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Simultaneous quadratic equations II

by

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1. Introduction. Let

$$f_i(x) = \sum_{j=1}^{13} a_{ij}x_j^2, \quad i = 1, 2, 3, \tag{1}$$

be diagonal quadratic forms with integer coefficients. We obtain sufficient conditions for the equations

$$f_1(x) = f_2(x) = f_3(x) = 0 \tag{2}$$

to have a non-trivial solution in integers. The method used here is a simple extension of that used in a previous paper [2]. An essential preliminary to the proof is a recent result of Mrs Ellison [5] on the solvability of the equations (2) in p-adic fields. I am grateful to Prof. D. J. Lewis for telling me of Mrs Ellison's work, and to Mrs Ellison for sending me a copy of her work.

THEOREM. *Suppose that*

(i) *For all real λ, μ, ν, not all zero, $\lambda f_1+\mu f_2+\nu f_3$ contains at least* 11 *variables explicitly*;

(ii) *There exist non-singular solutions of the equations* (2) *in the real and 2-adic fields*:

Then the equations (2) *have a non-trivial solution in integers.*

While it is known that 13 variables is best possible in this context, for example the equations

$$x_i^2+y_i^2-3z_i^2-3t_i^2 = 0, \quad i = 1, 2, 3,$$

have no non-trivial integer solutions, it is unlikely that condition (i) of the Theorem is best possible. However, some such condition is necessary. For example, the equations

$$\begin{aligned} x_1^2+\ldots+x_5^2+y_1^2-y_2^2+2y_3^2-2y_4^2+z_1^2-z_2^2+2z_3^2-2z_4^2 &= 0,\\ y_1^2+y_2^2-3y_3^2-3y_4^2 &= 0,\\ z_1^2+z_2^2-3z_3^2-3z_4^2 &= 0\end{aligned}$$

have non-singular real solutions but have no non-trivial solution in integers. Similarly, the equations

$$x_1^2+\ldots+x_7^2+y_1^2-y_2^2+2y_3^2-2y_4^2 = 0,$$
$$y_1^2+y_2^2-3y_3^2-3y_4^2 = 0$$

have non-singular real solutions but no non-trivial integer solutions, thus providing a counter-example to a Theorem of Swinnerton-Dyer [8].

2. Notation and preliminary lemmas.

LEMMA 2.1. *Let p be an odd prime, then the equations* (2) *have a non-trivial solution in the p-adic field.*

Proof. This result is due to Mrs Ellison [5].

LEMMA 2.2. *Let p be an odd prime, then the equations* (2) *have a non-singular solution in the p-adic field.*

Proof. This can readily be deduced from Lemma 2.1 and condition (i) of the Theorem by the method of Theorem 4 of Davenport and Lewis [4].

LEMMA 2.3. *The equations* (2) *have a non-singular real solution with none of the variables vanishing.*

Proof. This can be deduced from condition (ii) of the Theorem by a simple variational principle.

From such a real solution of (2) we have a solution χ of the linear equations

$$a_{i1}\chi_1+\ldots+a_{i13}\chi_{13} = 0, \quad i = 1, 2, 3,$$

such that $\chi_j > 0$ for $j = 1, \ldots, 13$. Then, choosing a suitable linear multiple of this solution, we may suppose that $\chi_j > 1$ for $j = 1, \ldots, 13$. We now choose $C > 1$ so that

$$(3) \qquad 1 < \chi_j < C^2 \quad \text{for} \quad j = 1, \ldots, 13.$$

Let

$$(4) \qquad \gamma_j = a_{1j}\alpha_1+a_{2j}\alpha_2+a_{3j}\alpha_3, \quad j = 1, \ldots, 13,$$

and

$$(5) \qquad T(\gamma_j) = \sum_{x=P}^{CP} e(\gamma_j x^2), \quad j = 1, \ldots, 13$$

where P is a large positive number, $e(\theta) = \exp(2\pi i\theta)$ and $e_q(\theta) = e(\theta/q)$.

Then the number of integer solutions of (2) in the box $\{x\colon P \leqslant x_j \leqslant CP\}$ is

$$(6) \qquad N(P) = \int_0^1 \prod_{j=1}^{13} T(\gamma_j)\, d\boldsymbol{a}$$

where the integral is threefold.

Let δ be a small positive constant. We take the major arc $\boldsymbol{M}(A, R)$ to consist of those $\boldsymbol{a}$ which have rational approximations

$$(7) \qquad |a_i - A_i/R| < P^{-2+\delta}$$

for $1 \leqslant R \leqslant P^\delta$, $1 \leqslant A_i \leqslant R$ where $(A_1, A_2, A_3, R) = 1$. We denote the union of the major arcs by $\boldsymbol{M}$, the minor arcs $\boldsymbol{m}$ consist of the rest of the unit cube. We use Vinogradov's $\ll$ notation where the implicit constants are independent of P.

3. The minor arcs.

LEMMA 3.1 (Dirichlet). *For any real numbers $\gamma, P \geqslant 1$ there exist integers a, q with*

$$(8) \qquad (a, q) = 1, \quad 1 \leqslant q \leqslant P^{1+\delta} \quad \text{and} \quad |q\gamma - a| < P^{-1-\delta}.$$

Proof. See, for example, Theorem 185 of Hardy and Wright [6].

