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Notes on small class numbers

by

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1. Introduction. In this paper we study the problem of obtaining lower bounds for the class number $h = h(-d)$ of an imaginary quadratic field $Q(\sqrt{-d})$, $d > 0$. We recall that Siegel [13] has shown that $h > d^{\frac{1}{2}-\varepsilon}$ for $d > d_0$, and that his argument does not permit one to determine all fields with given class number. Recently the problem of obtaining effective lower bounds for h has received considerable attention (see, for example, Baker [1], [3], and Stark [16]).

From the Deuring–Heilbronn formulae it may be shown that if $h < d^{\frac{1}{2}-\delta}$ then all non-trivial zeros of certain L-functions are on the critical line, at least up to a height depending on d, on δ, and on the L-function. If the class number is somewhat smaller, $h < d^{\frac{1}{4}-\delta}$, then the imaginary parts of these zeros can also be described; it is found that the zeros are quite evenly spaced, so that two zeros of the same L-function cannot be very close together. To state this more precisely, let $\varrho = \frac{1}{2}+i\gamma$ and $\varrho' = \frac{1}{2}+i\gamma'$ be consecutive zeros on the critical line of an L-function $L(s, \chi)$, where χ is a primitive character (mod k). Put

$$\lambda(K) = \min \frac{1}{2\pi}|\gamma-\gamma'|\log K,$$

where the minimum is over all $k \leqslant K$, all χ (mod k), and all $\varrho = \frac{1}{2}+i\gamma$ of $L(s, \chi)$ with $|\gamma| \leqslant 1$. In this range the average of $|\gamma-\gamma'|$ is $2\pi/\log k$, so trivially $\varlimsup\lambda(K) \leqslant 1$. Presumably $\lambda(K)$ tends to 0 as K increases; if this could be shown effectively then the effective lower bound $h > d^{\frac{1}{2}-\varepsilon}$ would follow. In fact the weak inequality $\lambda(K) < \frac{1}{4}-\delta$ for $K > K_0$ implies that $h > d^{4\delta-\varepsilon}$ for $d > C(K_0, \varepsilon)$; the function $C(K_0, \varepsilon)$ can be made explicit. Even $\lambda(K) < \frac{1}{2}-\delta$ has striking consequences.

The initial remark of the previous paragraph makes it clear that in bounding $\lambda(K)$ one may assume that all the zeros of the L-functions under consideration are on the critical line. In this situation the techniques of Montgomery [10] can be used to show $\lambda(K) < 0.68$ for all large K, but it is not clear that the method used there will produce an upper bound of the required sort. Hence we do not discuss the above observations in greater detail. Instead we illustrate our approach by using specifically known zeros to derive a modest result.

An examination of the formulae in Lemma 1 reveals that if h is small then real L-functions have no zeros near $\frac{1}{2}$; indeed zeros of real L-functions near $\frac{1}{2}$ have the same effect as correspondingly close pairs of zeros of arbitrary L-functions. We have found several zeros of real L-functions which are near $\frac{1}{2}$; using these zeros we demonstrate the following theorem.

THEOREM. *Let h denote the class number of an imaginary quadratic field with discriminant $-d$. If $10^{12} \leqslant d \leqslant 10^{1200}$ then $h \neq 2$, and if $10^{12} \leqslant d \leqslant 10^{2500}$ then $h \neq 3$.*

Lehmer, Lehmer, and Shanks [9] used a sieving technique to demonstrate that if $10^6 \leqslant d \leqslant 10^{12}$ then $h \neq 3$. Moreover, all imaginary quadratic fields with $h = 3$ and $d \leqslant 10^6$ are known, so our theorem enables one to list all such fields with class number $h = 3$ and $1 \leqslant d \leqslant 10^{2500}$.

Baker [2] and Stark [18] have given effective treatments of the $h = 2$ situation; in particular Stark showed that if $h = 2$ then $d < 10^{1100}$. Moreover, D. H. Lehmer used his delay-line sieve to show that $h = 2$ does not occur for $10^6 \leqslant d \leqslant 10^{12}$, while the fields with $h = 2$ and $d < 10^6$ were known previously. Combining these results we have the following

COROLLARY. *If an imaginary quadratic field $Q(\sqrt{-d})$ with discriminant $-d$ has class number 2, then d is one of the eighteen numbers $d = 15, 20, 24, 35, 40, 51, 52, 88, 91, 115, 123, 148, 187, 232, 235, 267, 403, 427$.*

Stark (to appear) has adopted a somewhat different approach to treat the range $10^{12} \leqslant d \leqslant 10^{1100}$ with $h = 2$. His method, however, depends on an extended computer calculation.

We are indebted to Dr. George Purdy for providing us with a list of real L-functions which could be expected to have zeros near $\frac{1}{2}$.

2. Notation. The number $-d$ is the discriminant of an imaginary quadratic field $Q(\sqrt{-d})$, with $d > 4$. We let $Q(x, y) = ax^2 + bxy + cy^2$ be a reduced quadratic form of discriminant $-d$, with first minimum a, and we let $\sum_Q$ denote a sum over all reduced quadratic forms with discriminant $-d$. We put $\chi_1(n) = \left(\frac{-d}{n}\right)$, in the notation of Kronecker, while $\chi(n)$ is a real primitive character modulo k, $k > 1$.

