}, \ldots, t_{\gamma-2}^{\nu, a_x}, t_{q-\gamma+1}^{\nu, a_x}, \ldots, t_{q-1}^{\nu, a_x}\}.$$

Put

$$C_x = \langle t_0^x, t_1^x, \ldots, t_{\gamma-1}^x \rangle = \begin{cases} \langle x, t_{q-\gamma+1}^{\nu, a_x} \ldots t_{q-1}^{\nu, a_x} \rangle \text{ if } t_0^{\nu, a_x} < \ldots < t_{q-1}^{\nu, a_x}, \\ \langle x \cdot t_{\gamma-2}^{\nu, a_x} \ldots t_0^{\nu, a_x} \rangle \text{ if } t_0^{\nu, a_x} > \ldots > t_{q-1}^{\nu, a_x}. \end{cases}$$

Thus the sequence C_x is increasing and satisfies (12). Besides $t_0^x = x$ and $\{t_0^x, \ldots, t_{\gamma-1}^x\} \subset A_{a_x}$.

For each $x \in Z_\varrho$ we define a new sequence $D_x = \langle r_0^x, \ldots, r_{l-2}^x \rangle$ where

$$(16) \qquad r_\nu^x = \max\{r;\ 2^r \leqslant t_{\nu+1}^x - t_0^x\} \quad \text{for} \quad \nu < l-1$$

(the definition is correct since by (8) $l \leqslant \gamma$).

Lemma 6. *The sequence D_x is increasing and has $|D_x| = l-1$ distinct elements. The mapping $x \to D_x$ is one-to-one on Z_ϱ.*

Proof. Let $x \in Z_\varrho$. We shall prove by induction that for $i+1 < \gamma$

$$(17) \qquad t_{i+1}^x - t_0^x \geqslant 2(t_i^x - t_0^x) + t_1^x - t_0^x.$$

For $i = 0$ (17) is obvious. Assume that (17) is true for certain i. Then, applying (12) we get

$$\begin{aligned} t_{i+2}^x - t_0^x &= (t_{i+2}^x - t_{i+1}^x) + (t_{i+1}^x - t_0^x) \geqslant (t_{i+2}^x - t_{i+1}^x) + 2(t_i^x - t_0^x) + t_1^x - t_0^x \\ &= 2(t_{i+1}^x - t_0^x) + t_1^x - t_0^x + (t_{i+2}^x - t_{i+1}^x) - 2(t_{i+1}^x - t_i^x) \\ &\geqslant 2(t_{i+1}^x - t_0^x) + t_1^x - t_0^x \end{aligned}$$

which completes the proof of (17). It follows that for $i < l-2$

$$2 \leqslant 2^{r_i^x} \leqslant t_{i+1}^x - t_0^x < 2^{r_i^x+1} \leqslant t_{i+2}^x - t_0^x$$

hence $r_i^x < r_{i+1}^x$ and $|D_x| = l-1$.

Suppose that $x, y \in Z_\varrho$, $x \neq y$ and $D_x = D_y$. Since $Z_\varrho \subset I_\varrho$ we have $|x-y| \leqslant K^{1/4}$. From (12) we get $t_1^y - t_0^y > \sqrt{K}$, hence

$$t_1^y > \sqrt{K} + t_0^y = \sqrt{K} + y \geqslant \sqrt{K} - K^{1/4} + x > x = t_0^x.$$

Put for $i < l-1$ $r_i = r_i^x = r_i^y$. It follows from (16) and (17) that

$$t_{i+2}^y - t_0^y \geqslant 2^{r_i+1} + \sqrt{K} > t_{i+1}^x - t_0^x + \sqrt{K} > t_{i+1}^x - t_0^x + (t_0^x - t_0^y)$$

hence

$$t_{i+2}^y > t_{i+1}^x.$$

Thus for all $i < l-1$ we have

$$t_{i+1}^y > t_i^x, \qquad t_{i+1}^x > t_i^y.$$

It follows that

$$\begin{aligned} t_0^x < t_1^y < t_2^x < \ldots < t_l^y \quad &\text{if} \quad l \text{ is odd}, \\ t_0^x < t_1^y < t_2^x < \ldots < t_l^x \quad &\text{if} \quad l \text{ is even}. \end{aligned}$$

Since $\{t_0^x, \ldots, t_{\gamma-1}^x\} \subset A_{a_x}$ we have in particular

$$a_x = a_{t_0^x} = a_{t_2^x} = \ldots, \qquad a_y = a_{t_1^y} = a_{t_3^y} = \ldots$$

For $x, y \in Z_\varrho$, $x \neq y$ implies $a_x \neq a_y$. Thus a contains an alternating $l+1$ sequence $\langle a_{t_0^x}, a_{t_1^y}, \ldots \rangle$ contrary to the assumption.

