ACTA ARITHMETICA
XXV (1973)

Some results on the distribution of additive arithmetic functions III

by

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0. Introduction. A real-valued arithmetic function h is said to *have a distribution* if there exists a distribution function $G(x)$ on the real line such that the density of the set $\{m \geqslant 1: h(m) < x\}$ of positive integers exists and is equal to $G(x)$ at each continuity point x of $G(x)$.

It is known that the distribution of a real-valued additive arithmetic function f, if it exists, is pure, that is, either discrete, continuous singular or absolutely continuous. It is also known that the distribution of a real-valued additive arithmetic function is discrete iff $\sum_{f(p)\neq 0} 1/p < \infty$.

In 1939 P. Erdös [3] has shown that, if f is a real additive arithmetic function given by

$$f(p) = O\left(\frac{1}{p^c}\right)$$

for all prime numbers p and for some positive constant c, then the distribution of f exists and is singular (i.e., either discrete or continuous singular). In this paper we show that if, for some $c > 0$,

$$\sum_{\substack{p>N \\ p\notin Q}} \frac{\{f(p)\}^2}{p} = O\left(\frac{1}{N^c}\right) \quad \text{as} \quad N\to\infty, \tag{1}$$

where Q is a set of prime numbers such that

$$\sum_{p\in Q} \frac{1}{p} < \infty,$$

then the distribution of $f(m)-f(m+1)$ exists and is singular (i.e., either discrete or continuous singular). From this result we shall deduce that if f satisfies (1) and f has a distribution, then the distribution of f is singular. In particular, every bounded real-valued additive arithmetic function has a singular distribution. We shall obtain similar results for the distribution of values of $f(F(m))$, where $F(m)$ is an integral polynomial taking positive values for $m = 1, 2, \ldots$

Most of the proofs depend on the following observation. Let f be a real-valued additive arithmetic function, having a distribution. The distribution of f is singular (absolutely continuous) iff the distribution function corresponding to the characteristic function $g(t)$ given by

$$\log g(t) = \sum_{\substack{p \\ |f(p)|<1}} \left(e^{itf(p)} - 1 - itf(p)\right)\frac{1}{p}$$

is singular (absolutely continuous).

1. Notations and definitions.

$p, q, p_1, p_2, \ldots$ always denote prime numbers.

N, k, t, m, n, etc., with or without suffixes always denote positive integers.

$\mathrm{P}\{\ldots\}$ denotes the probability of the event in $\{\ldots\}$. For any random variable X, $L(X)$ denotes the distribution function corresponding to the random variable X.

$\omega(m)$ denotes the number of distinct prime divisors of m.

$c_1, c_2, \ldots$ denote constants.

2. Results.

THEOREM 1. *If f is a real-valued additive arithmetic function satisfying* (1), *then the distribution of $f(m)-f(m+1)$ exists and is singular.*

THEOREM 2. *If f is a real-valued additive arithmetic function having an absolutely continuous distribution, then the distribution of $f(m)-f(m+1)$ is absolutely continuous.*

COROLLARY 1. *Suppose f is a real-valued additive arithmetic function having a distribution. If f satisfies condition* (1), *then the distribution of f is singular.*

COROLLARY 2. *The distribution of every bounded real-valued additive arithmetic function is singular. In particular, no additive arithmetic function can have a uniform distribution.*

THEOREM 3. *Suppose g is any real-valued additive arithmetic function for which there exists a constant K such that*

$$(2) \qquad |g(m)-g(m+1)| < K \quad \textit{for} \quad m = 1, 2, \ldots$$

Then the distribution of $g(m)-g(m+1)$ exists and is singular.

THEOREM 4. *Let f be a real-valued additive arithmetic function satisfying*

$$\liminf_{\varepsilon\to 0}(1/\varepsilon^2|\log\varepsilon|) \sum_{|f_p|<\varepsilon} \frac{f_p^2}{p} > 4,$$

where

$$f_p = \begin{cases} f(p) & \text{if} \quad |f(p)| < 1, \\ 1 & \text{if} \quad |f(p)| \geq 1. \end{cases}$$

Then the distribution of f is absolutely continuous.