We let $s = \sigma + it$ be a complex variable, while $\varrho = \beta + i\gamma$ is a nontrivial zero of the zeta function or of an L-function. To avoid confusion we let C denote Euler's constant, $C = 0.5771\ldots$ The letter θ denotes a real or complex number, possibly different on various appearances, such that $|\theta| \leqslant 1$.

As is customary we let

$$\Gamma(w, a) = \int_a^\infty e^{-y} y^{w-1}\, dy$$

be the partial gamma function, defined for $\operatorname{Re} a > 0$, and arbitrary complex w. Finally,

$$K_\nu(x) = \frac{1}{2}\int_1^\infty e^{-\frac{x}{2}(u+u^{-1})}(u^{\nu-1} + u^{-\nu-1})\, du$$

is the modified Bessel function of the second kind, where $\operatorname{Re} x > 0$, and ν is an arbitrary complex number.

3. Lemmas. In the following lemmas we assemble the mechanism that we use to obtain our result.

LEMMA 1. *If $(d, k) = 1$ then*

$$\left(\frac{kd^{1/2}}{2\pi}\right)^{s-\frac{1}{2}} \Gamma(s) L(s, \chi) L(s, \chi\chi_1) = T(s) + T(1-s) + U(s) \tag{1}$$

for all complex s, where

$$T(s) = \Gamma(s)\zeta(2s) P_k(s) A(s) \left(\frac{kd^{1/2}}{2\pi}\right)^{s-\frac{1}{2}},$$

$$U(s) = 4\pi^{1/2} k^{-1} \sum_Q a^{-1/2} \sum_{n=1}^{\infty} K_{s-\frac{1}{2}}\left(\frac{\pi n d^{1/2}}{ak}\right) n^{s-\frac{1}{2}} \sum_{y|n} y^{1-2s} \times$$
$$\times \operatorname{Re}\left\{\sum_{j=1}^{k} \chi(Q(j; y))\, e\left(\frac{jn}{ky}\right) e\left(\frac{bn}{2ak}\right)\right\},$$

$$P_k(s) = \prod_{p|k}(1 - p^{-2s}),$$

and

$$A(s) = \sum_Q \chi(a) a^{-s}.$$

Proof. This follows immediately from Corollary 2 of Stark [17] together with the functional equation for the zeta function in the form

$$\pi^{1/2}\Gamma(s-\tfrac{1}{2})\zeta(2s-1) = \pi^{2s-1}\Gamma(1-s)\zeta(2-2s).$$

We now multiply both sides of (1) by $s - \frac{1}{2}$, and we take $s = \frac{1}{2} + it$, $t \geqslant 0$, to obtain the identity

$$itL(\tfrac{1}{2}+it, \chi) L(\tfrac{1}{2}+it, \chi\chi_1) \Gamma(\tfrac{1}{2}+it) \left(\frac{kd^{1/2}}{2\pi}\right)^{it} = M(t)\sin\varphi(t) + \theta t E(t), \tag{2}$$

where

(3) $$M(t) = |2t\zeta(1+2it)\Gamma(\tfrac{1}{2}+it)P_k(1+2it)A(\tfrac{1}{2}+it)|,$$

(4) $$\varphi(t) = \arg\left(i\zeta(1+2it)\Gamma(\tfrac{1}{2}+it)P_k(1+2it)A(\tfrac{1}{2}+it)\left(\frac{kd^{1/2}}{2\pi}\right)^{it}\right),$$

and

(5) $$E(t) = 4\pi^{1/2}k^{-1}\sum_{Q} a^{-1/2}\sum_{n=1}^{\infty} K_0\left(\frac{\pi n d^{1/2}}{ak}\right)\sum_{y|n}\left|\sum_{j=1}^{k}\chi(Q(j,y))e\left(\frac{jn}{ky}\right)\right|.$$

In subsequent lemmas we give a lower bound for $M(t)$, we show that $\varphi(t)$ is almost linear for small t, and we bound $E(t)$ from above. When these estimates are sufficiently sharp we then deduce from (2) that the zeros of $L(s,\chi)L(s,\chi\chi_1)$ lie approximately in an arithmetic progression.

Lemma 2. *Let $F(s) = (2s-1)\zeta(2s)\Gamma(s)$. If $0 < t \leqslant 6$ then $|F(\frac{1}{2}+it)|$ is monotonically decreasing, and*

$$\arg F(\tfrac{1}{2}+it) = (C-\log 4)t + 30t^3.$$

Proof. From the definition of $F(s)$ it follows (see, for example, Section 12 of Davenport [5]) that

$$F(s) = \frac{\pi^s}{2s}e^{-2As}\prod_{\varrho}\left(1-\frac{2s}{\varrho}\right)e^{2s/\varrho},$$

where

$$A = \sum_{\varrho}\mathrm{Re}\frac{1}{\varrho} = \tfrac{1}{2}C+1-\tfrac{1}{2}\log 4\pi = 0.023\ldots,$$

and the product and sum are over all non-trivial zeros of the zeta function. On one hand

$$\frac{d}{dt}\mathrm{Re}\log F(\tfrac{1}{2}+it) = -\mathrm{Im}\frac{d}{d\sigma}\log F(\sigma+it)|_{\sigma=\frac{1}{2}}$$
$$= \mathrm{Im}\left((\tfrac{1}{2}+it)^{-1} - \sum_{\varrho}\left((1+2it-\varrho)^{-1}+(1+2it-\bar\varrho)^{-1}\right)\right) \leqslant 0$$