Corollary.

$$|\{x \in Z_\varrho;\ r_{l-2}^x \leqslant K^{1/8l}\}| < \frac{K^{1/8}}{4\sqrt{l}}.$$

Proof.

$$\begin{aligned} |\{x \in Z_\varrho;\ r_{l-2}^x \leqslant K^{1/8l}\}| &= \sum_{D_x, r_{l-2}^x \leqslant K^{1/8l}} 1 \\ &\leqslant |\{\langle a_0, \ldots, a_{l-2} \rangle;\ 1 \leqslant a_0 < \ldots < a_{l-2} \leqslant K^{1/8l}\}| \leqslant \binom{K^{1/8l}}{l-1} < \frac{K^{1/8}}{4\sqrt{l}}. \end{aligned}$$

Since $|Z_\varrho| \geqslant K^{1/8}/2\sqrt{l}$ we have

$$|\{x \in Z_\varrho;\ r_{l-2}^x > K^{1/8l}\}| > \frac{K^{1/8}}{4\sqrt{l}}$$

thus

$$|\{x \in Z_\varrho;\ t_{l-1}^x - t_0^x \geqslant 2^{K^{1/8l}}\}| > \frac{K^{1/8}}{4\sqrt{l}}.$$

Put

$$Z = \bigcup_{\varrho \in V} \{x \in Z_\varrho;\ t_{l-1}^x - t_0^x \geqslant 2^{K^{1/8l}}\}.$$

The sets Z_ϱ are disjoint since $Z_\varrho \subset I_\varrho$ hence

$$|Z| = \sum_{\varrho \in V} |\{x \in Z_\varrho;\ t_{l-1}^x - t_0^x \geqslant 2^{K^{1/8l}}\}| \geqslant |V| \frac{K^{1/8l}}{4\sqrt{l}} > \frac{1}{12\sqrt{l}} n \cdot K^{7/8}.$$

Put

$$Z' = \left\{x \in Z;\ 2^{(t_{j(l-1)}^x - t_0^x)^{1/2l}} < t_{(j+1)(l-1)}^x - t_0^x \text{ for all } j < 4(l-1)\right\}.$$

LEMMA 7. $|Z'| \geq \frac{1}{2}|Z|$.

Proof. Suppose that $|Z'| < \frac{1}{2}|Z|$. Let

$$S^j = \{x \in Z;\ 2^{(t^x_{j(l-1)} - t^x_0)^{1/2l}} \geq t^x_{(j+1)(l-1)} - t^x_0\}.$$

Then

$$\bigcup_{1 \leq j < 4(l-1)} S^j \supset Z \setminus Z'$$

hence

$$\max_{1 \leq j < 4(l-1)} |S^j| \geq \frac{1}{4l} \sum_{1 \leq j < 4(l-1)} |S^j| \geq \frac{1}{4l} \Big| \bigcup_{j=1}^{4l-4} S^j \Big| \geq \frac{1}{4l} |Z \setminus Z'|$$
$$= \frac{1}{4l}(|Z| - |Z'|) \geq \frac{1}{8l}|Z| \geq \frac{n \cdot K^{7/8}}{96 l \cdot \sqrt{l}} > n.$$

Thus there exists a positive integer $j < 4(l-1)$ such that $|S^j| > n$. Put

$$S^j_i = \{x \in S^j;\ 2^i \leq t^x_{j(l-1)} - t^x_0 < 2^{i+1}\}$$

for $i = 0, 1, \ldots$ Let $x \in S^j_i$; then $x \in Z$ and

$$2^{K^{1/8l}} \leq t^x_{l-1} - t^x_0 \leq t^x_{j(l-1)} - t^x_0 < 2^{i+1}$$

hence $i > K^{1/8l} - 1$. Thus

$$S^j = \bigcup_{i=0}^{\infty} S^j_i = \bigcup_{i \geq K^{1/8l}}^{\infty} S^j_i.$$

