Beweis von Korollar 1. $\varphi(z)$ genügt der Funktionalgleichung $\varphi(mz) = (1+z)\varphi(z)$. Für die Entwicklung

$$\varphi(z) = \sum_{\mu=0}^{\infty} \frac{\varphi^{(\mu)}(0)}{\mu!} z^{\mu}$$

gilt

$$(m^{\mu}-1)\varphi^{(\mu)}(0) = \mu\varphi^{(\mu-1)}(0), \quad \mu = 1, 2, \ldots$$

Mit $\varphi(0) = 1$ folgt hieraus, daß die Entwicklungskoeffizienten aus K sind und somit die Voraussetzungen von Satz 2 erfüllt werden.

Beweis der Korollare 2 und 3. Die Beweise ergeben sich sofort aus Satz 2, falls man beachtet, daß f der Funktionalgleichung $f(2z) = zf(z)+z$ bzw. $f(mz) = (z+r)f(z)+z$ genügt.

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Metric theorems on the approximation of zero by a linear combination of polynomials with integral coefficients

by

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1. In 1964 V. G. Sprindžuk ([8], [9], [10], [1]) proved Mahler's conjecture: let n be a fixed integer; then for almost every real t there are only finitely many polynomials $P(x) = a_0 + a_1x + \ldots + a_nx^n$ with integral coefficients satisfying $|P(t)| < h_p^{-n-\varepsilon}$ where $\varepsilon > 0$ is any number, $h_p = \max(|a_0|, |a_1|, \ldots, |a_n|)$. The proof was based on the special properties of polynomials with integral coefficients. In the present paper a generalization of this problem is considered. This is the problem of an approximation to zero by the linear combinations $\lambda_1P_1(x_1) + \ldots + \lambda_mP_m(x_m)$ where $P_1(x_1), \ldots, P_m(x_m)$ are polynomials with integral coefficients, $\lambda_1 \neq 0, \ldots, \lambda_m \neq 0$ are real numbers, i.e. $\|\lambda_1P_1(x_1) + \ldots + \lambda_mP_m(x_m)\|$ is studied where $\|x\|$ is the distance from x to the nearest integer. In the proof Sprindžuk's method introduced in [7], [13] and Vinogradov's mean value theorem are used. Four theorems are proved. Theorems 1 and 3 are concerned with polynomials $P_i(x_i)$ for which $P_i(0) = 0$. Theorems 2 and 4 are concerned with those $P_i(x_i)$ for which $P_i(0) \neq 0$.

Theorem 1. *Let $\lambda_1 \neq 0, \ldots, \lambda_m \neq 0$ be real numbers, and let $k_1 \geq 1, \ldots, k_m \geq 1$ be integers, $K = \max(k_1, \ldots, k_m)$, $k = \min(k_1, \ldots, k_m)$. Let w_0 be the least upper bound of those $w > 0$ for which there are infinitely many m-tuples of polynomials $P_1(x_1), \ldots, P_m(x_m)$ with integral coefficients*

$$P_i(x_i) = \sum_{n=1}^{k_i} a_{in}x^n \quad (i = 1, 2, \ldots, m)$$

which satisfy

$$(1) \qquad \|\lambda_1P_1(t_1) + \ldots + \lambda_mP_m(t_m)\| < h^{-w}, \qquad h = \max_{\substack{1\leq n\leq k_i\\ 1\leq i\leq m}} (|a_{in}|) \neq 0$$

when $k_1, \ldots, k_m$ are fixed integers, and $t_1, \ldots, t_m$ are real numbers, $h \to \infty$. Then for almost every $(t_1, \ldots, t_m) \in R^m$

$$w_0 = k_1 + \ldots + k_m,$$

if

$$(2)\qquad m \geqslant \begin{cases} 2, & K = 1,2, \\ [2K(K-1)^2 \ln \frac{1}{2}K(K-1) + \frac{1}{2}K^2(K-1) - 2]/k, & K \geqslant 3. \end{cases}$$