for $0 \leqslant t \leqslant 6$, since the zeros $\varrho = \beta+i\gamma$ all satisfy $|\gamma| \geqslant 14$. This gives the first result. On the other hand

$$\frac{d}{dt}\mathrm{Im}\log F(\tfrac{1}{2}+it) = \mathrm{Re}\frac{d}{d\sigma}\log F(\sigma+it)|_{\sigma=\frac{1}{2}}$$
$$= \log\pi - \frac{2}{1+4t^2} + 2\,\mathrm{Re}\sum_{\varrho}(\varrho+2it)^{-1}$$
$$= C + \frac{8t^2}{1+4t^2} - \log 4 + 2\,\mathrm{Re}\sum_{\varrho}\left((\varrho+2it)^{-1}-\varrho^{-1}\right).$$

We note that

$$\mathrm{Re}\left((\varrho+2it)^{-1}-\varrho^{-1}+(\bar\varrho+2it)^{-1}-\bar\varrho^{-1}\right) = \frac{8\beta t^2(3\gamma^2-\beta^2-4t^2)}{(\beta^2+\gamma^2)(\beta^2+(\gamma+2t)^2)(\beta^2+(\gamma-2t)^2)},$$

so that

$$0 < 2\,\mathrm{Re}\sum_{\varrho}\left((\varrho+it)^{-1}-\varrho^{-1}\right) < 24t^2\sum_{\varrho}\frac{\beta}{(\gamma^2-144)^2}$$

for $0 \leqslant t \leqslant 6$. It would not be difficult to estimate precisely the size of this last sum, but a crude upper bound suffices. Since $|\gamma| \geqslant 14$, we see that $(\gamma^2-144)^2 \geqslant 13(\gamma^2+1)$. Hence

$$\sum_{\varrho}\frac{\beta}{(\gamma^2-144)^2} < \frac{1}{13}A < \frac{1}{500}.$$

Thus

$$\arg F(\tfrac{1}{2}+it) = (C-\log 4)t + 8\int_0^t\frac{u^2}{1+4u^2}\,du + \frac{1}{3}\theta t^3.$$

Here the integral is $< \frac{1}{3}t^3$, so we have the result.

Lemma 3. *If $h = 2$ and $d \geqslant 10^{12}$, then $a = 1$ or $a \geqslant 3$. If h is odd then $a = 1$ or $a \geqslant (d/4)^{1/h}$.*

Proof. The first assertion was proved independently by Weinberger [20] and Kenku [7], following Stark [15], but in the spirit of Siegel [14]. To prove the second assertion we note that from the theory of genera it is clear that if h is odd then d is prime. Hence $(a,d) = 1$, so that in $Q(\sqrt{-d})$ the ideal (a) splits into two non-principal ideals. The desired result now follows from Lemma 5 of Weinberger [21] (see also Lemma 2 of Boyd and Kisilevsky [4]).

Lemma 4. *Let $M(t)$ be given by (3), and suppose that $d > 10^{12}$. If $0 < t \leqslant \frac{1}{20}$ and $h = 3$ then*

$$M(t) \geqslant \tfrac{3}{2}\prod_{p|k}(1-p^{-1}).$$

The above also holds if $0 < t \leqslant \frac{1}{4}$, $h = 2$, and if the first minimum a of the non-principal quadratic form is larger than 10^4. Finally, if $0 < t \leqslant \frac{1}{20}$ and $h = 2$ then

$$M(t) \geqslant \tfrac{2}{3}\prod_{p|k}(1-p^{-1}).$$

Proof. In Lemma 2 we established that $|F|$ is decreasing, so if $0 < t \leqslant \frac{1}{20}$ then

$$|F(\tfrac{1}{2}+it)| \geqslant |F(\tfrac{1}{2}+\tfrac{1}{20}i)| > \tfrac{7}{4};$$

if $0 < t \leqslant \frac{1}{4}$ then

$$|F(\tfrac{1}{2}+it)| \geqslant |F(\tfrac{1}{2}+\tfrac{1}{4}i)| > \tfrac{11}{7}.$$

The required values of $|\zeta(1+2it)|$ were computed by Haselgrove and Miller [6]. Clearly

$$|P_k(1+2it)| \geqslant \prod_{p|k}(1-p^{-1}),$$

so it remains to treat $A(s)$. If $h = 3$ then by the previous lemma $a = 1$ or $a \geqslant (d/4)^{1/3}$, so

$$|A(\tfrac{1}{2}+it)| \geqslant 1-2a^{-1/2} \geqslant 1-\tfrac{3}{100},$$

which suffices. If $h = 2$ and $a > 10^4$ then

$$|A(\tfrac{1}{2}+it)| \geqslant 1-a^{-1/2} \geqslant 1-\tfrac{1}{100},$$

and we obtain the second assertion. If $h = 2$ and $d \geqslant 10^{12}$ then by the previous lemma $a \geqslant 3$, so in any case

$$|A(\tfrac{1}{2}+it)| \geqslant 1-3^{-1/2} > \tfrac{2}{5}.$$

LEMMA 5. *Suppose that* $d > 10^{12}$, $0 \leqslant t \leqslant 6$, *and* $h = 2$ *or* $h = 3$. *Then*

$$\varphi(t) = t\left(C+\log\frac{kd^{1/2}}{8\pi}\right)+3\theta t^3+\theta t\left(c(h)+2\sum_{p|k}\frac{\log p}{p-1}\right),$$

where $c(2) = \frac{5}{3}$, *and* $c(3) = \frac{1}{300}$.