It follows that there exists $i > K^{1/8l} - 1$ such that

$$|S^j_i| \geq \frac{8n}{i^2}$$

since otherwise we had

$$|S^j| \leq \sum_{i > K^{1/8l} - 1} \frac{8n}{i^2} < \frac{8n}{K^{1/8l} - 2} < n.$$

For $\nu < n \cdot K/2^i$ let $J_\nu = \{t < N;\ 2^i \nu \leq t < 2^i(\nu+1)\}$. Put

$$M_\nu = \{t^x_{j(l-1)};\ t^x_{j(l-1)} \in J_\nu,\ x \in S^j_i\}.$$

Then $\bigcup_{\nu < n \cdot K/2^i} M_\nu = \{t^x_{j(l-1)};\ x \in S^j_i\}$ hence

$$\max_{\nu < n \cdot K/2^i} M_\nu > \frac{2^i}{n \cdot K + 2^i} \Big| \bigcup_{\nu < n \cdot K/2^i} M_\nu \Big| > \frac{2^{i-1}}{n \cdot K} |S^j_i| > \frac{2^{i+2}}{i^2 \cdot K} > \frac{2^{i+1}}{i^{8l+1}}.$$

Choose ν such that $|M_\nu| > \frac{2^{i+1}}{i^{8l+2}}$. Since $a(M_\nu) \subset a\big(\bigcup_{\varrho \in X} Z_\varrho\big)$ and $a(Z_\varrho) \cap T = \emptyset$ we have $a(M_\nu) \cap T = \emptyset$. It follows that for $m \in M_\nu$, $|A a_m| < K^{3/2}$. Choose one element from each set $M_\nu \cap A_i$ where $i \in a(M_\nu)$. In this way we get a set $M'_\nu \subset M_\nu$ such that $|M'_\nu| \geq |M_\nu|/K^{3/2}$. The function a is one-to-one on M'_ν.

For each $t^x_{j(l-1)} \in M'_\nu$ we define a sequence

$$\tilde{D}_x = \langle \tilde{r}^x_0, \ldots, \tilde{r}^x_{l-2} \rangle$$

where

$$\tilde{r}^x_s = \max\{r:\ 2^r \leq t^x_{j(l-1)+s+1} - t^x_{j(l-1)}\} \quad \text{for} \quad s < l-1.$$

This definition is correct since for $x \in Z$, $t^x_{j(l-1)}$ uniquely determines $t^x_{j(l-1)+s+1}$ and moreover by (8)

$$4(j+1)(l-1) \leq 4(l-1)^2 \leq \gamma - 1.$$

We prove like (17) that

(18) $$t^x_{j(l-1)+s+1} - t^x_{j(l-1)} \geq 2(t^x_{j(l-1)+s} - t^x_{j(l-1)}) + t^x_{j(l-1)+1} - t^x_{j(l-1)}.$$

It follows that for $s < l-2$

$$2 \leq 2^{\tilde{r}^x_s} \leq t^x_{j(l-1)+s+1} - t^x_{j(l-1)} < 2^{\tilde{r}^x_s + 1} \leq t^x_{j(l-1)+s+2} - t^x_{j(l-1)}$$

hence $\tilde{r}^x_s < \tilde{r}^x_{s+1}$ and $\tilde{D}_x$ has $l-1$ distinct elements. Suppose that $t^x_{j(l-1)}$, $t^y_{j(l-1)} \in M'_\nu$, $t^x_{j(l-1)} \neq t^y_{j(l-1)}$ and $\tilde{D}_x = \tilde{D}_y$. Since

$$M'_\nu \subset J_\nu$$

we have

$$|t^x_{j(l-1)} - t^y_{j(l-1)}| \leq 2^i.$$

From (17) and $x \in S^j_i$ we get

(19) $$t^x_{j(l-1)+1} - t^x_{j(l-1)} \geq t^x_{j(l-1)} - t^x_0 + t^x_1 > 2^i$$

hence

$$t^x_{j(l-1)+1} > 2^i + t^x_{j(l-1)} \geq t^y_{j(l-1)}.$$

Put for $s < l-1$ $\tilde{r}_s = \tilde{r}^x_s = \tilde{r}^y_s$. It follows from (18), (19) and the definition of $\tilde{r}_s$ that