THEOREM 2. *Let $k_1 \geqslant 1, \ldots, k_m \geqslant 1$ be integers, $K = \max(k_1, \ldots, k_m)$, $k = \min(k_1, \ldots, k_m)$. Suppose $\lambda_1 \neq 0, \ldots, \lambda_m \neq 0$ are real numbers for which there are only finitely many integers $a_1, \ldots, a_m$ with*

$$(3)\qquad \|\lambda_1 a_1 + \ldots + \lambda_m a_m\| < (a_1' \ldots a_m')^{-1-\gamma},$$

where $a_i' = \max(1, |a_i|)$ $(i = 1, 2, \ldots, m)$, γ is a fixed number and $0 < \gamma < (k_1 + \ldots + k_m)/m^2$. Let w_0 be the least upper bound of those $w > 0$ for which there are infinitely many m-tuples of polynomials $P_1(x_1), \ldots, P_m(x_m)$ with integral coefficients

$$P_i(x_i) = \sum_{n=0}^{k_i} a_{in} x^n$$

satisfying (1) *where $h = \max\limits_{\substack{0 \leqslant n \leqslant k_i \\ 1 \leqslant i \leqslant m}} (|a_{in}|)$. Then for almost every $(t_1, \ldots, t_m) \in R^m$*

$$w_0 = k_1 + \ldots + k_m + m,$$

if

$$(4)\qquad m \geqslant \begin{cases} 2, & K = 1,2, \\ [2K(K-1)^2 \ln \frac{1}{2}K(K-1) + \frac{1}{2}K^2(K-1) - 2]/(k+1), & K \geqslant 3. \end{cases}$$

The following theorems are equivalent to Theorems 1 and 2 in virtue of Khinchin's principle.

THEOREM 3. *Let $\lambda_1 \neq 0, \ldots, \lambda_m \neq 0$ be real numbers, and let $k_1 \geqslant 1, \ldots, k_m \geqslant 1$ be integers, $K = \max(k_1, \ldots, k_m)$, $k = \min(k_1, \ldots, k_m)$. Let v_1 be the least upper bound of those $v > 0$ for which there are infinitely many positive integral q with*

$$\max_{1 \leqslant i \leqslant m} (\|\lambda_i t_i q\|, \|\lambda_i t_i^2 q\|, \ldots, \|\lambda_i t_i^{k_i} q\|) < q^{-v}.$$

Then for almost every $(t_1, \ldots, t_m) \in R^m$

$$v_1 = 1/(k_1 + \ldots + k_m)$$

if m satisfies (2).

THEOREM 4. *Let $k_1 \geqslant 1, \ldots, k_m \geqslant 1$ be integers, $K = \max(k_1, \ldots, k_m)$, $k = \min(k_1, \ldots, k_m)$. Suppose that $\lambda_1, \ldots, \lambda_m$ are as in Theorem 2. Let v_2 be the least upper bound of those $v > 0$ for which there are infinitely many positive integral q with*

$$\max_{1 \leqslant i \leqslant m} (\|\lambda_i q\|, \|\lambda_i t_i q\|, \|\lambda_i t_i^2 q\|, \ldots, \|\lambda_i t_i^{k_i} q\|) < q^{-v}.$$

Then for almost every $(t_1, \ldots, t_m) \in R^m$

$$v_2 = 1/(k_1 + \ldots + k_m + m),$$

where m satisfies (4).

There are numbers satisfying (3), for example $\lambda_1, \ldots, \lambda_m$ being real algebraic numbers such that $1, \lambda_1, \ldots, \lambda_m$ are linearly independent over the field of rationals [6]. Almost all m-tuples $(\lambda_1, \ldots, \lambda_m) \in R^m$ are such. Theorems 1, 3 are valid for the polynomials $P_1(x_1), \ldots, P_m(x_m)$ such that $P_i(0) \neq 0$ $(i = 1, 2, \ldots, m)$ if $\lambda_1, \ldots, \lambda_m$ are integers or $\lambda_i = a_i/b_i$ are rationals with $b_i | a_{i0}$ $(i = 1, 2, \ldots, m)$.

The case where the number m of polynomials satisfies the condition

$$2 \leqslant m \leqslant [2K(K-1)^2 \ln \tfrac{1}{2}K(K-1) + \tfrac{1}{2}K(K^2 + K + 1) - 2]/k$$

is not investigated.

The problem under consideration is concerned with the metric theory of the Diophantine approximations to dependent values. This theory has been developed by J. P. Kubilius ([2], [3]), V. G. Sprindžuk [7]–[10] and W. M. Schmidt [5] and is elaborated by V. G. Sprindžuk [13], [12] for a wide class of dependent values.

