Corollary. *The set $\Gamma(D_\mu)$ has Lebesgue measure* 0.

Proof. This follows immediately from Theorems 1 and 2.

5. Some closing remarks. It is not hard to show D_g always has the cardinality of the continuum. It is also easy to show that $D_\mu \subset D_\tau$ properly. In fact in some ways it seems to be a very small subset.

These theorems would seem to have much possibility for generalization. For instance they suggest similar theorems for invariant means or perhaps even the possibility that the "number" of Lebesgue measurable sets is very small as compared to the class of all subsets of the reals.

References

[1] E. Borel, *Les probabilitiés dénombrables et leurs applications arithmétiques*, Rend. Circ. Mat. Palermo 27 (1909), pp. 247–271.

[2] R. C. Buck, *The measure theoretic approach to density*, Amer. J. Math. 68 (1946), pp. 560–580.

[3] R. Bumby and E. Ellentuck, *Finitely additive measures and the first digit problem*, Fund. Math. 65 (1969), pp. 33–42.

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An extension of Schur's theorem on sum-free partitions

by

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1. Introduction. A set $\mathscr{S}$ of integers is said to be *sum-free* if

$$a \in \mathscr{S},\ b \in \mathscr{S} \Rightarrow a+b \notin \mathscr{S}.$$

a and b need not be distinct.

The following is a well-known theorem of Schur [5]:

Theorem (Schur). *Given a positive integer k, there exists a greatest positive integer $N = N(k)$ with the property that the set $\{1, 2, \ldots, N\}$ can be partitioned into k sum-free sets. Further,*

$$\tfrac{1}{2}(3^k-1) \leqslant N(k) \leqslant [k!\, e]-1 \tag{1}$$

where $[x]$ denotes the greatest integer not exceeding x.

The upper bound in (1) has recently been improved slightly by Whitehead [8] whose results show that

$$N(k) \leqslant [k!\,(e-\tfrac{1}{24})]-1.$$

Abbott and Hanson [1] have recently proved

$$N(k) \geqslant c\cdot 89^{\frac{1}{4}k}$$

for some absolute constant c, so improving an earlier result of Abbott and Moser [2].

A natural extension of the concept of a sum-free set is contained in the following definition:

A set $\mathscr{S}$ of integers is said to be *r-sum-free* if

$$a_1, a_2, \ldots, a_r \in \mathscr{S} \Rightarrow a_1+a_2+\ldots+a_r \notin \mathscr{S},$$

where the a_i need not be distinct.

It follows from results of Rado ([4], Theorems 3 and 4), that, given positive integers k and r, $r \geqslant 2$, there exists a greatest positive integer $N = N(r, k)$ with the property that the set $\{1, 2, \ldots, N\}$ can be partitioned into k r-sum-free sets. Clearly $N(2, k) = N(k)$.

Znám [9] gave a lower bound for $N(r, k)$ generalizing that of (1), viz.

$$N(r,k) \geqslant \left(\frac{r-1}{r}\right)\{(r+1)^k-1\}. \tag{2}$$

Further, implicit in [9] is the upper bound $N(r,k) \leqslant R(r,k)-2$, where $R(r,k)$ is the Ramsey number that is the smallest integer n such that in any colouring of the edges of the complete graph on n vertices, K_n, using k colours, some subgraph K_{r+1} has all of its edges the same colour. Hence,

$$N(r,k) \leqslant (kr)!/(r!)^k-2 \tag{3}$$

using a well-known result of Greenwood and Gleason [3].

Znám also showed [10] that equality holds in (2) in the case $k = 2$.

The main result of this paper is an upper bound for $N(r, k)$ which is a generalization of that in (1) and a considerable improvement upon that of (3).

2. The main result.

THEOREM 1.

$$N(r,k) \leqslant \left[k!(r-1)^k \exp\left(\frac{1}{r-1}\right)\right]-1.$$

Most of the proof of Theorem 1 is contained in the following lemma.