$$t^x_{j(l-1)+s+2} - t^x_{j(l-1)} \geq 2^{r_s+1} + 2^i > t^y_{j(l-1)+s+1} - t^y_{j(l-1)} + t^y_{j(l-1)} - t^x_{j(l-1)}$$

hence

$$t^x_{j(l-1)+s+2} > t^y_{j(l-1)+s+1}.$$

Thus for all $s < l-1$ we have

$$t^y_{j(l-1)+s+1} > t^x_{j(l-1)+s}, \qquad t^x_{j(l-1)+s+1} > t^y_{j(l-1)+s}$$

and a contains an alternating $l+1$ sequence

$$\langle a^x_{t_{j(l-1)}}, a^y_{t_{j(l-1)+1}}, \ldots \rangle$$

contrary to the assumption (cf. the proof of Lemma 6). Thus the mapping $t^x_{j(l-1)} \to \tilde{D}_x$ is one-to-one on M'_ν. Let $\tilde{r}^x_{l-2}$ be the greatest among the last terms of $\tilde{D}_x$'s. It follows that $|M'_\nu| \leqslant \binom{\tilde{r}^x_{l-2}}{l-1}$ hence

$$\tilde{r}^x_{l-2} \geqslant |M'_\nu|^{1/l}.$$

Thus

$$t^x_{(j+1)(l-1)} - t^x_{j(l-1)} \geqslant 2^{\tilde{r}^x_{l-2}} > 2^{|M'_\nu|^{1/l}}.$$

On the other hand since $x \in S^j_i$ we have $2^{i+1} > t^x_{j(l-1)} - t^x_0$ thus

$$|M'_\nu| \geqslant \frac{|M_\nu|}{K^{3/2}} > \frac{2^{i+1}}{i^{8l+2}} > \frac{t^x_{j(l-1)} - t^x_0}{i^{8l+2} \cdot K^{3/2}}.$$

Hence

$$t^x_{(j+1)(l-1)} - t^x_{j(l-1)} > 2^{(t^x_{j(l-1)} - t^x_0)} \frac{1}{i^{8l+2} \cdot K^{3/2}} > 2^{(t^x_{j(l-1)} - t^x_0)^{1/2l}}$$

(for

$$\frac{(t^x_{j(l-1)} - t^x_0)^{1/2}}{i^{8l+2} \cdot K^{3/2}} \geqslant \frac{\sqrt{2}^{K^{1/8l}}}{i^{8l+2} \cdot K^{3/2}} > \frac{\sqrt{2}^{K^{1/8l}}}{10 K^{5/2}} > 1)$$

contrary to $x \in S^j$. This completes the proof of Lemma 7.

For each $x \in Z'$ put

$$P^x_j = t^x_{4j(l-1)}, \quad j = 0, \ldots, l-1.$$

Easy computations show that

$$2^{2^{(P^x_j - P^x_0)}} + 2^{K^{1/8l}} < P^x_{j+1} - P^x_0 \quad \text{for} \quad j < l-1. \tag{20}$$

Indeed for $j = 0$ (20) follows from $x \in Z$. Let $j \geqslant 1$. Since

$$t^x_{(4j+2)(l-1)} - t^x_0 > t^x_{4j(l-1)} - t^x_0 \geqslant t^x_{l-1} - t^x_0 > 2^{K^{1/8l}}$$

we have

$$2^{\frac{1}{2l}(t^x_{(4j+2)(l-1)} - t^x_0)^{1/2l}} > t^x_{(4j+2)(l-1)} - t^x_0,$$

$$2^{\frac{1}{2l}(t^x_{4j(l-1)} - t^x_0)^{1/2l}} > t^x_{4j(l-1)} - t^x_0 + K.$$

Hence by the definition of Z' we get

$$P^x_{j+1} - P^x_0 = t^x_{(4j+4)(l-1)} - t^x_0 > 2^{(t^x_{(4j+3)(l-1)} - t^x_0)^{1/2l}} > 2^{2^{\frac{1}{2l}(t^x_{(4j+2)l-1} - t^x_0)^{1/2l}}}$$
$$> 2^{t^x_{(4j+2)(l-1)} - t^x_0} > 2^{2^{(t^x_{(4j+1)(l-1)} - t^x_0)^{1/2l}}} > 2^{2^{2^{\frac{1}{2l}(t^x_{4j(l-1)} - t^x_0)^{1/2l}}}}$$
$$> 2^{2^{t^x_{4j(l-1)} - t^x_0 + K^{1/8l}}} = 2^{2^{P^x_j - P^x_0}} + 2^{K^{1/8l}}.$$

For each $x \in Z'$ we define a sequence

$$E_x = \langle h^x_0, \ldots, h^x_{l-2} \rangle$$

where $h^x_j = k(P^x_{j+1} - P^x_0)$ for $j < l-1$.