Proof. The first two terms arise from the definition of $\varphi(t)$ in (4), and from Lemma 2. In addition

$$\left|\frac{d}{dt}\arg P_k(1+2it)\right| = 2\left|\operatorname{Im}\frac{P_k'}{P_k}(1+2it)\right| \leqslant 2\sum_{p|k}\frac{\log p}{p-1},$$

so

$$\arg P_k(1+2it) = 2\theta t\sum_{p|k}\frac{\log p}{p-1}.$$

We use Lemma 3 to bound $A(s)$. If $h = 2$ and $d \geqslant 10^{12}$ then $a \geqslant 3$, so

$$\left|\frac{A'}{A}(\tfrac{1}{2}+it)\right| \leqslant \frac{\log a}{a^{1/2}-1} < \frac{5}{3}.$$

If $h = 3$ and $d > 10^{12}$ then $a \geqslant (d/4)^{1/3}$, so

$$\left|\frac{A'}{A}(\tfrac{1}{2}+it)\right| \leqslant \frac{2\log a}{a^{1/2}-2} < \frac{1}{300}.$$

The following lemmas are useful in bounding the error term $E(t)$.

LEMMA 6. *Let p be an odd prime, let χ be the quadratic character* (mod p), *and suppose that $p \nmid d$. Then*

$$\left|\sum_{j=1}^{p}\chi(Q(j,y))e\left(\frac{jm}{p}\right)\right| \leqslant \begin{cases} 2p^{1/2} & \text{if} \quad (y,m,p) = 1,\\ p & \text{if} \quad (y,m,p) = p.\end{cases}$$

Proof. The result is obvious if $p|y$. Otherwise the argument of Weil [19] (see his inequality (5)) gives the desired bound.

LEMMA 7. *Let q be an odd square-free number, and put $k = q$ or $k = 4q$. Let χ be the primitive quadratic character modulo k, and suppose that $(d,k) = 1$. Then*

$$\sum_{y|n}\left|\sum_{j=1}^{k}\chi(Q(j,y))e\left(\frac{jn}{ky}\right)\right| \leqslant 2^{\omega(k)}k^{1/2}\sum_{y|n}\prod_{p|k}\delta(n,y,p),$$

where $\delta(n,y,p) = \frac{1}{2}p^{1/2}$ *if* $p|y$ *and* $p|\frac{n}{y}$, *and* $\delta(n,y,p) = 1$ *otherwise.*

Proof. For given y the inner sum on the left can be written as a product over $p|k$ of sums of the sort in Lemma 6, with m so that $(m,p) = \left(\frac{n}{y},p\right)$. If $k = 4$ then the bound is trivial, for then the sum on the left is never larger than 2.

LEMMA 8. *Let*

$$f(a) = \sum_{n=1}^{\infty}d(n)\int_1^{\infty}e^{-anu}\frac{du}{u}$$

for $a > 0$. Then $f(a)$ is monotonically decreasing, and $f(a) \leqslant g(a)$, where

$$g(a) = a^{-1}e^{-a}\big(1+\log(1+a^{-1})\big).$$

Proof. The first assertion is trivial. Clearly

$$f(a) = \sum_{n=1}^{\infty}d(n)\int_{na}^{\infty}e^{-u}\frac{du}{u} = \int_a^{\infty}e^{-u}\Big(\sum_{n\leqslant\frac{u}{a}}d(n)\Big)\frac{du}{u}.$$

But

$$\sum_{n\leqslant x}d(n) = \sum_{n\leqslant x}\left[\frac{x}{n}\right] \leqslant x\sum_{n\leqslant x}\frac{1}{n} \leqslant x(1+\log x)$$

for $x \geqslant 1$, so

$$f(a) \leqslant \int_a^{\infty}e^{-u}\left(\frac{u}{a}\left(1+\log\frac{u}{a}\right)\right)\frac{du}{u} = a^{-1}\left(e^{-a}+\int_a^{\infty}e^{-u}\frac{du}{u}\right).$$

It is easily shown that

$$\int_a^\infty e^{-u}\frac{du}{u} - e^{-a}\log(1+a^{-1}) \nearrow 0,$$

so we have the result.