LEMMA 1. *Let k and r be positive integers, $r \geqslant 2$, and let*

$$N = \left[k!\,(r-1)^k \exp\left(\frac{1}{r-1}\right)\right].$$

If $a_0 < a_1 < \ldots < a_N$ is a sequence of non-negative integers and if the set of differences $a_j - a_i$ $(0 \leqslant i < j \leqslant N)$ is partitioned in any way into k classes, then at least one class contains a sequence of differences of the form $a_{i_r} - a_{i_{r-1}}$, $a_{i_{r-1}} - a_{i_{r-2}}, \ldots, a_{i_1} - a_{i_0}$, for some $i_0, i_1, \ldots, i_r$, with $0 \leqslant i_0 < i_1 < \ldots < i_r \leqslant N$, together with $a_{i_r} - a_{i_0}$.

Proof. Let M be a positive integer such that there exists a sequence $a_0 < a_1 < \ldots < a_M$ of non-negative integers for which the statement of the lemma is false. We shall say that such a sequence has property $\mathscr{P}$. It is sufficient to show $M < N$.

Let $a_0 < a_1 < \ldots < a_M$ be a sequence of non-negative integers having property $\mathscr{P}$, let $\mathscr{D} = \{a_j - a_i\colon 0 \leqslant i < j \leqslant M\}$ and let $\mathscr{D} = \mathscr{Z}_1 \cup \mathscr{Z}_2 \cup \ldots \cup \mathscr{Z}_k$ be a partition of the required kind.

Consider the set of integers $a_i - a_0$ $(1 \leqslant i \leqslant M)$. Choose a class of the partition, say $\mathscr{Z}_1$, that contains as many as possible, say n_1, of these

integers. Then

$$kn_1 \geqslant M$$

by the pigeon hole principle. Denote the integers of this type in $\mathscr{Z}_1$ by $b_i - a_0$ $(i = 1, 2, \ldots, n_1)$ where $b_i < b_j$ $(1 \leqslant i < j \leqslant n_1)$.

We now partition the set $\mathscr{B} = \{b_i\colon 1 \leqslant i \leqslant n_1\}$ into $r-1$ subsets

$$\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \ldots \cup \mathscr{B}_{r-1}$$

according to the following rules.

(i) $b_i \in \mathscr{B}_{r-1}$ if and only if there exist integers $j_1, j_2, \ldots, j_{r-2}$, where $1 \leqslant j_1 < j_2 < \ldots < j_{r-2} < i$, such that

$$\begin{aligned} &b_i - b_{j_{r-2}} \in \mathscr{Z}_1,\\ &b_{j_{r-2}} - b_{j_{r-3}} \in \mathscr{Z}_1,\\ &\ldots\ldots\ldots\\ &b_{j_2} - b_{j_1} \in \mathscr{Z}_1. \end{aligned}$$

Then successively for $h = r-2, r-3, \ldots, 3, 2$:

(ii) $b_i \in \mathscr{B}_h$ if and only if $b_i \notin \bigcup_{s=h+1}^{r-1} \mathscr{B}_s$ and there exist integers $j_1, j_2, \ldots, j_{h-1}$, where $1 \leqslant j_1 < j_2 < \ldots < j_{h-1} < i$, such that

$$\begin{aligned} &b_i - b_{j_{h-1}} \in \mathscr{Z}_1,\\ &b_{j_{h-1}} - b_{j_{h-2}} \in \mathscr{Z}_1,\\ &\ldots\ldots\ldots\\ &b_{j_2} - b_{j_1} \in \mathscr{Z}_1. \end{aligned}$$

(iii) $b_i \in \mathscr{B}_1$ if and only if $b_i \notin \bigcup_{s=2}^{r-1} \mathscr{B}_s$.

For convenience, in what follows we let

$$\varrho = \frac{1}{r-1}.$$

We now choose a set, which we denote by $\mathscr{B}^*$, being a set with maximum cardinality among the $\mathscr{B}_i$ $(1 \leqslant i \leqslant r-1)$. Then

$$n_1^* = |\mathscr{B}^*| \geqslant \varrho n_1,$$

where $|\ |$ denotes cardinality. We denote the members of $\mathscr{B}^*$ by b_i^* $(i = 1, 2, \ldots, n_1^*)$, where $b_i^* < b_j^*$ $(1 \leqslant i < j \leqslant n_1^*)$. We now have

$$b_j^* - b_i^* \notin \mathscr{Z}_1 \quad (1 \leqslant i < j \leqslant n_1^*).$$

For, suppose that $b_i^* \in \mathscr{B}_m$ $(1 \leqslant m \leqslant r-2)$, and that $b_s - b_i^* \in \mathscr{Z}_1$ for some s. Then $b_s \in \mathscr{B}_h$ for some $h > m$. If $b_i^* \in \mathscr{B}_{r-1}$, then $b_s - b_i^* \notin \mathscr{Z}_1$ for all s, otherwise property $\mathscr{P}$ would be violated.