Let $\boldsymbol{P}$ denote the set of all polynomials F with integral coefficients satisfying the following conditions:

(i) $F(m) > 0$ for $m = 1, 2, \ldots$

(ii) F is not divisible by the square of any irreducible polynomial.

THEOREM 5. *Let $F \in \boldsymbol{P}$ and let s denote the degree of the polynomial F. Let f be a real-valued additive arithmetic function satisfying*

$$(3) \qquad f(p^t)r(F, p) \to 0 \quad \textit{as} \quad p \to \infty \quad \textit{for} \quad t = 1, \ldots, s-1, \ \textit{if} \ s \geq 2,$$

$$(4) \qquad \sum_{\substack{p>N \\ p \notin Q}}' \frac{\{f(p)\}^2 r(F, p)}{p} = O\left(\frac{1}{N^c}\right) \quad \textit{as} \quad N \to \infty$$

where $r(F, t)$ denotes the number of incongruent solutions of the congruence relation

$$F(m) \equiv 0 \ (\mathrm{mod}\ t),$$

c is a positive constant and Q is a set of primes such that

$$\sum_{p \in Q}' \frac{r(F, p)}{p} < \infty.$$

Then the distribution of $f(F(m))$, if it exists, is singular.

THEOREM 6. *Under the conditions of Theorem* 5, $f(F(m))-f(F(m+1))$ *has a singular distribution.*

THEOREM 7. *With the same notation as above, suppose*

$$\liminf_{\varepsilon\to 0}(1/\varepsilon^2|\log\varepsilon|) \sum_{|f(p)|<\varepsilon} \frac{(f(p))^2 r(F, p)}{p} > 4,$$

then the distribution of $f(F(m))$ is absolutely continuous.

3. Preliminary results.

LEMMA 1. *If $\{X_n\}$ and $\{Y_n\}$ are two sequences of discrete and independent random variables defined on the same probability space satisfying*

$$\sum_n \mathrm{P}\{X_n \neq Y_n\} < \infty,$$

then $\sum_n X_n$ converges almost everywhere and $L(\sum_n X_n)$ is absolutely continuous (singular) iff $\sum_n Y_n$ converges almost everywhere and $L(\sum_n Y_n)$ is absolutely continuous (singular).

The proof of this lemma is well-known.

Lemma 2. *Let f be a real-valued additive arithmetic function satisfying*

$$\sum_p \frac{\{f_p\}^2}{p} < \infty .$$

Let $\{X_p\}$ be a sequence of independent random variables with

(5)
$$P\{X_p = x\} = P\{X_p = -x\} = \left(1-\frac{1}{p}\right)\left(\sum_{\substack{k \\ |f(p^k)|=x}} \frac{1}{p^k}\right) \quad if \quad x > 0,$$
$$P\{X_p = 0\} = 1-\frac{2}{p}+2\left(1-\frac{1}{p}\right)\left(\sum_{\substack{k \\ f(p^k)=0}} \frac{1}{p^k}\right).$$

Then $\sum_p X_p$ converges almost everywhere and the distribution of $f(m)-f(m+1)$ is $L(\sum_p X_p)$.

This result is contained in the proof of Proposition 1 [2].

Lemma 3. *If $F \in \boldsymbol{P}$, there exists a p_0 such that $p > p_0$ implies $r(F, p^t) = r(F, p)$ for any $t \geq 1$. Also*

$$f(F, a\cdot b) = r(F, a)r(F, b) \quad if \quad (a, b) = 1$$

and

$$r(F, p^t) \leq k \text{ for some integer } k \text{ depending only on } F.$$

See [6].

Lemma 4. *For any positive integer $k \geq 2$ and if $x \geq 3$ we have*

$$\sum_{m\leq x} k^{\omega(m)} = \left(\varphi(k)+O\left(\frac{1}{\log x}\right)\right)x(\log x)^{k-1}$$

where the constant induced by $O\left(\frac{1}{\log x}\right)$ depends only on k, and

$$\varphi(k) = \frac{1}{(k-1)!}\prod_p \left(1-\frac{1}{p}\right)^k\left(1+\frac{k}{p-1}\right).$$

See Kubilius [5], p. 140.