2. LEMMA 1. *Let P, Q, k be integers, $P \geqslant 1$, $k \geqslant 3$,*

$$S = \sum_{Q < x \leqslant Q+P} e^{2\pi i (a_1 x^k + a_2 x^{k-1} + \ldots + a_{k+1})},$$

where $a_1, \ldots, a_{k+1}$ are real numbers. Suppose M, l are positive integers, and u is such a positive integer that

$$2u > \tfrac{1}{4}(k-1)k + l(k-1),$$

$$\delta_l = \tfrac{1}{2}(k-1)k\left(1 - \frac{1}{k-1}\right)^l.$$

Then

$$S \ll P^{1-\frac{1}{2u}} M^{-\frac{k-1-\delta_l}{4u}+\varepsilon} \left(M^{k-1}P + P \sum_{z=1}^{P} \min\left(M^{k-1}, \frac{1}{\|a_1 z\|}\right)\right)^{\frac{1}{2u}} + M,$$

where $\ll$ is the Vinogradov symbol.

Proof. This is Vinogradov's theorem [14] in the form given by K. Prachar [4].

LEMMA 2. *Let $P \geqslant 1$, P, Q being integers. Let $f(x) = ax^2 + a_1 x + a_2$ be a polynomial with real coefficients. Suppose*

$$S = \sum_{Q < n \leqslant Q+P} e^{2\pi i f(n)}.$$

Then

$$|S|^2 < 16\left\{P + \sum_{1\leqslant n\leqslant P} \min\left(P, \frac{1}{\|2\alpha n\|}\right)\right\},$$

where $\min(P, 0^{-1}) = P$.

Proof. See [4].

Lemma 3. *Suppose n, q, Q are positive integers, and $g_{ij}, r_i > 0$ are real numbers $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, Q)$. Let $N(q, Q)$ be the number of those j $(1 \leqslant j \leqslant Q)$ for which the numbers $g_{1j}, g_{2j}, \ldots, g_{nj}$ satisfy the inequalities*

$$\|g_{1j}\| \leqslant q^{-r_1}, \quad \ldots, \quad \|g_{nj}\| \leqslant q^{-r_n}$$

simultaneously. Then

$$N(q, Q) \leqslant q^{-r} \sum_{|c_1|<q^{r_1}} \cdots \sum_{|c_n|<q^{r_n}} \left|\sum_{j=1}^{Q} e^{2\pi i(c_1 g_{1j}+\ldots+c_n g_{nj})}\right|,$$

where $r = r_1 + \ldots + r_n$.

Proof. See Lemma 4 of Sprindžuk [13].

Lemma 4. *Let $\tau(a)$ be a number of divisors of a positive integer a. Then*

$$\tau(a) \leqslant a^{\varepsilon}.$$

Proof. See [14].

Lemma 5. *Let Q be a positive integer, $0 < \xi_i < 1$ $(i = 1, 2, \ldots, m)$. Suppose $\lambda_1 \neq 0, \ldots, \lambda_m \neq 0$ are real numbers for which there are only finitely many integers $a_1, \ldots, a_m$ with*

$$\|\lambda_1 a_1 + \ldots + \lambda_m a_m\| < (a_1' \ldots a_m')^{-1-\gamma},$$

where $a_i' = \max(1, |a_i|)$ $(i = 1, 2, \ldots, m)$, $\gamma > 0$ is a fixed number. Let $N_Q(\xi_1, \ldots, \xi_m)$ be the number of vectors $(\|\lambda_1 q\|, \ldots, \|\lambda_m q\|)$ falling into the set $[0, \xi_1) \times \ldots \times [0, \xi_m)$ when $q = 1, 2, \ldots, Q$. Then

$$N_Q(\xi_1, \ldots, \xi_m) = 2\xi_1 \ldots \xi_m Q + O(Q^{\frac{\gamma m}{\gamma m+1}+\varepsilon}).$$

Proof. The lemma is proved by Vinogradov's trigonometrical sums method [14].

3. We shall prove Theorems 3 and 4 simultaneously. We apply Sprindžuk's method [13]. Let E^m be a unit cube in the m-dimensional real space R^m. We consider the following inequality for points $(t_1, \ldots, t^m) \in E^m$

(5) $$\max_{1\leqslant i\leqslant m}(\|\lambda_i q\|, \|\lambda_i t_i q\|, \|\lambda_i t_i^2 q\|, \ldots, \|\lambda_i t_i^{k_i} q\|) < q^{-v_0-\varepsilon},$$

where $v_0 = v_2$ if (5) contains the values $\|\lambda_i q\|$ $(i = 1, 2, \ldots, m)$ and $v_0 = v_1$ if (5) does not. We fix the integer q for which (5) is valid. It follows from (5) that

