Proof. By Lemma 6, we have

$$J_2 \ll NP^{1-\delta/2h}\Big[\int\limits_{\substack{E_2\\ |S(a_1\alpha)|\leqslant|S(a_2\alpha)|}}\!\!\cdots\!\int \prod_{j=2}^{s}|S(a_j\alpha)|L(\alpha)\{d\alpha\} + \int\limits_{\substack{E_2\\ |S(a_2\alpha)|\leqslant|S(a_1\alpha)|}}\!\!\cdots\!\int \prod_{j\neq 2}|S(a_j\alpha)|L(\alpha)\{d\alpha\}\Big].$$

Applying Hölder's inequality and Körner's theorem ([2], Satz 5), we obtain, as in [3], for $s-1\geqslant 2^m$ that $J_2 \ll NP^{1-\delta/2h}NP^{s-1-m+\delta/8h^3}NP^{\delta/4h}$, provided that P is large enough (to ensure that $P^{\delta/8h^2}$ exceeds a certain power of $\log P$) and Lemma 7 is proved.

As mentioned on p. 501 Lemmas 3, 4 and 7 together with (3) for $\delta\leqslant 1/4$ and large P, prove our Theorem.

Remark. It seems reasonable to expect that the condition $s\geqslant 2^m+1$ in the Theorem may be improved to $s\geqslant c'm\log m$ (for large m) as in Davenport–Roth [1].

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On the theorem of Gauss-Kusmin-Lévy and a Frobenius-type theorem for function spaces

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1. Introduction. If one wants to investigate the distribution of values of a_n in the regular continued-fraction expansion

$$a = [0;\ a_1, a_2, \ldots] := \cfrac{1}{a_1+\cfrac{1}{a_2+\ldots}},$$

where a varies randomly through the interval $(0, 1)$, one is readily led to considering the (Lebesgue-) measure $m_n(x)$ of the set

$$\{a;\ [0, a_{n+1}, a_{n+2}, \ldots] < x\},$$

where $0\leqslant x\leqslant 1$ (see for instance Khintchine [3]). Gauss [2], in a letter to Laplace, stated that

$$m_n(x)\to\frac{\log(1+x)}{\log 2}\quad\text{as}\quad n\to\infty.$$

The first one to publish a proof of this theorem was Kusmin [4] in 1928. Actually he proved that if we put

$$m_n(x) = \frac{\log(1+x)}{\log 2}+r_n(x)$$

then $r_n(x) = O(q^{\sqrt{n}})$ as $n\to\infty$, where q is some constant, $0<q<1$. Lévy [5] independently proved

$$r_n(x) = O(q^n)$$

by a different method (using probabilistic notions). As Szüsz [6] has shown this same result can also be obtained by Kusmin's approach. Szüsz' proof is easier than the two earlier ones and appears to give a smaller value ($q = 0.485$) than Lévy's $q = 0.7$ if one accepts the trouble of some calculation. He does not give all details though.

In the present paper we shall first show that Szüsz' proof itself can be simplified considerably. Without any numerical work we shall see that $|r_n(x)| \leqslant c_1 2^{-n}$ or rather

$$|r_n(x)| \leqslant \frac{c_1}{2^n} x(1-x),$$

finding at the same time a corresponding lower estimate

$$|r_n(x)| \geqslant \frac{c_2}{5^n} x(1-x),$$

which limits the range for possible improvements on the value of q. By more effective choices of a certain auxiliary function we can narrow the gap to

$$c_2(0,29)^n x(1-x) \leqslant (-1)^{n+1} r_n(x) \leqslant c_1(0,31)^n x(1-x)$$

and

$$c_2(0,302)^n x(1-x) \leqslant (-1)^{n+1} r_n(x) \leqslant c_1(0,305)^n x(1-x)$$

respectively, the first line being within reach of paper and pencil while for the second one a small computer is more adequate.

Given these results an obvious target emerges: To actually close rather than narrow the gap. The tool that serves this purpose is a theorem (formulated in 3.0) about the spectrum of certain positive linear operators on spaces of functions. Here we call an operator positive if it takes nonnegative functions into nonnegative functions. The corresponding theorem for finite dimensional spaces, due to Frobenius [1], states that a matrix M of positive elements has an eigenvector of positive components with a positive eigenvalue λ and that all other eigenvalues of M are of lesser modulus than λ. The straightforward generalization of Frobenius' theorem to infinite dimension is false. Our theorem therefore necessarily contains further conditions. These are boundedness below of the operator by some positive linear functional and the existence of a sufficiently good "approximate eigenfunction". On the other hand the theorem gives an explicit (and best possible) bound for the remaining spectrum. In this respect it may be new even in the finite dimensional case, where the extra conditions always hold.

For our problem the theorem yields (see 4.0)

$$r_n(x) = (-\lambda)^n \Psi(x) + O\big(x(1-x)\mu^n\big), \tag{1}$$

where λ and μ are constants, $0 < \mu < \lambda$, and Ψ in the first instance is a realvalued function on $[0,1]$ with a continuous second derivative and zeros at 0 and 1.