Lemma 5. *For any $n \geq 3$ and $k \geq 2$, we have*

$$\sum_{m\leq n} \frac{k^{\omega(m)}}{m} = \frac{\varphi(k)}{k}(\log n)^k + O(\log n)^{k-1}$$

where the constant induced by $O(\log n)^{k-1}$ depends only on k.

Proof. Partial summation gives

$$\sum_{m=1}^{n}\left(\frac{k^{\omega(m)}}{m}\right) = \frac{1}{n}\sum_{m=1}^{n} k^{\omega(m)} + \int_1^n \frac{1}{x^2}\left(\sum_{m\leq x} k^{\omega(m)}\right)dx$$
$$= O(\log n)^{k-1} + \int_2^n \frac{\varphi(k)(\log x)^{k-1}+O(\log x)^{k-2}}{x}\,dx$$
$$= O(\log n)^{k-1} + \frac{\varphi(k)}{k}(\log n)^k .$$

Lemma 6. *Suppose $\{a_p\}$ is a sequence of real numbers satisfying*

(6)
$$\sum_{p>N} \frac{a_p^2}{p} = O\left(\frac{1}{N^c}\right) \quad as \quad N\to\infty \quad for\ some\ c.$$

Further, suppose $g(t)$ is defined by

$$g(t) = \exp\left\{\sum_p (e^{ita_p}-1-ita_p)\frac{1}{p}\right\}.$$

Then $g(t)$ is a characteristic function of a distribution function and the distribution function corresponding to $|g(t)|^{2k}$ is singular for any integer $k \geq 1$.

Proof. It is clear that $g(t)$ is a characteristic function. Fix an integer $k \geq 1$. Let $\{X_p, Y_q$: p and q are primes greater than $2k\}$ be a set of independent random variables satisfying for each $p > 2k$

$$P\{X_p = t\} = P\{Y_p = -t\} = \left(\frac{k}{p}\right)^t\left(1-\frac{k}{p}\right) \quad \text{for} \quad t \geq 0.$$

Note that for any integer $t \geq 0$

$$P\{X_p+Y_p = -t\} = P\{X_p+Y_p = t\} = \left(\frac{k}{p}\right)^t\left(1+O\left(\frac{1}{p^2}\right)\right)e^{-2k/p}.$$

In view of (6) and Lemma 1, it follows that $\sum_p a_p(X_p+Y_p)$ converges almost everywhere and $L(\sum_p a_p(X_p+Y_p))$ is singular iff the distribution corresponding to $|g(t)|^{2k}$ is singular.

Without loss of generality we can assume $c < 1$, in (6). Let N be a large integer. Let $m < N^{c/6}$ and

$$m = p_{1m}^{k_{m1}}\ldots p_{t_m m}^{k_{mt_m}}, \quad k_{mt} \geq 1.$$

Consider the set

$$D_{mN} = \left\{\sum_{i=1}^{t_m} \varepsilon_i k_{mi} a_{p_{im}}:\ \varepsilon_i = +1 \text{ or } -1,\ i = 1,\ldots,t_m\right\}.$$

Put $D_N = \bigcup_{m<N^{c/6}} D_{mN}$.

Since there are $2^{\omega(m)}$ sequences $(\varepsilon_1, \ldots, \varepsilon_{t_m})$ of $+1$ and -1, and

$$\mathrm{P}\{X_{p_{im}} + Y_{p_{im}} = \varepsilon_i k_{mi}, \; i = 1, \ldots, t_m \text{ and } X_p + Y_p = 0 \text{ if } p \leqslant N \text{ and } (p, m) = 1\}$$

is the same for any sequence $(\varepsilon_1, \ldots, \varepsilon_{t_m})$ of $+1$ and -1, we have

$$\mathrm{P}\Big\{\sum_{p\leqslant N} a_p(X_p + Y_p) \in D_N\Big\}$$
$$\geqslant \sum_{m<N^{c/6}} \sum_{(\varepsilon_1,\ldots,\varepsilon_{t_m})} \mathrm{P}\{X_{p_{im}} + Y_{p_{im}} = \varepsilon_i k_{mi}, \; i = 1, \ldots, t_m \text{ and } X_p + Y_p = 0 \text{ if } p \leqslant N \text{ and } (p, m) = 1\}$$
$$\geqslant \sum_{m<N^{c/6}} 2^{\omega(m)} \frac{k^{\omega(m)}}{m} \prod_{p\leqslant N}\Big(1 + O\Big(\frac{1}{p^2}\Big)\Big) \exp\Big\{-2k \sum_{p\leqslant N} \frac{1}{p}\Big\}.$$

Since $\sum_{p\leqslant N} 1/p = \log\log N + O(1)$, by Lemma 5 it follows that

$$\mathrm{P}\Big\{\sum_{p\leqslant N} a_p(X_p + Y_p) \in D_N\Big\} \geqslant a > 0$$

for some constant a and for all large N.