(6) $$|t_i - a_i/\lambda_i q| < q^{-1-v_0-\varepsilon}/\lambda_i \quad (i = 1, 2, \ldots, m)$$

where a_i are integers. Since $(t_1, \ldots, t_m) \in E^m$,

(7) $$|a_i| \leqslant q \quad (i = 1, 2, \ldots, m).$$

Since $\lambda_i t_i^j = \lambda_i(a_i/\lambda_i q)^j + O(q^{-1-v_0-\varepsilon})$ and $\|q\lambda_i t_i^j\|$ satisfies (5) $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, k_i)$, a_i satisfying (6), (7) is the solution of the system

(8) $$\|q\lambda_i(a_i/\lambda_i q)^j\| \leqslant q^{-v_0-\varepsilon} \quad (j = 1, 2, \ldots, k_i;\ i = 1, 2, \ldots, m).$$

Consequently the measure of those $(t_1, \ldots, t_m) \in E^m$ for which (5) is valid under the fixed q is estimated by

(9) $$\leqslant q^{-m(1+v_0)-\varepsilon} \cdot N(q),$$

where $N(q)$ is the number of the integer solution $a_1, \ldots, a_m$ of system (8) satisfying (7).

Now we estimate $N(q)$. System (8) is broken up into m independent systems

$$\|a_i^2/\lambda_i q\| \leqslant q^{-v_0-\varepsilon},$$
$$\cdots\cdots\cdots\cdots$$
$$\|a_i^{k_i}/(\lambda_i q)^{k_i-1}\| \leqslant q^{-v_0-\varepsilon}$$

$(i = 1, 2, \ldots, m)$.

Let $N_i(q)$ be the number of solutions of the ith system; then

(10) $$N(q) = N_1(q) \ldots N_m(q).$$

Hence it is enough to estimate $N_i(q)$. We shall write k, a, λ instead of k_i, a_i, λ_i in the course of estimating $N_i(q)$. By Lemma 3

(11) $$N_i(q) \leqslant q^{-(k-1)v_0-\varepsilon} \sum_{\substack{c_2, \ldots, c_k \\ \max|c_i| \leqslant q^{v_0}}} \left|\sum_{0\leqslant a\leqslant q} e^{2\pi i\left(c_2 \frac{a^2}{\lambda q} + \ldots + c_k \frac{a^k}{(\lambda q)^{k-1}}\right)}\right|.$$

We split the internal sum in (11) into special sums as follows: a sum in (11) in which $c_k \neq 0$ is denoted by Σ_k, a sum in (11) in which $c_k = 0$ and $c_{k-1} \neq 0$ is denoted by Σ_{k-1}, and so on. Then

(12) $$N_i(q) \leqslant q^{1-(k-1)v_0-\varepsilon} + \Sigma_2 + \ldots + \Sigma_k.$$

We apply Weyl's theorem to estimate Σ_2 and Vinogradov's theorem to estimate the remaining sums. By Lemma 2

$$\Sigma_2 \leqslant q^{-(k-1)v_0-\varepsilon} \sum_{1\leqslant c_2\leqslant q^{v_0}} \left|\sum_{0\leqslant a\leqslant q} e^{2\pi i c_2 \frac{a^2}{\lambda q}}\right|$$
$$\leqslant q^{-(k-1)v_0+v_0+1/2-\varepsilon} + q^{-(k-1)v_0-\varepsilon} \sum_{c_2} \left(\sum_{1\leqslant z\leqslant q} \min\left(q, \frac{1}{\|2c_2 z/\lambda q\|}\right)\right)^{1/2}.$$

Applying Lemma 4 and the Cauchy–Bunjakovski inequality, we obtain

$$(13)\qquad \Sigma_2 \ll q^{1-(k-1)v_0+v_0-1/2-\varepsilon}+q^{-(k-1)v_0+v_0/2-\varepsilon}\Big(\sum_{1\leqslant z\leqslant 2q^{1+v_0}} \min\Big(q, \frac{1}{\|z/\lambda q\|}\Big)\Big)^{1/2}.$$

Next we estimate the inner sum in (13). The interval of values of z is split into subintervals $I_0, I_1, \ldots, I_n, \ldots, I_q$ so that if $z\in I_n$ then $n < z/\lambda q \leqslant n+1$. The number of these subintervals equals $O(q^{v_0})$. We estimate the inner sum in (13) as follows:

$$\sum_{\lambda qn<z\leqslant\lambda q(n+\frac12)} \min\Big(q, \frac{1}{z/\lambda q-n}\Big)+\sum_{\lambda q(n+\frac12)<z\leqslant\lambda q(n+1)} \min\Big(q, \frac{1}{n+1-z/\lambda q}\Big) \ll q\ln q.$$

Consequently it is valid

$$\sum_z \ll q^{1+v_0}\ln q$$

for the inner sum in (13). Then

$$(14)\qquad \Sigma_2 \ll q^{1-v_0(k-1)-\varepsilon}$$

since $v_0\leqslant\frac12$.