Hence, the set of integers $\{b_i^* - b_1^*: 2 \leqslant i \leqslant n_1^*\}$ has cardinality at least $\varrho n_1 - 1$, and none of these integers belongs to $\mathscr{X}_1$. They must therefore be distributed among the remaining $k-1$ classes. Choose a class, $\mathscr{X}_2$ say, that contains as many as possible, n_2 say, of these integers. Then

$$(k-1)n_2 \geqslant \varrho n_1 - 1.$$

Denote the integers of this type in $\mathscr{X}_2$ by $c_i - b_1^*$ $(1 \leqslant i \leqslant n_2)$, where $c_i < c_j$ $(1 \leqslant i < j \leqslant n_2)$.

We now partition the set $\mathscr{C} = \{c_i: 1 \leqslant i \leqslant n_2\}$ into $r-1$ subsets

$$\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2 \cup \ldots \cup \mathscr{C}_{r-1},$$

according to rules analogous to those used above for set $\mathscr{B}$ and class $\mathscr{X}_1$. Choose a set $\mathscr{C}^*$ having maximum cardinality among the $\mathscr{C}_i$ $(1 \leqslant i \leqslant r-1)$. Then

$$n_2^* = |\mathscr{C}^*| \geqslant \varrho n_2.$$

We denote the members of $\mathscr{C}^*$ by c_i^* $(i = 1, 2, \ldots, n_2^*)$, where $c_i^* < c_j^*$ $(1 \leqslant i < j \leqslant n_2^*)$. Consider the set of integers $\{c_i^* - c_1^*: 2 \leqslant i \leqslant n_2^*\}$. By an argument identical to that used above, none of these integers can belong to $\mathscr{X}_2$, and since each c_i^* is a b_j^*, none of them can belong to $\mathscr{X}_1$. Hence they must be distributed among the remaining $k-2$ classes. Choose a class, $\mathscr{X}_3$ say, that contains as many as possible, n_3 say, of these integers. Then

$$(k-2)n_3 \geqslant \varrho n_2 - 1.$$

Continuing in this way, we obtain a sequence of integers n_μ $(\mu = 1, 2, \ldots, k)$ satisfying the inequalities

$$\varrho n_\mu - 1 \leqslant n_{\mu+1}(k-\mu) \qquad (\mu = 1, 2, \ldots, k-1). \tag{4}$$

From (4) we obtain

$$\frac{\varrho n_\mu}{(k-\mu)!} \leqslant \frac{n_{\mu+1}}{(k-\mu-1)!} + \frac{1}{(k-\mu)!} \qquad (\mu = 1, 2, \ldots, k-1). \tag{5}$$

Also, we must clearly have

$$n_k \leqslant r-1 \tag{6}$$

since, in our partition of the set of differences into $r-1$ subsets at the kth stage, no subset can contain more than one member.

Now, if we multiply the μth inequality in (5) by $(r-1)^\mu$, and add, we obtain, using (6),

$$\frac{n_1}{(k-1)!} \leqslant (r-1)\left\{\frac{1}{(k-1)!} + \frac{(r-1)}{(k-2)!} + \ldots + \frac{(r-1)^{k-2}}{1!} + (r-1)^{k-1}\right\}.$$

Therefore,

$$\frac{n_1}{(k-1)!} \leqslant (r-1)^k\left\{1 + \frac{\varrho}{1!} + \frac{\varrho^2}{2!} + \ldots + \frac{\varrho^{k-1}}{(k-1)!}\right\} < (r-1)^k\left\{e^\varrho - \frac{\varrho^k}{k!}\right\}.$$

Hence,

$$n_1 < (k-1)!(r-1)^k e^\varrho - \frac{1}{k},$$

and so

$$M \leqslant kn_1 < k!\,(r-1)^k e^\varrho - 1 < N.$$

This completes the proof of the lemma.