Put $h = [c/3] + 1$. For all sufficiently large N, we have,

$$\mathrm{P}\Big\{\Big|\sum_{p>N} a_p(X_p + Y_p)\Big| > N^{-c/3}\Big\}$$
$$\leqslant \mathrm{P}\Big\{\sum_{p>N} |a_p(X_p + Y_p)| > N^{-c/3} \text{ and for all } p > N, \; |X_p + Y_p| < h + 2\Big\} + O\Big(\sum_{p>N} \frac{1}{p^{h+2}}\Big)$$
$$= O\Big(\sum_{p>N} \sum_{t=1}^{h+1} \frac{t^2 a_p^2 k^t}{p^t}\Big) N^{2c/3} + O\Big(\sum_{p>N} \frac{1}{p^2}\Big) N^{-h}$$
$$= O(N^{-c/3}) < bN^{-c/3} \quad \text{for some} \quad b > 0.$$

So

$$\mathrm{P}\Big\{\sum_{p} a_p(X_p + Y_p) \in G_N\Big\} \geqslant a - bN^{-c/3} > a/2 > 0.$$

for all sufficiently large N, where $G_N = \bigcup_{d\in D_N} [d - N^{-c/3}, d + N^{-c/3}]$. By Lemma 4, the Lebesgue measure of the set G_N tends to zero as $N \to \infty$. Hence $L(\sum_p a_p(X_p + Y_p))$ is singular. This completes the proof of the lemma.

4. Proofs of the theorems.

Proof of Theorem 1. By (1) and Lemma 2, we can find a sequence $\{X_p\}$ of independent random variables satisfying (5) and the distribution of $f(m) - f(m+1)$ is $L(\sum_p X_p)$. It is not hard to find a sequence $\{Y_p\}$ of independent random variables defined on the same probability space on which X_p are defined and satisfying the following condition:

(7) $\qquad \mathrm{P}(Y_p = 0) = 1 \quad \text{if} \quad p \in Q,$

(8) the characteristic function of Y_p is

$$|\exp\{(e^{itf_p} - 1 - itf_p)/p\}|^2$$

and

(9) $\qquad P(X_p \neq Y_p) = O(1/p^2).$

Now the theorem follows from Lemmas 1 and 6.

Proof of Theorem 2. Let $\{\eta_p\}$ be a sequence of independent random variables with

$$\mathrm{P}(\eta_p = x) = \Big(1 - \frac{1}{p}\Big) \sum_{\substack{t \\ f(p^t) = x}} \frac{1}{p^t}.$$

It is easy to see that ([5]) if f has a distribution, then $\sum_p \eta_p$ converges almost everywhere and the distribution of f coincides with $L(\sum_p \eta_p)$. Let $\{Y_p, Z_q\colon p, q \text{ primes}\}$ be a set of independent random variables defined on the same probability space on which $\{\eta_p\}$ are defined and satisfying the following conditions:

$$\mathrm{P}\{f(p)Y_p \neq \eta_p\} = O\Big(\frac{1}{p^2}\Big),$$

$$\mathrm{P}\{Y_p = k\} = \mathrm{P}\{Z_p = -k\} = \frac{1}{p^k}\Big(1 - \frac{1}{p}\Big), \quad k = 0, 1, \ldots$$

Since $L(\sum_p \eta_p)$ is absolutely continuous, by Lemma 1, it follows that $L(\sum_p f(p)Y_p)$ is absolutely continuous. Consequently $L(\sum_p f(p)(Y_p + Z_p))$ is absolutely continuous. Again by Lemma 1 and from the proof of Theorem 1 it follows that the distribution of $f(m) - f(m+1)$ is absolutely continuous. This completes the proof of Theorem 2.