For each r with $r=3,4,\ldots,k$ we estimate Σ_r in (12). We present this sum in the form

$$(15)\qquad \Sigma_r = \sum_{\substack{c_r\neq 0\\ c_2=\ldots=c_{r-1}=0}} + \sum_{\substack{c_r\neq 0,\, c_l\neq 0\\ c_j=0\\ (2\leqslant j\leqslant r-1,\, j\neq l)}} + \ldots + \sum_{\substack{c_l\neq 0\\ (2\leqslant l\leqslant r)}}.$$

We consider the second sum. We do the same with the remaining sums. By Lemma 1

$$S_1 = q^{-(k-1)v_0}\sum_{\substack{c_r\neq 0,\, c_l\neq 0\\ c_j=0\\ (2\leqslant j\leqslant r-1,\, j\neq l)}}\Big|\sum_{0\leqslant a\leqslant q} e^{2\pi i\left(c_r\frac{a^r}{(\lambda q)^{r-1}}+c_l\frac{a^l}{(\lambda q)^{l-1}}\right)}\Big|$$

$$\ll q^{-(k-1)v_0}\sum_{\substack{1\leqslant c_r\leqslant q^{v_0}\\ 1\leqslant c_l\leqslant q^{v_0}}}\Big[q^{1-\frac{1}{4u_r}}\cdot M_r^{\frac{\delta_{l(r)}}{4u_r}+\varepsilon}+$$

$$+q^{1-\frac{1}{4u_r}}\cdot M_r^{-\frac{r-1-\delta_{l(r)}}{4u_r}}\Big(\sum_{1\leqslant z\leqslant q}\min\Big(M_r^{r-1}, \frac{1}{\|c_r z/(\lambda q)^{r-1}\|}\Big)\Big)^{\frac{1}{4u_r}}+M_r\Big],$$

where the parameters $M_r, u_r, \delta_{l(r)}$ denote M, u, δ for the sum Σ_r and they are not chosen yet. Suppose $l(r)=l$ for brevity. Applying Lemma 4

and Hölder's inequality to S_1 we obtain

$$S_1 \ll q^{-(k-3)v_0}\Big[M_r+q^{1-\frac{1}{4u_r}}M_r^{\frac{\delta_l}{4u_r}+\varepsilon}+$$

$$+q^{1-\frac{1+v_0}{4u_r}+\varepsilon}M_r^{-\frac{r-1-\delta_l}{4u_r}}\Big(\sum_{1\leqslant z\leqslant q^{1+v_0}}\min\Big(M_r^{r-1}, \frac{1}{\|z/(\lambda q)^{r-1}\|}\Big)\Big)^{\frac{1}{4u}}\Big].$$

Since $1\leqslant z\leqslant q^{1+v_0}$, we have $|z/(\lambda q)^{r-1}|\leqslant 1/|\lambda|^{r-1}\cdot q^{r-2-v_0}\leqslant\frac12$. If $r\geqslant 3$ and $q\geqslant(2/|\lambda|^{r-1})^{1/(1-v_0)}$ then $1/|\lambda|^{r-1}q^{r-2-v_0}\leqslant\frac12$. Since $3\leqslant r\leqslant k$, we consider

$$(16)\qquad q\geqslant\begin{cases}(2/|\lambda|^{k-1})^{1/(1-v_0)}=B_1 & \text{if}\quad |\lambda|<1,\\ (2/|\lambda|^2)^{1/(1-v_0)}=B_2 & \text{if}\quad |\lambda|\geqslant 1.\end{cases}$$

We demand that

$$(17)\qquad (|\lambda|q)^{r-1}/z < M_r^{r-1}.$$

This is valid for $(|\lambda|q/M_r)^{r-1}<z\leqslant q^{1+v_0}$. Then

$$\sum_{1\leqslant z\leqslant q^{1+v_0}}\min\Big(M_r^{r-1}, \frac{1}{\|z/(\lambda q)^{r-1}\|}\Big)=\sum_{1\leqslant z\leqslant q^{1+v_0}}\min\Big(M_r^{r-1}, \frac{(|\lambda|q)^{r-1}}{z}\Big)$$

$$=\sum_{1\leqslant z\leqslant(|\lambda|q/M_r)^{r-1}}M_r^{r-1}+\sum_{(|\lambda|q/M_r)^{r-1}<z\leqslant q^{1+v_0}}\frac{(|\lambda|q)^{r-1}}{z}\ll q^{r-1}\ln q.$$