Proof of Theorem 1. Put $a_i = i$ $(0 \leqslant i \leqslant N)$ in the lemma. Then the differences $a_j - a_i$ $(0 \leqslant i < j \leqslant N)$ are precisely the integers $1, 2, \ldots, N$. The theorem now follows on observing that

$$(a_{i_r} - a_{i_0}) = (a_{i_r} - a_{i_{r-1}}) + (a_{i_{r-1}} - a_{i_{r-2}}) + \ldots + (a_{i_1} - a_{i_0}).$$

3. A related problem. In Schur's theorem and its extension given above, the sum-free property is concerned with sums of integers that are not necessarily distinct. We can ask how the situation is affected when sums of distinct integers only are considered.

We define a set $\mathscr{S}$ of integers to be *weakly r-sum-free* if

$$a_1, a_2, \ldots, a_r \in \mathscr{S} \Rightarrow a_1 + a_2 + \ldots + a_r \notin \mathscr{S},$$

where the a_i are all distinct.

The case $r = 2$ of this problem is discussed in Sierpiński ([6], p. 409).

THEOREM 2. *Given positive integers k and r, $r \geqslant 2$, there exists a greatest positive integer $M = M(r, k)$ with the property that the set $\{1, 2, \ldots, M\}$ can be partitioned into k weakly r-sum-free sets. Further,*

$$M(r, k) \leqslant \left[\tfrac{1}{2}k!(r-1)^k(rk+1)\exp\left(\frac{1}{r-1}\right) + \frac{1}{r-1}\right].$$

Our proof follows similar lines to the proof of Theorem 1. Most of the work is contained in the lemma.

LEMMA 2. *Let k and r be positive integers, $r \geqslant 2$, and let*

$$M = \left[\tfrac{1}{2}k!(r-1)^k(rk+1)\exp\left(\frac{1}{r-1}\right) + \frac{r}{r-1}\right].$$

Then if $a_0 < a_1 < \ldots < a_M$ is a sequence of non-negative integers, and if the set of differences $a_j - a_i$ $(0 \leqslant i < j \leqslant M)$ is partitioned in any way into k classes, at least one of these classes contains a set of differences of the form

$$a_{i_r} - a_{i_{r-1}},\ a_{i_{r-1}} - a_{i_{r-2}},\ \ldots,\ a_{i_1} - a_{i_0},\ a_{i_r} - a_{i_0},$$

for some $i_0 < i_1 < \ldots < i_r$, with no two of these $r+1$ differences equal.

Proof. Let $a_0 < a_1 < \ldots < a_N$ be a sequence of non-negative integers, and suppose that $\mathscr{D} = \{a_j - a_i\colon 0 \leqslant i < j \leqslant N\}$ has been partitioned into k classes, $\mathscr{D} = \mathscr{Z}_1 \cup \mathscr{Z}_2 \cup \ldots \cup \mathscr{Z}_k$, none of which contains a set of integers such as is described in the statement of the lemma. It is sufficient to show that $N < M$.

Consider the set of differences of the form $a_i - a_0$ $(1 \leqslant i \leqslant N)$. Choose a class of the partition, $\mathscr{Z}_1$ say, that contains as many as possible, n say, of these differences. Then

$$kn \geqslant N.$$

Denote the differences of this type in $\mathscr{Z}_1$ by $b_i - a_0$ $(i = 1, 2, \ldots, n)$, where $b_i < b_j$ $(1 \leqslant i < j \leqslant n)$.

We now partition the set $\mathscr{B} = \{b_i\colon 1 \leqslant i \leqslant n\}$ into $r-1$ subsets $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \ldots \cup \mathscr{B}_{r-1}$ according to the following rules:

(i) $b_i \in \mathscr{B}_{r-1}$ if and only if there exist integers $j_1, j_2, \ldots, j_{r-2}$, where $1 \leqslant j_1 < j_2 < \ldots < j_{r-2} < i$, such that

$$\begin{gathered} b_i - b_{j_{r-2}} \in \mathscr{Z}_1, \\ b_{j_{r-2}} - b_{j_{r-3}} \in \mathscr{Z}_1, \\ \cdots\cdots\cdots \\ b_{j_2} - b_{j_1} \in \mathscr{Z}_1, \end{gathered}$$

with no two of these $r-2$ differences equal, and none of them equal to $b_{j_1} - a_0$.