Corollary 1 now follows easily from the above two theorems.

Proof of Corollary 2. Since $f(m)$ is bounded, $\sum_p f(p)$ converges absolutely and hence $|f(p)| < 1$ for all sufficiently large p. So for N sufficiently large, we have

$$\sum_{p>N} \frac{f(p)^2}{p} \leqslant \Big(\sum_{p>N} f(p)^4\Big)^{1/2} \Big(\sum_{p>N} \frac{1}{p^2}\Big) = O\Big(\Big(\sum_{p>N} \frac{1}{p^{1+\frac{1}{2}}}\Big)^{1/2}\Big) \cdot \frac{1}{N^{1/4}} = O(N^{-1/4}).$$

Now this corollary follows from Corollary 1.

Proof of Theorem 3. If a real additive arithmetic function g satisfies (2), then by a result of Wirsing [7] there exists a constant D and a bounded real additive arithmetic function f such that

$$g(m) = D\log m + f(m) \quad \text{for} \quad m = 1, 2, \ldots$$

Since f is bounded, it satisfies condition (1). So the distribution of $f(m) - f(m+1)$ is singular. But $g(m) - g(m+1) = f(m) - f(m+1) + o(1)$ as $m \to \infty$. Hence the distribution of $g(m) - g(m+1)$ is the same as that of $f(m) - f(m+1)$. Consequently the distribution of $g(m) - g(m+1)$ is singular.

Proof of Theorem 4. Define a function $g(t)$ by

$$\log g(t) = \sum_p (e^{itf_p} - 1 - itf_p)\frac{1}{p}.$$

Since $\left\{\exp\left(\frac{1}{p}(e^{itf_p} - 1 - itf_p)\right)\right\}$ is the characteristic function of a centred Poisson random variable and since

$$\sum_p \frac{f_p^2}{p} < \infty$$

and $h_n(t) = \exp\left\{\sum_{p\leq n}(e^{itf_p} - 1 - itf_p)\frac{1}{p}\right\}$ is a characteristic function for each n, $h_n(t)$ converges absolutely and uniformly to $g(t)$ in every bounded interval, and $g(t)$ is an infinitely divisible characteristic function. As in the proof of Theorem 1, it is sufficient to show that the distribution corresponding to $g(t)$ is absolutely continuous.

Note that since

$$|\sin y| \geq |y|/2 \quad \text{if} \quad |y| \leq \tfrac{1}{2},$$

we have for any $\varepsilon > 0$

$$\frac{1}{|\log 2\varepsilon|}\sum_p \frac{(\sin(f_p/2\varepsilon))^2}{p} \geq \frac{1}{6\varepsilon^2|\log 2\varepsilon|}\left(\sum_{|f_p|<\varepsilon}\frac{f_p^2}{p}\right),$$

i.e.

$$\liminf_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\sum_p\left\{\frac{(\sin(f_p/\varepsilon))^2}{p}\right\} > \frac{1}{4}.$$

We have

$$\log|g(2u)|^2 = \left\{\sum_p (e^{i2uf_p} - 1 - i2uf_p)\frac{1}{p}\right\} + \left\{\sum_p (e^{-i2uf_p} - 1 + i2uf_p)\frac{1}{p}\right\}$$

$$= \sum_p (e^{i2uf_p} + e^{-i2uf_p} - 2)\frac{1}{p} = \sum_p (2\cos 2uf_p - 2)\frac{1}{p} = -2\sum_p \frac{2(\sin uf_p)^2}{p}.$$

So,

$$-\frac{1}{2}\log|g(2u)| = \sum_p \frac{(\sin uf_p)^2}{p}.$$

Hence,

$$\liminf_{u\to\infty}\left\{-\frac{1}{2}(\log|g(2u)|)\frac{1}{|\log u|}\right\} > \frac{1}{4},$$

i.e.

$$\liminf_{u\to\infty}\left\{-(\log|g(u)|)\frac{1}{|\log u|}\right\} > \frac{1}{2}.$$

Hence for some $\delta > 0$,

$$|g(u)| = O(|u|^{-\frac{1}{2}-\delta}) \quad \text{as} \quad u \to \infty.$$

So $g(u)$ is square integrable, consequently, by Plancherel's theorem, it follows that the distribution function corresponding to g is absolutely continuous. This completes the proof of Theorem 4.