Consequently

$$S_1\ll q^{-(k-3)v_0}[M_r+q^{1-\frac{1}{4u_r}}M_r^{\frac{\delta_l}{4u_r}+\varepsilon}+q^{1+\frac{r-2-v_0}{4u_r}+\varepsilon}M_r^{-\frac{r-1-\delta_l}{4u_r}}].$$

The least summand of (15) is estimated as

$$\sum_{1\leqslant c_2,\ldots,c_r\leqslant q^{v_0}} \ll q^{-(k-r)v_0}[M_r+q^{1-\frac{1}{4u_r}}M_r^{\frac{\delta_l}{4u_r}+\varepsilon}+q^{1+\frac{r-2-v_r}{4u_r}}M_r^{-\frac{r-1-\delta_l}{4u_r}}].$$

Thus

$$(18)\qquad \Sigma_r\ll q^{-(k-r)v_0}[M_r+q^{1-\frac{1}{4u_r}}M_r^{\frac{\delta_l}{4u_r}+\varepsilon}+q^{1+\frac{r-2-v_0}{4u_r}+\varepsilon}M_r^{-\frac{r-1-\delta_l}{4u_r}}]$$

for $3\leqslant r\leqslant k$.

It follows from (12), (14), (18) that

$$N_i(q)\ll q^{1-(k-1)v_0-\varepsilon}+$$

$$+\sum_{3\leqslant r\leqslant k}q^{-(k-r)v_0}[M_r+q^{1-\frac{1}{4u_r}}M_r^{\frac{\delta_l}{4u_r}}+q^{1+\frac{r-2-v_0}{4u_r}}M_r^{-\frac{r-1-\delta_l}{4u_r}}].$$

The values of parameters u_r, M_r, δ_l $(3 \leqslant r \leqslant k)$ are chosen by comparing values depending on them with the first summand in the preceding inequality assuming (17). We have

$$M_r = q^{1-(r-1)v_0}, \quad u_r = \tfrac{1}{4}r(r-1)+l(r-1)+1, \quad \delta_l = \tfrac{1}{2}(r-1)r\left(1-\frac{1}{r-1}\right)^l$$

where $l = l(r) = \theta(r-1)\ln\frac{1}{2}r(r-1)+1$, $1 < \theta < 2$ $(3 \leqslant r \leqslant k)$. This is possible provided that

$$m \geqslant [2K(K-1)^2\ln\tfrac{1}{2}K(K-1)+\tfrac{1}{2}K^2(K-1)-2]/k,$$

where $K = \max(k_1, \ldots, k_m)$, $k = \min(k_1, \ldots, k_m)$. Thus

$$N(q) \ll q^{1-(k_i-1)v_0-\varepsilon} \quad (i = 1, 2, \ldots, m).$$

It follows from (10) that

$$N(q) \ll q^{m-v_0(k_1+\ldots+k_m-m)-\varepsilon}.$$

We return to (9). The preceding implies that the measure of those $(t_1, \ldots, t_m) \in E^m$ for which (5) holds when q is fixed is estimated as

$$\ll q^{-v_0(k_1+\ldots+k_m)-\varepsilon}.$$

If $P_1(0) = \ldots = P_m(0) = 0$ then $v_0 = 1/(k_1+\ldots+k_m)$. Now we apply the Borel–Cantelly Lemma. Then the corresponding series

$$\sum_{q=B}^{\infty} q^{-v_0(k_1+\ldots+k_m)-\varepsilon}$$

converges where

$$B = \begin{cases} B_1 & \text{if} \quad |\lambda| < 1, \\ B_2 & \text{if} \quad |\lambda| \geqslant 1, \end{cases}$$

B_1 and B_2 being determined from (16). Theorem 3 is proved for the points $(t_1, \ldots, t_m) \in E^m$. The property of the enumerable additivity of the Lebesgue measure implies that Theorem 3 holds for almost every $(t_1, \ldots, t_m) \in R^m$.

If $P_1(0) \neq 0, \ldots, P_m(0) \neq 0$ then $v_0 = 1/(k_1+\ldots+k_m+m)$, and applying the Borel–Cantelly Lemma, we must sum over those integers q for which (5) holds, i.e.