LEMMA 9. *Let $E(t)$ be as in (5), let $0 < t \leqslant \frac{1}{4}$, and let k be as in Lemma 7. Then*

$$E(t) \leqslant \left(\frac{8}{3^{1/4}\pi^{1/2}}\right)(h-1+e^{-d^{1/2}k^{-1}})\, d^{-1/4}k^{1/2}(\log k)\prod_{p|k}(2+3p^{-3/2}).$$

Proof. For $n \geqslant 1$ write $n = uv$, where u is the largest divisor of n such that (u, k^2) is square-free. In this notation the bound of Lemma 7 is

$$= 2^{\omega(k)}k^{1/2}(\log k)\, d(u)\prod_{p^a\|v}\left(1+\tfrac{1}{2}(a-1)p^{1/2}\right) \leqslant 2^{\omega(k)}k^{1/2}d(n)\prod_{p|v}(\tfrac{1}{2}p^{1/2}),$$

since $1+\frac{1}{2}(a-1)p^{1/2} \leqslant \frac{1}{2}p^{1/2}(a+1)$. Thus by Lemma 7 we see that the sum over n in $E(t)$ is

$$\leqslant 2^{\omega(k)}k^{1/2}\sum_{n=1}^\infty K_0\left(\frac{\pi n d^{1/2}}{ak}\right)d(n)\prod_{\substack{p|k\\ p^2|n}}(\tfrac{1}{2}p^{1/2})$$

$$= 2^{\omega(k)}k^{1/2}\sum_{r|k}\mu^2(r)\left(\prod_{p|r}(\tfrac{1}{2}p^{1/2})\right)\sum_{\substack{n=1\\ r^2|n}}^\infty d(n)K_0\left(\frac{\pi n d^{1/2}}{ak}\right).$$

From the definition of $K_0(x)$ and the fact that $d(n) \leqslant d(r^2)d(n/r^2)$ if $r^2|n$, it follows that the above is

$$\leqslant 2^{\omega(k)}k^{1/2}\sum_{r|k}\mu^2(r)f\left(\frac{\pi r^2 d^{1/2}}{2ak}\right)\prod_{p|r}(\tfrac{3}{2}p^{1/2}).$$

In Lemma 8 we note that $ag(a)$ is monotonically decreasing, so the above is

$$\leqslant 2^{\omega(k)}k^{1/2}g\left(\frac{\pi d^{1/2}}{2ak}\right)\sum_{r|k}\mu^2(r)\prod_{p|r}(\tfrac{3}{2}p^{-3/2}) = g\left(\frac{\pi d^{1/2}}{2ak}\right)k^{1/2}\prod_{p|k}(2+3p^{-3/2}).$$

Hence altogether

$$E(t) \leqslant 4\left(\frac{\pi}{k}\right)^{1/2}\left(\prod_{p|k}(2+3p^{-3/2})\right)\sum_Q a^{-1/2}g\left(\frac{\pi d^{1/2}}{2ak}\right).$$

Again $a^{1/2}g(a)$ is monotonically decreasing, and $a \leqslant (d/3)^{1/2}$, so we can replace the $a > 1$ by $(d/3)^{1/2}$. Thus the above sum over Q is

$$\leqslant 2k(\log k)(h-1+e^{-d^{1/2}/k})/(\pi(3d)^{1/4}),$$

and we are done.

4. Proof of the theorem. In this section we make use of information concerning the location of certain zeros $\frac{1}{2}+i\gamma$ of L-functions $L(s,\chi)$, where χ is the primitive quadratic character $(\mathrm{mod}\, k)$. In the next section we discuss how these zeros were calculated.

Table I

k	γ	$L(\frac{1}{2},\chi)$	$\log\frac{e^c k}{8\pi}$	π/γ
163	0.202901	6.8532×10^{-2}	2.44679	15.4834
427	0.249925	1.4376×10^{-1}	3.40983	12.5701
2683	0.156679	8.6867×10^{-2}	5.24774	20.0511
17923	0.030986	6.0171×10^{-3}	7.14688	101.387
28963	0.033774	5.8873×10^{-3}	7.62682	93.018
30895	0.018494	6.9746×10^{-3}	7.69139	169.875
37427	0.019505	3.1815×10^{-3}	7.88319	161.066
115147	0.003158	6.0362×10^{-5}	9.00701	994.931
123204	0.010650	2.5269×10^{-3}	9.07464	294.985
139011	0.012930	3.0484×10^{-3}	9.19535	242.966
145412	0.017312	5.8390×10^{-3}	9.24037	181.469
151419	0.021347	6.5712×10^{-3}	9.28085	147.168
188995	0.026513	5.5608×10^{-3}	9.50252	118.493

The values of γ printed above are correct to within one unit in the last place. We require only a few of the above k. We do not use the values of $L(\frac{1}{2},\chi)$; the quantities tabulated in the last two columns arise in the course of our proof. Note that we do not claim that $\frac{1}{2}+i\gamma$ is thenea rest zero to $\frac{1}{2}$, although we have no doubt that it is.

Suppose that $\frac{1}{2}+i\gamma$, $\gamma > 0$, is a zero of $L(s,\chi)$. We put $t = \gamma$ in (2), so that the left hand side vanishes. Hence if $(d,k) = 1$ then

$$M(\gamma)|\sin\varphi(\gamma)| \leqslant \gamma E(\gamma). \tag{6}$$

We consider $h = 3$ first. Here d must be prime, so $(d,k) = 1$ if $k < 10^{12} \leqslant d$. We first put $k = 17923$, which is prime, and let γ be the corresponding value in Table I. From Lemmas 4 and 9 we see that (6) requires that $|\sin\varphi(\gamma)| < \frac{2}{5}$. On the other hand, if $10^{12} \leqslant d \leqslant 10^{52}$, then from Lemma 5 we see that $\frac{1}{6}\pi \leqslant \varphi(\gamma) \leqslant \frac{5}{6}\pi$, so $\sin\varphi(\gamma) > \frac{1}{2}$. Thus there is no such d. If