Then successively for $h = r-2, r-3, \ldots, 3, 2$:

(ii) $b_i \in \mathscr{B}_h$ if and only if $b_i \notin \bigcup_{s=h+1}^{r-1} \mathscr{B}_s$ and there exist integers $j_1, j_2, \ldots, j_{h-1}$ where $1 \leqslant j_1 < j_2 < \ldots < j_{h-1} < i$, such that

$$\begin{gathered} b_i - b_{j_{h-1}} \in \mathscr{Z}_1, \\ b_{j_{h-1}} - b_{j_{h-2}} \in \mathscr{Z}_1, \\ \cdots\cdots\cdots \\ b_{j_2} - b_{j_1} \in \mathscr{Z}_1, \end{gathered}$$

with no two of these $h-1$ differences equal, and none of them equal to $b_{j_1} - a_0$.

(iii) $b_i \in \mathscr{B}_1$ if and only if $b_i \notin \bigcup_{s=2}^{r-1} \mathscr{B}_s$.

Let $|\mathscr{B}_i| = x(i)$ $(1 \leqslant i \leqslant r-1)$. Then

$$\sum_{i=1}^{r-1} x(i) = n.$$

For a given integer m_1 $(1 \leqslant m_1 \leqslant r-1)$, denote the members of $\mathscr{B}_{m_1}$ by b_i^* $(i = 1, 2, \ldots, x(m_1))$, where $b_i^* < b_j^*$ $(1 \leqslant i < j \leqslant x(m_1))$, and consider the set of integers $b_i^* - b_1^*$ $(i = 2, 3, \ldots, x(m_1))$. At most m_1 of these integers can belong to $\mathscr{Z}_1$, so that the remainder, numbering at least $x(m_1) - 1 - m_1$, must be distributed among the remaining $k-1$ classes. Choose a class, $\mathscr{Z}_2$ say, that contains as many as possible, $n(m_1)$ say, of these integers. Then

$$(k-1)\, n(m_1) \geqslant x(m_1) - 1 - m_1. \tag{7}$$

This presupposes $x(m_1) - 1 - m_1 > 0$, but if this is not the case, then $n(m_1) = 0$ and relation (7) continues to hold.

Denote the integers of this type in $\mathscr{Z}_2$ by $c_i - b_1^*$ $(i = 1, 2, \ldots, n(m_1))$, where $c_i < c_j$ $(1 \leqslant i < j \leqslant n(m_1))$. We now partition the set $\mathscr{C} = \{c_i\colon 1 \leqslant i \leqslant n(m_1)\}$ into $r-1$ subsets, $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2 \cup \ldots \cup \mathscr{C}_{r-1}$, according to rules analogous to those used above for set $\mathscr{B}$ and class $\mathscr{Z}_1$. Let $|\mathscr{C}_i| = x(i, m_1)$ $(1 \leqslant i \leqslant r-1)$. Then

$$\sum_{i=1}^{r-1} x(i, m_1) = n(m_1).$$

For a given integer m_2 $(1 \leqslant m_2 \leqslant r-1)$, denote the members of $\mathscr{C}_{m_2}$ by c_i^* $(i = 1, 2, \ldots, x(m_2, m_1))$, and consider the set of differences $c_i^* - c_1^*$ $(i = 2, 3, \ldots, x(m_2, m_1))$. At most m_2 of these integers can belong to $\mathscr{Z}_2$, and at most m_1 can belong to $\mathscr{Z}_1$. Hence the remainder, numbering at least $x(m_2, m_1) - 1 - m_1 - m_2$, must be distributed among the remaining $k-2$ classes. Choose a class, $\mathscr{Z}_3$ say, that contains as many as possible, $n(m_2, m_1)$ say, of these integers. Then

$$(k-2)\, n(m_2, m_1) \geqslant x(m_2, m_1) - 1 - (m_1 + m_2). \tag{8}$$

Again (8) is valid when $x(m_2, m_1) - 1 - (m_1 + m_2) \leqslant 0$, in which case $n(m_2, m_1) = 0$.

Continuing in this way, we obtain sequences of integers

$$\begin{gathered} n(m_{i-1}, m_{i-2}, \ldots, m_1) \quad (1 \leqslant i \leqslant k), \\ x(m_i, m_{i-1}, \ldots, m_1) \quad (1 \leqslant i \leqslant k-1), \end{gathered}$$

such that

$$(k-\mu)\, n(m_\mu, \ldots, m_1) \geqslant x(m_\mu, \ldots, m_1) - 1 - \sum_{i=1}^{\mu} m_i \tag{9}$$

for $\mu = 1, 2, \ldots, k-1$.