LEMMA 3.2 (Weyl). *Suppose that*

$$(9) \qquad |q\gamma - a| < P^{-1-\delta}, \quad P^{1-\delta} \leqslant q \leqslant P^{1+\delta}, \quad (a, q) = 1$$

then

$$(10) \qquad |T(\gamma)| \ll P^{1/2+\delta}.$$

Proof. See, for example, Lemma 1 of Davenport [3].

LEMMA 3.3. *If $(a, q) = 1$, $1 \leqslant q \leqslant P^{1-\delta}$, $\gamma = a/q+\varphi$ and $|q\varphi| < P^{-1-\delta}$ then*

$$(11) \qquad |T(\gamma)| \ll q^{-1/2}\min(P, P^{-1}|\varphi|^{-1}).$$

Proof. This follows from the corollary to Lemma 9 of Birch and Davenport [1].

LEMMA 3.4. *If γ_1, γ_2 and γ_3 are independent linear forms and $\boldsymbol{a} \in \boldsymbol{m}$ then*

$$(12) \qquad \prod_{i=1}^{3} |T(\gamma_i)| \ll P^{3-\delta/3}.$$

Proof. This follows from Lemmas 3.2 and 3.3 by using the method of Lemma 19 of Davenport and Lewis [4].

LEMMA 3.5. *Any 12 of the forms γ_i can be arranged into 4 sets of 3 independent forms.*

Proof. Condition (i) of the theorem implies that any 3 distinct forms γ_i are linearly independent so this result follows immediately.

LEMMA 3.6. *Let $\gamma_1, \gamma_2, \gamma_3$ be independent linear forms. Then*

$$(13) \qquad \int_0^1 \prod_{i=1}^{3} |T(\gamma_i)|^4\, d\boldsymbol{a} \ll P^{6+\varepsilon}.$$

Proof. Since $\gamma_1, \gamma_2, \gamma_3$ are independent the determinant of their coefficients is non-zero, and bounded. In the range of integration for α each γ_i is bounded. Changing the variables of integration to $\gamma_1, \gamma_2, \gamma_3$ we have

$$\int_0^1 \prod_{i=1}^3 |T(\gamma_i)|^4 d\alpha \ll \int_0^1 \prod_{i=1}^3 |T(\gamma_i)|^4 d\gamma_1 d\gamma_2 d\gamma_3 = \prod_{i=1}^3 \int_0^1 |T(\gamma_i)|^4 d\gamma_i \ll P^{6+3\varepsilon}$$

by a lemma of Hua [7].

LEMMA 3.7.

(14) $$\int_{m} \prod_{j=1}^{13} |T(\gamma_j)| \, d\alpha \ll P^{7+\varepsilon-\delta/9}.$$

Proof. The forms $\gamma_1, \gamma_2, \gamma_3$ are independent so from Lemma 3.4

$$\max_{m} \min_{j} |T(\gamma_j)| \ll P^{1-\delta/9}$$

and any 12 of the forms γ_i can be arranged as in Lemma 3.5. The result now follows from Lemma 3.6.

4. The major arcs. Since the treatment of the major arcs closely follows that in Davenport and Lewis [4] only outline details are given. For $\alpha \in M(A, R)$ let

(15) $$d_j = \text{g.c.d.}(A_1 a_{1j} + A_2 a_{2j} + A_3 a_{3j}, R), \quad j = 1, \ldots, 13,$$

(16) $$R = R_j d_j, \quad j = 1, \ldots, 13,$$

(17) $$\varphi_i = \alpha_i - A_i/R, \quad i = 1, 2, 3,$$

and

(18) $$\beta_j = a_{1j}\varphi_1 + a_{2j}\varphi_2 + a_{3j}\varphi_3, \quad j = 1, \ldots, 13.$$

We choose C_j so that $(C_j, R_j) = 1$ and

(19) $$C_j/R_j = (A_1 a_{1j} + A_2 a_{2j} + A_3 a_{3j})/R.$$

We take

(20) $$S(a, q) = \sum_{x=1}^{q} e_q(ax^2)$$

and

(21) $$I(\varphi) = \int_P^{CP} e(\varphi\xi^2)\, d\xi.$$

LEMMA 4.1. *The contribution of the $M(A, R)$ to $N(P)$ is*

(22) $$G(P^\delta) J(P) + O(P^{6+1/2})$$

where

(23) $$G(P^\delta) = \sum_{R \leqslant P^\delta} \sum_{A} \prod_{j=1}^{13} R_j^{-1} S(C_j, R_j)$$

and

(24) $$J(P) = \int \prod_{j=1}^{13} I(\beta_j)\, d\varphi$$

the integration being over $|\varphi_j| \leqslant P^{\delta-2}$, $j = 1, \ldots, 13$.

Proof. This can be proved in the same way as Lemma 28 of Davenport and Lewis [4].

LEMMA 4.2.

(25) $$J(P) \sim KP^7 \quad \text{as } P \to \infty \text{ where } K > 0.$$

Proof. This can be proved in the same way as Lemma 30 of Davenport and Lewis [4].

LEMMA 4.3.

(26) $$G(P^\delta) = G + o(1) \quad \text{as } P \to \infty \text{ where } G > 0.$$

Proof. This follows from Lemma 2.2 using arguments similar to Lemmas 29 and 31 of Davenport and Lewis [4].

Thus

$$N(P) = KGP^7 + o(P^7) \quad \text{as } P \to \infty$$

where $KG > 0$, and the theorem has been proved.

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