$d > 10^{52}$ then we put $k = 115147 = 113 \cdot 1019$, and we let γ be the corresponding value in Table I. From (6) and Lemmas 4 and 9 we conclude that

$$|\sin\varphi(\gamma)| < 10^{-10}.$$

From Lemma 5 we see that if $10^{52} \leqslant d \leqslant 10^{2500}$ then $\frac{1}{15} \leqslant \varphi(\gamma) \leqslant 3\pi - \frac{1}{11}$, so if the above is to hold then

$$|\varphi(\gamma) - \pi| < 10^{-8} \quad \text{or} \quad |\varphi(\gamma) - 2\pi| < 10^{-8}.$$

These inequalities hold only if

$$(7) \qquad 10^{850} \leqslant d \leqslant 10^{860} \quad \text{or} \quad 10^{1705} \leqslant d \leqslant 10^{1735}.$$

To treat these remaining "short" intervals we take $k = 123204 = 4 \cdot 3 \cdot 10267$, and we let γ be the corresponding value in Table I. If $d > 10^{840}$ then $\gamma E(\gamma) M(\gamma)^{-1} < 10^{-200}$. For d in the intervals (7), we find that $(3+\frac{1}{4})\pi < \varphi(\gamma) < (3+\frac{1}{2})\pi$ or $(6+\frac{1}{2})\pi < \varphi(\gamma) < (6+\frac{7}{8})\pi$. Thus (6) does not hold for d satisfying (7), and we are done.

By continuing in the above manner one can clearly improve on the bound 10^{2500}; one should achieve 10^{10000} easily. If it became necessary one might push the bound to 10^{10^5}.

If $h = 2$ then our approach is more complicated because d is no longer prime, but instead $\omega(d) = 2$. However, if $d \geqslant 10^{12}$ then d can have at most one prime divisor $< 10^6$. Hence if $(k_1, k_2) = 1$, $k_i < 10^6$, $i = 1, 2$, then at least one of k_1 and k_2 is relatively prime to d.

Suppose that $h = 2$ and $10^{12} \leqslant d \leqslant 10^{52}$. We first suppose that $(d, 17923) = 1$, we put $k = 17923$, and let γ be the value in Table I. The bound provided by Lemma 4 is a little weaker than before, but the class number is smaller, so we find that $\gamma E(\gamma) M(\gamma)^{-1} < 3/7$. From Lemma 5 we see that we still have $\frac{1}{6}\pi < \varphi(\gamma) < \frac{5}{6}\pi$, so $\sin\varphi(\gamma) > \frac{1}{2}$. Thus (6) does not hold, and we conclude that if $10^{12} \leqslant d \leqslant 10^{52}$ then $17923 \mid d$. Hence $(d, 427) = (d, 37427) = 1$, and the first minimum of the non-principal quadratic form is $a = 17923 > 10^4$. We first put $k = 427 = 7 \cdot 61$, and let γ be the value in Table I. If $d \geqslant 10^{12}$ then $\gamma E(\gamma)\, M(\gamma)^{-1} < \frac{3}{8}$, but if $10^{12} \leqslant d \leqslant 10^{15}$ then $\pi + \frac{2}{3} < \varphi(\gamma) < 2\pi - \frac{2}{3}$, so (6) fails. We now assume that $10^{15} \leqslant d \leqslant 10^{52}$; we put $k = 37427 = 13 \cdot 2879$, and let γ be the value in Table I. We still have $17923 \mid d$, so $\gamma E(\gamma)\, M(\gamma)^{-1} < \frac{1}{10}$. On the other hand $\frac{1}{5} < \varphi(\gamma) < \frac{1}{2}\pi$ for $10^{15} \leqslant d \leqslant 10^{52}$, so (6) fails. This completes our treatment of the case $10^{12} \leqslant d \leqslant 10^{52}$.

We now suppose that $h = 2$ and $10^{52} \leqslant d \leqslant 10^{1200}$. We choose three values of k that are pairwise coprime; each k eliminates certain ranges of d for which $(d, k) = 1$. It is easy to check that each d, $10^{52} \leqslant d \leqslant 10^{1200}$,

is treated by at least two of these k, so $(d, k) = 1$ for one of them, as required. We first put $k = 115147$, and use the same zero as before. We find that $\gamma E(\gamma) M(\gamma)^{-1} < 10^{-9}$, so we must have $10^{850} < d < 10^{860}$. Next we take $k = 123204$ and use the same zero as before. Then $\gamma E(\gamma) M(\gamma)^{-1} < 10^{-8}$, so (6) holds only if $10^{244} < d < 10^{253}$, $10^{500} < d < 10^{509}$, $10^{756} < d < 10^{765}$, or $10^{1013} < d < 10^{1021}$. Finally we put $k = 37427 = 13 \cdot 2879$ and we use the zero given in Table I. Here $\gamma E(\gamma)^{-1} M(\gamma)^{-1} < 10^{-10}$, and we find that (6) holds only if $10^{129} < d < 10^{137}$, $10^{269} < d < 10^{277}$, $10^{409} < d < 10^{417}$, $10^{549} < d < 10^{557}$, $10^{689} < d < 10^{697}$, $10^{829} < d < 10^{837}$, $10^{969} < d < 10^{977}$, or $10^{1109} < d < 10^{1117}$.

