ACTA ARITHMETICA
XXV (1973)

On the measure of measurable sets of integers*

by

MILTON PARNES (Amherst, N.Y.)

1. Introduction. Let N be the set of natural numbers and 2^N the set of all subsets of N. E. Borel [1] introduced the function Γ from 2^N onto $[0,1]$; where $\Gamma(A) = \sum \chi_A(n)2^{-n}$, $A \in 2^N$, and χ_A is the characteristic function of A, in order to study measure theoretic questions on sequences. In particular, if $S \subset 2^N$, we may ask what is the value of $\lambda(\Gamma(S))$ where λ is the Lebesgue measure on $[0,1]$. This notion makes particular sense when we note that Γ is bijective except for a countable set of elements of 2^N (namely, those sets of elements which are finite or that have finite complements). In this paper we show that a class of sets, the g-measurable sets as defined by R. Bumby and E. Ellentuck [3], are under a mild condition of measure zero. As a corollary we answer a question raised by R. C. Buck [2], p. 580.

2. g-invariant measure. We are interested in measures ν on the natural numbers N with the following properties:

1) $\nu(N) = 1$,
2) If $A \in 2^N$; $\nu(A)$ is defined and lies between 0 and 1,
3) ν is finitely additive,
4) ν is non-atomic, i.e. $\nu(\{n\}) = 0$,

where $n \in N$.

We shall denote the collection of all such measures by M.

If $C \subset M$ and $A \subset N$ we define $C(A) = \{\nu(A) \mid \nu \in C\}$.

If $C(A)$ is a single point we say that A is *C-measurable.*

Now let g be any function from N into N. A *measure ν is g-invariant* if for all sets $A \in 2^N$, $\nu(A) = \nu(g^{-1}(A))$.

Let J_g be the set of measures of M that are g-invariant. The set J_g is not empty [3].

Our main theorem is the following:

THEOREM 1. *Let D_g be the set of all J_g-measurable sets, where g has no finite orbits. Then $\lambda(\Gamma(D_g)) = 0$.*

---

* This research was partially supported by the State University of New York Research Foundation.

3. The proof of the main theorem. Let $g_0(n) = g(n)$ and $g_k(n) = g(g_{k-1}(n))$ for $n = 1, 2, \ldots$ Then Theorem 1 of [3], p. 36, implies that A is J_g-measurable if and only if

$$(1)\qquad \liminf_m \Big(\operatorname{glb}_n \frac{1}{m}\sum_{k=0}^{m-1} \chi_A(g_k(n))\Big) = \limsup_m \Big(\operatorname{lub}_n \frac{1}{m}\sum_{k=0}^{m-1} \sum \chi_A(g_k(n))\Big).$$

Now suppose that the set A is J_g-measurable and that

$$(2)\qquad 0 \leqslant \frac{k-1}{L} < J_g(A) < \frac{k}{L} \leqslant 1$$

where k and L are integers, $L > 2$. (A slight change is necessary in case $J_g(A) = 0$ or $J_g(A) = 1$.) Then by (1) for M large enough and all n

(3) the cardinality of

$$(A \cap \langle g_0(n), g_1(n), \ldots, g_{LM-1}(n)\rangle)$$

lies between $(k-1)M$ and kM.

Now let $D(n, LM, (k-1)M, kM)$ be the set of all sets A with the properties listed under (3) (for a particular n). Then

$$(4)\qquad \lambda\big[\Gamma\big(D(n, LM, (k-1)M, kM)\big)\big] = p$$

where $p = 2^{-LM}\sum\limits_{(k-1)M}^{kM}\binom{LM}{i} < 1.$

Let $n_1 = 1$, $n_2 = g_{LM}(1)$, $n_3 = g_{2LM}(1), \ldots$ Since g has no finite orbits the sets $G_i = \{g_k(n_i)\}$, $k = 0, \ldots, LM-1$; $i = 1, 2, \ldots$ are disjoint and we have

$$\lambda\big[\Gamma\big(\bigcap_{i=1}^{\infty} D(n_i, LM, (k-1)M, kM)\big)\big] = \lim_{T\to\infty} p^T = 0.$$

The set of all J_g-measurable sets with measure between $(k-1)/L$ and k/L is contained in

$$\bigcup_M \big[\bigcap_i D(n_i, LM, (k-1)M, kM)\big]$$

and is of measure zero. Clearly D_g is contained in a countable union of such sets obtained by varying k and L. We can therefore conclude that $\Gamma(D_g)$ is of measure zero.

Remark 1. Let $\tau(n) = n+1$, then the J_τ is the set of translation invariant measures.

Remark 2. It is not hard to show that these same g's have uncountably many J_g-measurable sets.

Remark 3. We have not attempted to prove Theorem 1 in its greatest generality. It should be noted that we only used one infinite orbit.

4. On a measure introduced by R. C. Buck. In [2] Buck introduced the following outer measure. Let D_0 be the class of subsets of N expressible as the disjoint union of arithmetic progressions $\{KN+j\}$ all with a common fixed K or a set which differs from the above type of set by a finite number of elements.