(19) $$\max(\|\lambda_1 q\|, \ldots, \|\lambda_m q\|) < q^{-v_0-\varepsilon}.$$

Let A be a set of those $q > 0$ for which (19) is valid. Then the corresponding series is

(20) $$\sum_{q \in A} q^{-v_0(k_1+\ldots+k_m)-\varepsilon} = \sum_{q \in A} q^{-(1-mv_0)-\varepsilon}.$$

We find the conditions of the convergence of this series. We consider its partial sum

$$S(Q) = \sum_{\substack{q=B \\ q \in A}}^{Q} q^{-(1-mv_0)-\varepsilon}, \quad B < Q.$$

In order to estimate $S(Q)$ Sprindžuk's method for calculating the solutions of the Diophantine inequalities ([11]) is used. The segment $[B, Q]$ is split into the partial segments

$$[B, Q] = \sum_{k=B^{1/\beta}}^{k_0-1} [k^\beta, (k+1)^\beta] + [k_0^\beta, Q],$$

where k runs through integers,

(21) $$\beta > 1, \quad k_0^\beta \leqslant Q < (k_0+1)^\beta.$$

Then

(22) $$S(Q) = \sum_{k=B^{1/\beta}}^{k_0-1} \Big(\sum_{\substack{k^\beta \leqslant q < (k+1)^\beta \\ q \in A}} q^{-(1-mv_0)-\varepsilon} \Big) + \sum_{\substack{k_0^\beta \leqslant q \leqslant Q \\ q \in A}} q^{-(1-mv_0)-\varepsilon}.$$

The sum is estimated on every partial segment as follows. The common member of the sum satisfies

$$q^{-(1-mv_0)-\varepsilon} \leqslant k^{-\beta(1-mv_0)-\varepsilon}.$$

Let $N(k)$ be the number of those $q \in [k^\beta, (k+1)^\beta]$ for which (19) holds. Then

(23) $$\sum_{\substack{k^\beta \leqslant q < (k+1)^\beta \\ q \in A}} q^{-(1-mv_0)-\varepsilon} \leqslant N(k)\, k^{-\beta(1-mv_0)-\varepsilon}.$$

Now we estimate $N(k)$. It is equal to the number of the solutions of the system

$$\begin{cases} \|\lambda_1 q\| < q^{-v_0-\varepsilon}, \\ \cdots\cdots\cdots \\ \|\lambda_m q\| < q^{-v_0-\varepsilon}, \end{cases} \quad k^\beta \leqslant q < (k+1)^\beta.$$

We introduce two numbers: let $N_1(k)$ be the number of the solutions of the system

$$\begin{cases} \|\lambda_1 q\| < (k^\beta)^{-v_0-\varepsilon}, \\ \cdots\cdots\cdots \\ \|\lambda_m q\| < (k^\beta)^{-v_0-\varepsilon}, \end{cases} \quad k^\beta \leqslant q < (k+1)^\beta$$

and let $N_2(k)$ be the number of the solutions of the system

$$\begin{cases} \|\lambda_1 q\| < (k+1)^{\beta(-v_0-\varepsilon)}, \\ \cdots\cdots\cdots \\ \|\lambda_m q\| < (k+1)^{\beta(-v_0-\varepsilon)}, \end{cases} \quad k^\beta \leqslant q < (k+1)^\beta.$$

Then $N_2(k) \leqslant N(k) \leqslant N_1(k)$. Consequently

$$N(k) = N_1(k) - \theta\big(N_1(k) - N_2(k)\big). \tag{24}$$

By Lemma 5

$$N_1(k) = 2(k^\beta)^{-mv_0-\varepsilon}[(k+1)^\beta - k^\beta] + O\Big([(k+1)^\beta - k^\beta]^{\frac{\gamma m}{\gamma m+1}+\varepsilon}\Big) \tag{25}$$
$$\ll (k^\beta)^{-mv_0-\varepsilon} k^{\beta-1} + k^{(\beta-1)\frac{\gamma m}{\gamma m+1}+\varepsilon},$$

$$N_1(k) - N_2(k) \tag{26}$$
$$= [(k^\beta)^{-v_0-\varepsilon} - (k+1)^{\beta(-v_0-\varepsilon)}]^m [(k+1)^\beta - k^\beta] + O\Big([(k+1)^\beta - k^\beta]^{\frac{\gamma m}{\gamma m+1}+\varepsilon}\Big)$$
$$\ll k^{-(\beta v_0+1+\varepsilon)m} k^{\beta-1} + k^{(\beta-1)\frac{\gamma m}{\gamma m+1}+\varepsilon} = k^{-\beta v_0 m - m+\beta-1-\varepsilon} + k^{(\beta-1)\frac{\gamma m}{\gamma m+1}+\varepsilon}.$$

It follows from (24), (25), (26) that

$$N(k) \ll k^{-\beta(mv_0-1)-1-\varepsilon} + k^{(\beta-1)\frac{\gamma m}{\gamma m+1}+\varepsilon}. \tag{27}$$