We had some freedom in choosing our triple of pairwise coprime k. Among other triples that work, we note (30895, 115147, 139011), and (37427, 115147, 145412). If we had based our argument on close pairs of zeros of the zeta function then the complications arising from the condition $(d, k) = 1$ would not have arisen. The close pairs of zeros of found by Rosser, Schoenfeld, and Yohe [12] would in principle be suitable in this connection. When working with the zeta function with large t, a van der Corput bound for $K_u(x)$ provides a natural t analogue of the estimate of Lemma 7, so that in place of Lemma 9 one has an estimate of the sort

$$E(\gamma) \ll hd^{-1/4} t^{1/2} \log t.$$

5. The zeros. When calculating values of the zeta function it is customary to use an approximate functional equation. There is an approximate functional equation for L-functions, but the error terms are difficult to bound without undertaking extended calculations. Lavrik [8] has given a formula which meets our needs; in the present situation it takes the following form:

$$(8) \qquad \xi(s, \chi)$$
$$= \left(\frac{k}{\pi}\right)^{\frac{s+a}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s, \chi) = \left(\frac{k}{\pi}\right)^{\frac{s+a}{2}} \sum_{n=1}^{\infty} \chi(n) n^{-s} \Gamma\left(\frac{s+a}{2}, \pi n^2 z/k\right) +$$
$$+ \left(\frac{k}{\pi}\right)^{\frac{1-s+a}{2}} \sum_{n=1}^{\infty} \chi(n) n^{s-1} \Gamma\left(\frac{1-s+a}{2}, \pi n^2/(kz)\right).$$

Here $\operatorname{Re} z > 0$ and $a = \frac{1}{2}(1 - \chi(-1))$. If s is small then we simply take $z = 1$, but if t is large then it is useful to be able to take z so $\arg z$ is near $\pm\pi/2$. Lavrik derived his formula from the functional equation, but in fact one can easily deduce the above from the usual theta function relation, by slightly modifying the usual derivation of the functional equation.

The Dedekind zeta function of an imaginary quadratic field has an approximate functional equation similar to that in (8). If the field has discriminant $-d$ and $d > 4$ then we find that

$$(9)\quad \left(\frac{d^{1/2}}{2\pi}\right)^s \Gamma(s)\zeta(s)L(s,\chi_1) = \frac{h}{2s(s-1)} + \left(\frac{d^{1/2}}{2\pi}\right)^s \sum_{n=1}^{\infty} r(n)n^{-s}\Gamma(s, 2\pi nz/d^{1/2}) + \left(\frac{d^{1/2}}{2\pi}\right)^{1-s} \sum_{n=1}^{\infty} r(n)n^{s-1}\Gamma\big(1-s, 2\pi n^2/(d^{1/2}z)\big).$$

Here again $\operatorname{Re} z > 0$, and

$$r(n) = \sum_{m|n} \chi_1(m).$$

If $\chi(-1) = -1$ then we can put $d = k$, $\chi_1 = \chi$, and use (9) to compute $L(s,\chi)$. We note that $r(n) = 0$ for many values of n so that a calculation based on (9) instead of (8) involves the computation of rather fewer of the numbers $\Gamma(w, a)$. The k we consider are all such that $\chi(-1) = -1$, so we abandon $\chi(\text{mod } k)$, and instead consider $\chi_1(\text{mod } d)$.

We now set $z = 1$ in (9); in this case (9) is obtained from an identity of Siegel [13] by interchanging the order of summation and integration. If we put $1-s$ for s in (9) then the right hand side is unchanged; hence

$$\left(\frac{d^{1/2}}{2\pi}\right)^s \Gamma(s)\zeta(s)L(s,\chi_1) = \left(\frac{d^{1/2}}{2\pi}\right)^{1-s} \Gamma(1-s)\zeta(1-s)L(1-s,\chi_1).$$

Here we put $s = \frac{1}{2}+it$, and write the left hand side as $Z(t, d)$. Clearly $Z(t, d)$ is real. Hence if $0 < t_1 < t_2 \leqslant 14$, and if $Z(t_1, d)$ and $Z(t_2, d)$ are of opposite signs then there is at least one zero $\frac{1}{2}+i\gamma$ of $L(s,\chi_1)$ with $t_1 < \gamma < t_2$. Ancillary to his thesis research, Purdy [11] found several values of d for which $Z(0, d)$ is small. For some of these d, and for a few others, we have located a nearby change of sign for $Z(t, d)$. These are the γ in Table I.

In order to be sure that $Z(t, d)$ has changed sign it is necessary to know how small a calculated value of $|Z(t, d)|$ can, with certainty, be distinguished from zero. The errors in the calculated values of $Z(t, d)$ arise from truncating the infinite series, and from approximating the various terms, chiefly the partial gamma function, in the remaining finite sum. We estimate the truncation error first. Trivially $|\Gamma(\frac{1}{2}+it, a)| \leqslant \Gamma(\frac{1}{2}, a) < a^{-1/2}e^{-a}$, and $r(n) \leqslant d(n) < 2n^{1/2}$, so

$$\Big|\sum_{n>N} r(n)n^{-1/2-it}\Gamma(\tfrac{1}{2}+it, 2\pi n/d^{1/2})\Big| < \frac{d^{1/2}}{\pi}\left(\frac{d^{1/2}}{2\pi N}\right)^{1/2} e^{-2\pi N/d^{1/2}}.$$

We take N so that the right hand side above is $< \frac{1}{2}\cdot 10^{-7}$.