D_0 is closed under complementation, finite unions and intersections and the functional $\mu\big(\bigcup_{i\in L}\{KN+j_i\} \pm \text{finite set}\big) = (\text{cardinality of } L)/K$ is finitely additive on D_0. It can therefore be used to define an outer measure on N, by defining for each $S \subset N$

$$\mu^*(S) = \inf \mu(A) \qquad (S \subset A \in D_0).$$

A set S is *measurable* if and only if $\mu^*(S) + \mu^*(S') = 1$ where S' is the complement of S or equivalently, if and only if for every $\varepsilon > 0$, there exist sets A and B in D_0 such that

$$A \subset S \subset B \quad \text{and} \quad \mu(B-A) < \varepsilon.$$

Buck denoted the set of all measurable sets by D_μ, proved that there are uncountably many members of D_μ and then asked whether the set $\Gamma(D_\mu)$ is measurable ([2], p. 580). We show that $\Gamma(D_\mu)$ is measurable by showing that $D_\mu \subset D_\tau$, where D_τ was defined at the end of the last section and is therefore of measure zero.

THEOREM 2. *If $S \in D_\mu$, then $S \in D_\tau$ and $J_\tau(S) = \mu(S)$.*

Proof. Let ν be any τ-invariant measure in M. Then $\nu\{KN+j\} = 1/K$, since the sets $\tau_0\{KN+j\}$, $\tau_1\{KN+j\}, \ldots, \tau_{k-1}\{KN+j\}$ are disjoint and have equal ν-measure and their union, $\bigcup_{i=0}^{K-1}\{KN+j+i\} = N - \{\text{finite set}\}$. By the finite additivity of ν any member D_0 such as the A defined at the beginning of this section has ν-measure (cardinality of L)/K and is J_τ-measurable. Suppose S is in D_μ then there exist A_i and B_i such that

$$A_i \subset S \subset B_i \quad \text{and} \quad \mu(B_i - A_i) < 1/i.$$

Then

$$\nu(A_i) = \mu(A_i) \leqslant \mu(S) \leqslant \mu(B_i) = \nu(B_i) \leqslant \mu(A_i) + 1/i$$

but

$$\nu(A_i) \leqslant \nu(S) \leqslant \nu(B_i).$$

The result follows by letting i go to ∞.

Corollary. *The set $\Gamma(D_\mu)$ has Lebesgue measure* 0.

Proof. This follows immediately from Theorems 1 and 2.

5. Some closing remarks. It is not hard to show D_g always has the cardinality of the continuum. It is also easy to show that $D_\mu \subset D_\tau$ properly. In fact in some ways it seems to be a very small subset.

These theorems would seem to have much possibility for generalization. For instance they suggest similar theorems for invariant means or perhaps even the possibility that the "number" of Lebesgue measurable sets is very small as compared to the class of all subsets of the reals.

References

[1] E. Borel, *Les probabilitiés dénombrables et leurs applications arithmétiques*, Rend. Circ. Mat. Palermo 27 (1909), pp. 247–271.

[2] R. C. Buck, *The measure theoretic approach to density*, Amer. J. Math. 68 (1946), pp. 560–580.

[3] R. Bumby and E. Ellentuck, *Finitely additive measures and the first digit problem*, Fund. Math. 65 (1969), pp. 33–42.

Received on 5. 7. 1972 (303)

ACTA ARITHMETICA
XXV (1973)

An extension of Schur's theorem on sum-free partitions

by

Robert W. Irving (Glasgow)

1. Introduction. A set $\mathscr{S}$ of integers is said to be *sum-free* if

$$a \in \mathscr{S},\ b \in \mathscr{S} \Rightarrow a+b \notin \mathscr{S}.$$

a and b need not be distinct.

The following is a well-known theorem of Schur [5]:

Theorem (Schur). *Given a positive integer k, there exists a greatest positive integer $N = N(k)$ with the property that the set $\{1, 2, \ldots, N\}$ can be partitioned into k sum-free sets. Further,*

$$\tfrac{1}{2}(3^k-1) \leqslant N(k) \leqslant [k!\,e]-1 \tag{1}$$

where $[x]$ denotes the greatest integer not exceeding x.

The upper bound in (1) has recently been improved slightly by Whitehead [8] whose results show that

$$N(k) \leqslant [k!\,(e-\tfrac{1}{24})]-1.$$

Abbott and Hanson [1] have recently proved

$$N(k) \geqslant c\cdot 89^{\frac{1}{4}k}$$

for some absolute constant c, so improving an earlier result of Abbott and Moser [2].

A natural extension of the concept of a sum-free set is contained in the following definition:

A set $\mathscr{S}$ of integers is said to be *r-sum-free* if

$$a_1, a_2, \ldots, a_r \in \mathscr{S} \Rightarrow a_1+a_2+\ldots+a_r \notin \mathscr{S},$$

where the a_i need not be distinct.

It follows from results of Rado ([4], Theorems 3 and 4), that, given positive integers k and r, $r \geqslant 2$, there exists a greatest positive integer $N = N(r, k)$ with the property that the set $\{1, 2, \ldots, N\}$ can be partitioned into k r-sum-free sets. Clearly $N(2, k) = N(k)$.