It follows from (20) that the sequence E_x is increasing. Notice that $a(Z') \cap T = \emptyset$. It follows that for $x \in Z'$, $|A_{a_x}| \leqslant K^{3/2}$. We choose one element from each set $Z' \cap A_i$, where $i \in a(Z')$. In this way we obtain a set $Z'' \subset Z'$ such that $|Z''| > |Z'| \cdot K^{-3/2}$. The function a is one-to-one on Z''. Put $I^*_\nu = \{t < N, 2^{K^{1/8l}}\nu \leqslant t < 2^{K^{1/8l}}(\nu+1)\}$ for $\nu < N/2^{K^{1/8l}}$. Since

$$\bigcup_{\nu < N/2^{K^{1/8l}}} I^*_\nu \cap Z'' = Z''$$

we have

$$\max_{\nu < N/2^{K^{1/8l}}} |I^*_\nu \cap Z''| \geqslant \frac{2^{K^{1/8l}}}{2N} |Z''| > \frac{2^{K^{1/8l}}}{4NK^{3/2}} |Z|$$
$$> \frac{2^{K^{1/8l}} n \cdot K^{7/8}}{48\sqrt{l} \cdot K^{3/2}} = \frac{2^{K^{1/8l}}}{48\sqrt{l} \cdot K^{13/8}}.$$

Choose ν so that $S = I^*_\nu \cap Z''$ satisfies

$$|S| > \frac{2^{K^{1/8l}}}{48\sqrt{l} \cdot K^{13/8}}.$$

LEMMA 8. *The mapping $x \to E_x$ is one-to-one on S.*

Proof. Suppose that $x, y \in S$, $x \neq y$ and $E_x = E_y$. Since $S \subset I^*_\nu$ we have $|x-y| \leqslant 2^{K^{1/8l}}$. Since $S \subset Z$, $t^y_{l-1} - t^y_0 > 2^{K^{1/8l}}$. Hence

$$P^y_1 = t^y_{4(l-1)} > (t^y_{l-1} - t^y_0) + t^y_0 > 2^{K^{1/8l}} + y \geqslant x = P^x_0.$$

Put for $j < l-1$; $h_j = h^x_j = h^y_j$. It follows from (20) and the definition of h_j that

$$P^y_{j+2} - P^y_0 > 2^{2^{\exp h_{j-1}(1)}} + 2^{K^{1/8l}} > \exp h^{(1)}_j + 2^{K^{1/8l}} > P^x_{j+1} - P^x_0 + (P^x_0 - P^y_0)$$

hence

$$P^y_{j+2} > P^x_{j+1}.$$

Thus for all $j < l$ we have

$$P^y_{j+1} > P^x_j, \quad P^x_{j+1} > P^y_j$$

and a contains an alternating $(l+1)$ sequence (cf. the proof of Lemma
contrary to the assumption.

Let x be that element of S for which h^x_{l-2} is the greatest. It follow from Lemma 9 that

$$\binom{h^x_{l-2}}{l-1} \geqslant |S| > \frac{2^{K^{1/8l}}}{48\sqrt{l}\cdot K^{13/8}}.$$

Hence

$$k(N) \geqslant k(P^x_{l-2}-P^x_0) = h^x_{l-2} > \left(\frac{2^{K^{1/8l}}}{48\sqrt{l}\cdot K^{13/8}}\right)^{1/l} > K+1 = A\cdot k(n)+1.$$

On the other hand by (7_1) we have $N < ln(n-1)+1 < ln^2$ thus by (

$$k(N) \leqslant k(ln^2) \leqslant k(n)+1 < A\cdot k(n)+1.$$

The contradiction obtained proves the theorem.

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