We finally reach a set of $n(m_{k-1}, \ldots, m_1)$ differences which must belong to $\mathscr{Z}_k$. We denote these by

$$w_i - v_1^* \quad (i = 1, 2, \ldots, n(m_{k-1}, \ldots, m_1)),$$

where $w_i < w_j$ $(1 \leqslant i < j \leqslant n(m_{k-1}, \ldots, m_1))$. We partition the set $\mathscr{W} = \{w_i\colon 1 \leqslant i \leqslant n(m_{k-1}, \ldots, m_1)\}$ into $r-1$ subsets $\mathscr{W} = \mathscr{W}_1 \cup \mathscr{W}_2 \cup \ldots \ldots \cup \mathscr{W}_{r-1}$, in the usual way, and let $|\mathscr{W}_i| = x(i, m_{k-1}, \ldots, m_1)$ $(i = 1, 2, \ldots \ldots, r-1)$. Clearly we must have

$$x(m_k, \ldots, m_1) \leqslant \sum_{j=1}^{k} m_j + 1 \qquad (m_k = 1, 2, \ldots, r-1).$$

Hence,

$$n(m_{k-1}, \ldots, m_1) = |\mathscr{W}| = \sum_{m_k=1}^{r-1} |\mathscr{W}_{m_k}| \leqslant \sum_{m_k=1}^{r-1} \Big\{\sum_{j=1}^{k} m_j + 1\Big\}$$
$$= (r-1)\Big(\tfrac{1}{2}r + 1 + \sum_{j=1}^{k-1} m_j\Big).$$

Now we sum over possible values of $m_1, m_2, \ldots, m_{k-1}$. All unspecified sums are from 1 to $r-1$.

$$(10) \qquad \sum_{m_1} \ldots \sum_{m_{k-1}} n(m_{k-1}, \ldots, m_1) \leqslant (r-1) \sum_{m_1} \ldots \sum_{m_{k-1}} \Big(\sum_{i=1}^{k-1} m_i + \tfrac{1}{2}r + 1\Big)$$
$$= (r-1)^k(\tfrac{1}{2}rk + 1).$$

We now sum (9) over possible values of $m_1, \ldots, m_\mu$. Again, all unspecified sums are from 1 to $r-1$.

$$\sum_{m_1} \ldots \sum_{m_\mu} (k-\mu)\, n(m_\mu, \ldots, m_1) \geqslant \sum_{m_1} \ldots \sum_{m_\mu} \Big\{x(m_\mu, \ldots, m_1) - 1 - \sum_{i=1}^{\mu} m_i\Big\},$$

i.e.

$$(11) \qquad \frac{1}{(k-\mu)!} \sum_{m_1} \ldots \sum_{m_{\mu-1}} n(m_{\mu-1}, \ldots, m_1)$$
$$\leqslant \frac{1}{(k-\mu-1)!} \sum_{m_1} \ldots \sum_{m_\mu} n(m_\mu, \ldots, m_1) + \frac{(r-1)^\mu(1 + \frac{1}{2}\mu r)}{(k-\mu)!}$$

for $\mu = 1, 2, \ldots, k-1$.

Combining the $k-1$ inequalities in (11), we obtain

$$\frac{n_1}{(k-1)!} \leqslant \sum_{m_1} \ldots \sum_{m_{k-1}} n(m_{k-1}, \ldots, m_1) + \sum_{\mu=1}^{k-1} \frac{(r-1)^\mu(1 + \frac{1}{2}\mu r)}{(k-\mu)!},$$

and using (10)

$$\frac{n_1}{(k-1)!} \leqslant (\tfrac{1}{2}rk + 1)(r-1)^k + \sum_{\mu=1}^{k-1} \frac{(r-1)^\mu}{(k-\mu)!} + \tfrac{1}{2}r \sum_{\mu=1}^{k-1} \frac{\mu(r-1)^\mu}{(k-\mu)!}$$
$$= T_1 + T_2 + T_3.$$

Again we let $\varrho = 1/(r-1)$. It is not difficult to show that

$$T_2 < (r-1)^k(e^\varrho - 1) - \frac{1}{k!},$$

$$T_3 < \tfrac{1}{2}kr(r-1)^k(e^\varrho - 1) + \frac{r}{(r-1)k!} - \tfrac{1}{2}r(r-1)^{k-1}e^\varrho,$$

so that we have

$$\frac{n_1}{(k-1)!} < \tfrac{1}{2}(rk+1)(r-1)^k e^\varrho + \frac{1}{(r-1)k!}.$$

Therefore,

$$N \leqslant kn_1 < \tfrac{1}{2}k!(rk+1)(r-1)^k e^\varrho + \varrho < M.$$

The proof of Lemma 2 is now complete.