The proof of Theorem 7 is similar to the proof of Theorem 4.

Proof of Theorem 5. Define

$$f(p) = \begin{cases} a_p & \text{if} \quad p \notin Q,\ r(p) \neq 0 \text{ and } |f(p)| < 1, \\ 0 & \text{otherwise}. \end{cases}$$

Let $r(F, p) \leq k$ for all p.

In view of Theorem 1 of [1] and Lemma 1, it follows that the distribution of $f(F(m))$, if it exists, is singular iff the distribution function corresponding to the characteristic function

$$h(t) = \exp\left\{\sum_p (e^{ita_p} - 1 - ita_p)\frac{r(F,p)}{p}\right\}$$

is singular.

Define

$$s(t) = \exp\left\{\sum_{\substack{p \\ r(F,p)<k}} (e^{ita_p} - 1 - ita_p)\frac{(k - r(F,p))}{p}\right\}.$$

Since $\sum_p a_p^2/p$ is finite, $s(t)$ defines a characteristic function and $|g(t)|^{2k} = |h(t)s(t)|^2$. Clearly (4) implies that

$$\sum_{p>N} a_p^2/p = O\left(\frac{1}{N^c}\right) \quad \text{as} \quad N \to \infty.$$

So the distribution function corresponding to $|g(t)|^{2k}$ is singular by Lemma 6 and hence the distribution corresponding to $h(t)$ is singular. This completes the proof of Theorem 5.

Proof of Theorem 6. It is not difficult to show from the proofs of Theorems 1, 2 of [1] and from Lemma 1 that the distribution of $f(F(m))-f(F(m+1))$ exists and is singular iff the distribution function corresponding to the characteristic function

$$g^*(t) = \exp\Big\{\sum_p (e^{ita_p}+e^{-ita_p}-2)\big(r^*(F,p)/p\big)\Big\}$$

is singular, where a_p is as defined in Theorem 5 and $r^*(F,p) \leq r(F,p)$. ($r^*(F,p) < r(F,p)$ for some p if there exist two factors $P(m)$ and $Q(m)$ of $F(m)$ such that $Q(m) = P(m+1)$ for all m.)

From Lemma 6 and from the proof of Theorem 5 it easily follows that the distribution corresponding to $g^*(t)$ is singular. This completes the proof of Theorem 6.

Remark. Let $F \in \boldsymbol{P}$ and let the degree of F be > 1. Suppose that condition (3) is satisfied and there exists a set Q of prime numbers such that

$$(10) \qquad \sum_{p \in Q} \frac{1}{p} < \infty \text{ and } p \notin Q \text{ implies either } r(F,p) \neq 0$$
$$\text{or } f(p) = 0 \text{ and } r(F,p) = 0.$$

Then we have the following

PROPOSITION. *If the distribution of $f(n)-f(n+1)$ exists and is absolutely continuous, then the distribution of $f(F(m))-f(F(m+1))$ also exists and is absolutely continuous.*

The proof of this proposition is similar to the proof of Proposition 3 of [2].

Note that condition (10) is satisfied if F is divisible by a linear polynomial. Condition (10) cannot be replaced, since if (10) is violated, then the following example shows that the distribution of $f(m)-f(m+1)$ is absolutely continuous but $f(F(m))-f(F(m+1)) = 0$ for all $m \geq 1$.

Define a strongly additive arithmetic function by

$$f(p) = \begin{cases} \dfrac{1}{(\log\log p)^{3/2}} & \text{if} \quad p < e^e \text{ and } p \equiv 3 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Take $F(m) = m^2+1$.

It is known ([2]) that if $p \equiv 3 \pmod 4$ then $p \nmid m^2+1$ for any m, so $f(F(m))-f(F(m+1)) = 0$ for all m. But, on the other hand, it is not difficult to show that the distribution of $f(m)-f(m+1)$ is absolutely continuous (see [2]).

Acknowledgements. The author expresses his indebtedness to Prof. J. K. Ghosh, whose suggestions and careful editing helped this article to assume its present form. He also wishes to thank Prof. E. M. Paul for the various discussion he had with him.

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Received on 13. 7. 1972 (295)