We conclude with a brief description of our calculations; the complete error analysis will appear elsewhere [22]. The computations were done on an IBM/360 model 67 computer in long floating arithmetic. We may

conservatively think of the computer as performing 15 digit decimal arithmetic. To calculate $\Gamma(\frac{1}{2}+it, a)$ for the values of a required in (9), we first calculated the

$$(10)\quad \int_{j/2}^{(j+1)/2} x^{-\frac{1}{2}+it}e^{-x}\,dx \qquad (6 \leqslant j \leqslant 25),$$

using six-point Gauss–Legendre integration, and the integral

$$(11)\quad \int_{13}^{\infty} x^{-\frac{1}{2}+it}e^{-x}\,dx,$$

using eight point Gauss–Laguerre integration. The error in each computed value is less than 10^{-12}. We now compute $\Gamma(\frac{1}{2}+it, a)$ in various ways, depending on the size of a. If $a > 13$, we simply compute the integral in the same way that we computed (11). If $3 \leqslant d \leqslant 13$ we calculate $\int_a^{j/2}$ for the j for which $j/2$ is closest to a, in the same way that we computed the integrals (10). Finally, if $0 < a < 3$ then we write simply

$$\int_a^3 x^{s-1}e^{-x}\,dx = \sum_{j=0}^{J-1} \frac{(-1)^j x^{j+s}}{j!(j+s)}\Big|_a^3 + \theta\left(\frac{x^{J+\frac{1}{2}}}{J!(J+\frac{1}{2})}\right)\Big|_a^3.$$

Here we took $J = 22$, so that the truncation error is 2.2×10^{-12} when $0 < a < 3$. Altogether we found that each value of $\Gamma(\frac{1}{2}+it, a)$ is known to within 10^{-10}. Thus the error in computing $Z(t, d)$ is substantially

$$2\left(\frac{d^{1/2}}{2\pi}\right)^{1/2} 10^{-10} \sum_{n\leqslant N} r(n)n^{-1/2}.$$

From Table II below we deduce that if $|Z(t, d)| > 10^{-7}$ then we can distinguish $Z(t, d)$ from 0.

Table II

d	h	$\sum_{n\leqslant N} r(n)n^{-1/2}$	$\sum_{n\leqslant N,\ r(n)>0} 1$
163	1	2.76	7
427	2	4.89	16
2683	5	8.15	32
17923	15	14.6	74
28963	16	13.7	92
30895	52	44.8	156
37427	38	31.4	137
115147	32	19.8	180
123204	92	56.1	422
139011	78	46.0	315
145412	92	54.2	368
151419	68	39.0	317
188995	48	26.2	270

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On Siegel's theorem

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1. Linnik [1] proved in an elementary way the famous theorem of Siegel, according to which the class number $h(-k)$ of the imaginary quadratic field belonging to the fundamental discriminant $-k<0$ satisfies $h(-k)>k^{1/2-\varepsilon}$ if $\varepsilon>0$ and $k>K_0(\varepsilon)$, where $K_0(\varepsilon)$ denotes an ineffective constant depending on ε.

The proof of Linnik is composed of the following 4 parts:

Definition. A real primitive character χ_k has the *property* $A(\beta)$ if there is such a constant $C_0(\chi_k,\beta)$, that for all $N>C_0(\chi_k,\beta)$ there exists an $N_1\epsilon[\sqrt{N},N]$ with the property

$$\Big|\sum_{n\leqslant N_1}\chi_k(n)\mu(n)\Big|>N_1^{\beta}.$$

1. If $h(-k)<k^{1/2-\varepsilon_1}$ with $0<\varepsilon_1<0.01$ for a sufficiently large k depending on ε_1, then $L(s,\chi_k)$ vanishes somewhere in the interval $[1-0.001\varepsilon_1, 1+0.001\varepsilon_1]$.

2. If $L(s,\chi_k)$ vanishes somewhere in the interval $[1-0.001\varepsilon_1, 1+ +0.001\varepsilon_1]$ then χ_k possesses the property $A(\beta)$ for $\beta=1-0.08\varepsilon_1$.

3. If χ_k possesses the property $A(\beta)$ with a $\beta>1/2$ and ε is an arbitrary number with $0<\varepsilon<\beta-1/2$, then there is such a constant $C_1(\beta,\varepsilon,\chi_k)$, that for all $N_1>C_1(\beta,\varepsilon,\chi_k)$ there exists an $N_2\epsilon[N^{2\beta-1-2\varepsilon},N_1]$, such that $\big|\sum_{n\leqslant N_2}\chi_k(n)\lambda(n)\big|>N_2^{\beta-\varepsilon}$.

4. If χ_k possesses the property expressed in the assertion of the 3rd statement for $\beta\geqslant 3/4$, then there is such a constant $C_2(\chi_k,\eta_0)$ that in case of $k>C_2(\chi_k,\eta_0)$ $h(-k)>k^{1/2-\eta(\beta)}$, where $\eta(\beta)=10.5(1-\beta)+\eta_0$ and η_0 is an arbitrary positive number.

In this paper, based on the above sketched order of ideas of Linnik, we give a simpler elementary proof of Siegel's theorem. Our proof will be completely elementary, we shall prove, that for arbitrary positive