Proof of Theorem 2. Theorem 2 now follows at once from Lemma 2 in the same way as Theorem 1 was a consequence of Lemma 1.

Remark. G. W. Walker [7] stated without proof that

$$2M(2, k) < M(2, k+1) \leqslant 3M(2, k).$$

While the first inequality is trivial to prove, the second is false, as can easily be shown by the use of the result of Abbott and Hanson [1] discussed in § 1.

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References

[1] H. L. Abbott and D. Hanson, *A problem of Schur and its generalizations*, Acta Arith. 20 (1972), pp. 175–197.
[2] — and L. Moser, *Sum-free sets of integers*, Acta Arith. 11 (1966), pp. 393–396.
[3] R. E. Greenwood and A. M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math. 7 (1955), pp. 1–7.
[4] R. Rado, *Studien zur Kombinatorik*, Math. Zeitschr. 36 (1933), pp. 424–480.
[5] I. Schur, *Über die Kongruenz* $x^m + y^m \equiv z^m \pmod p$, Jber. Deutsch. Math.-Verein. 25 (1916), pp. 114–117.
[6] W. Sierpiński, *Elementary Theory of Numbers*, Warszawa 1964.
[7] G. W. Walker, *A problem in partitioning*, Amer. Math. Monthly 59 (1952), p. 253.

[8] E. G. Whitehead, *The Ramsey number* $N(3,3,3,3;2)$, Discrete Mathematics, 4(1973), pp. 389–396.

[9] S. Znám, *Generalization of a number-theoretical result*, Mat.-Fyz. Časopis Sloven. Akad. Vied. 16 (1966), pp. 357–361.

[10] — *On k-thin sets and n-extensive graphs*, Mat. -Fyz. Časopis Sloven. Akad. Vied. 17(1967), pp. 297–307.

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Factorization of irreducible polynomials over a finite field with the substitution $x^{q^r}-x$ for x

by

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1. Introduction. Let $GF(q)$ denote the finite field of order $q = p^n$, where p is an arbitrary prime and $n \geqslant 1$. $Q(x)$ will denote an irreducible polynomial of degree s over $GF(q)$. For convenience we assume $Q(x)$ monic throughout the paper.

It is well known ([3], p. 34) that if $Q(x)$ is irreducible of degree s over $GF(q)$, then $Q(x^p - x)$ is also irreducible over $GF(q)$ if the coefficient β of x^{s-1} in $Q(x)$ satisfies

$$\sum_{j=0}^{n-1} \beta^{p^j} \neq 0. \tag{1.1}$$

On the other hand if the sum in (1.1) is equal to zero, $Q(x^p - x)$ is the product of p irreducible factors each of degree s over $GF(q)$. It has also been shown ([4], p. 307) that $Q(x^{q^s} - x)$ is the product of p^{ns-1} irreducibles each of degree ps over $GF(q)$ with no restrictions on β. The purpose of this present paper is to describe the irreducible factors of $Q(x^{q^r} - x)$ over $GF(q)$ for an arbitrary positive integer r. The principal results are contained in the following two theorems from § 5:

Let

$$N(s, q) = \sum_{ij=s} \mu(i) q^j$$

where μ is the Möbius function, and let

$$\varrho_s(x) = \sum_{j=0}^{s-1} x^{q^{dj}}$$

where $d = (r, s)$.

THEOREM I. *Let $Q(x)$ be irreducible of degree s over* $GF(q)$. *Let $(r, s) = d$, and let $s/d = s'$ and $r/d = r'$. If $Q(x) | \varrho_{s'}(x)$ then $Q(x^{q^r} - x)$ is the product over* $GF(q)$ *of irreducibles of degree st, $t|r'$. The number of irreducibles of*