It follows from (22), (23), (27) that

$$S(Q) \ll \sum_{k=B^{1/\beta}}^{k_0-1} [k^{-\beta(1-mv_0)-\varepsilon}(k^{-\beta(mv_0-1)-1-\varepsilon} + k^{\frac{(\beta-1)\gamma m}{\gamma m+1}+\varepsilon})] +$$
$$+ k_0^{-\beta(1-mv_0)-\varepsilon}(k_0^{-\beta mv_0-\varepsilon} k_0^{\beta-1} + k_0^{\frac{(\beta-1)\gamma m}{\gamma m+1}+\varepsilon})$$
$$\ll \int_{B^{1/\beta}}^{k_0} (x^{-1-\varepsilon} + x^{-\beta(1-mv_0)+\frac{(\beta-1)\gamma m}{\gamma m+1}+\varepsilon})\,dx$$
$$\ll k_0^{-\varepsilon} + k_0^{1-\beta(1-mv_0)+(\beta-1)\frac{\gamma m}{\gamma m+1}+\varepsilon}.$$

if

$$[1 - mv_0 - \gamma m/(\gamma m+1)] > 0. \tag{28}$$

In conjunction with (21) we have

$$S(Q) \ll Q^{\frac{1}{\beta}\left(1-\beta(1-mv_0)+\frac{(\beta-1)\gamma m}{\gamma m+1}+\varepsilon\right)}.$$

To make the series (20) convergent we demand that an exponent of the preceding estimation be equal to zero, i.e.

$$1 - \beta(1-mv_0) + \frac{(\beta-1)\gamma m}{\gamma m+1} + \varepsilon = 0,$$
$$\beta\left(1 - mv_0 - \frac{\gamma m}{\gamma m+1}\right) = 1 - \frac{\gamma m}{\gamma m+1} + \varepsilon,$$

$$\beta = \frac{1 - \frac{\gamma m}{\gamma m+1} + \varepsilon}{1 - mv_0 - \frac{\gamma m}{\gamma m+1}} = \frac{1+\varepsilon}{(1-mv_0)(\gamma m+1) - \gamma m}.$$

Since $\beta > 1$, we are to demand that

$$\frac{1+\varepsilon}{(1-mv_0)(\gamma m+1) - \gamma m} > 1,$$

i.e.

$$0 < (1-mv_0)(\gamma m+1) - \gamma m < 1+\varepsilon.$$

The condition

$$(1-mv_0)(\gamma m+1) - \gamma m < 1+\varepsilon$$

is valid. The second condition

$$(1-mv_0)(\gamma m+1) - \gamma m > 0$$

is equivalent to (28) and it implies that

$$\gamma < \frac{1-mv_0}{m^2 v_0} = \frac{k_1 + \ldots + k_m}{m^2}.$$

Thus Theorem 4 is proved for points $(t_1, \ldots, t_m) \in E^m$. By the property of the enumerable additivity of the measure in R^m, Theorem 4 holds for almost all $(t_1, \ldots, t_m) \in R^m$.

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Investigations in the powersum theory II

by

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To the memory of L. J. Mordell

1. The second named author, partly in collaboration with S. Knapowski based a number of applications on the following theorem. Let be $b_1, \ldots, b_n$ complex numbers, m nonnegative integer, further

$$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n| > 0 \tag{1.1}$$

and

$$g(\nu) = \sum_{j=1}^{n} b_j z_j^{\nu}. \tag{1.2}$$

Then the theorem in question asserts the existence of an integer ν_0 satisfying

$$m+1 \leqslant \nu_0 \leqslant m+n \tag{1.3}$$

for which the inequality

$$|g(\nu_0)| \geqslant \left(\frac{n}{8e(m+n)}\right)^n \min_{l=1,\ldots,n} |b_1 + \ldots + b_l| \tag{1.4}$$

holds.

In (1.1) the fact that $|z_1| = 1$ is of course only a normalization. It means "essentially" that $|g(\nu)|$ is estimated for a proper choice of ν in a "narrow" interval from below by its maximal term (which explains its applicability).

In the case when the b_j numbers are in a half-plane then the application of the above theorem goes smoothly; this holds of course in the important case when all b_j's are 1. In the first paper of this series(1) (quoted as I in the sequel) we have seen how one can reduce very considerably the range of l in (1.4) which helps a lot in the applications since it permits to replace the last inconvenient factor essentially by $\left|\sum_{j=1}^{n} b_j\right|$. In many

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