Both λ and Ψ can be computed by an iterative procedure but for neither can we give an explicit expression in terms of known functions or constants. We have

$$\lambda = 0.3036630029\ldots$$

In the rest of the paper (5.1–5.3) we show that Ψ is in fact holomorphic and that it can be extended holomorphically into the whole complex plane with a cut along the negative real axis from -1 to ∞. If z approaches any rational point on the cut, $\Psi(x)$ becomes logarithmically infinite. The cut is therefore the natural boundary of Ψ. Furthermore Ψ is subject to the functional equation

$$\Psi(z) - \Psi(z+1) = \frac{1}{\lambda}\Psi\left(\frac{1}{z+1}\right).$$

The following problem is left open. Is it possible to extract further "main terms" from the "error term" $O(\mu^n)$?

2.0. Like Kusmin and Szüsz we base our proof on the recursion

$$m_{n+1}(x) = \sum_{k=1}^{\infty}\left(m_n\left(\frac{1}{k}\right) - m_n\left(\frac{1}{k+x}\right)\right), \tag{2}$$

which follows easily from the definition of m_n. If we introduce a linear operator S by

$$(Sm)(x) := \sum_{k=1}^{\infty}\left(m\left(\frac{1}{k}\right) - m\left(\frac{1}{k+x}\right)\right) \tag{3}$$

we can simply write

$$m_{n+1} = Sm_n.$$

One sees easily that S has the eigenfunction $\log(1+x)$ with the eigenvalue 1. If m' exists and is bounded then (3) can be differentiated term by term and $(Sm)'$ is also bounded. Since $m_0(x) = x$ has these properties so, by induction, do all m_n. For the new functions

$$f_n(x) := (1+x)m_n'(x)$$

one finds in this way

$$f_{n+1} = Tf_n,$$

where

$$(Tf)(x) := \sum_{k=1}^{\infty} \frac{1+x}{(k+x)(k+1+x)} f\left(\frac{1}{k+x}\right). \tag{4}$$

This substitution improves convergence and simplifies the main term by taking $\log(1+x)$ into 1. So the constants, as can be checked immediately, are eigenfunctions of T. We can exploit this by differentiating our recursion again, because if the constants are eigenfunctions of an operator T then $(Tf)'$ is linear not only in f but even in f' (assuming that f has a continuous derivative):

$$(Tf)' = \frac{d}{dx} T\Big(f(0) + \int_0^x f'(y)\,dy\Big) = \frac{d}{dx} T \int_0^x f'(y)\,dy.$$

(This was already used in passing from S to T.) Hence, if we set

$$g_n(x) := f_n'(x)$$

we have

$$g_{n+1} = -Ug_n,$$

where U can be defined by

(5) $$(Tf)' = -U(f').$$

(We write $-U$ rather than U because this way U will turn out to be positive.) The essential point now is that by our sequence of substitutions the main term $\frac{\log(1+x)}{\log 2}$ is taken into zero and only the remainders are left in the recursion:

(6) $$g_n(x) = \big((1+x)m_n'(x)\big)' = \big((1+x)r_n'(x)\big)'.$$

Though g_n is essentially the second derivative of r_n estimating g_n is as good as estimating r_n since $m_n(0) = 0$ and $m_n(1) = 1$ (see the definition of m_n or (2)), whence

(7) $$r_n(0) = r_n(1) = 0.$$

To obtain an explicit expression for U we set $f' =: g$ and insert (4) into (5) noting that

$$\frac{1+x}{(k+x)(k+1+x)} = \frac{k}{k+1+x} - \frac{k-1}{k+x}.$$

We find

$$(Ug)(x) = -\frac{d}{dx}(Tf)(x)$$
$$= -\sum_{k=1}^{\infty}\Big(\frac{k-1}{(k+x)^2} - \frac{k}{(k+1+x)^2}\Big) f\Big(\frac{1}{k+x}\Big) +$$
$$+\sum_{k=1}^{\infty} \frac{1+x}{(k+x)^3(k+1+x)}\, g\Big(\frac{1}{k+x}\Big).$$

The first sum after partial summation becomes

$$\sum_{k=1}^{\infty} \frac{k}{(k+1+x)^2}\Big(f\Big(\frac{1}{k+x}\Big) - f\Big(\frac{1}{k+1+x}\Big)\Big),$$

hence

(8) $(Ug)(x)$
$$= \sum_{k=1}^{\infty}\Big\{\frac{k}{(k+1+x)^2}\int_{1/k+1+x}^{1/k+x} g(y)\,dy + \frac{1+x}{(k+x)^3(k+1+x)}\, g\Big(\frac{1}{k+x}\Big)\Big\}.$$

The representation shows that U is positive, as stated above. This implies, of course, that

$$U\varphi_1 \leqslant U\varphi_2 \quad \text{if} \quad \varphi_1 \leqslant \varphi_2.$$

2.1. In order to estimate $g_n = (-U)^n g_0$ we need a function φ with a positive lower and some upper bound,

$$0 < \alpha \leqslant \varphi(x) \leqslant \beta,$$

and such that

(9) $$s\varphi \leqslant U\varphi \leqslant t\varphi$$

with positive constants s and t. Given such a function we note that $m_0(x) = x$ and therefore

$$g_0(x) = 1, \quad \frac{1}{\beta}\varphi \leqslant g_0 \leqslant \frac{1}{\alpha}\varphi,$$

and apply U repeatedly:

$$\frac{1}{\beta} U^n\varphi \leqslant (-1)^n g_n \leqslant \frac{1}{\alpha} U^n\varphi,$$

(10) $$\frac{1}{\beta} s^n\varphi \leqslant (-1)^n g_n \leqslant \frac{1}{\alpha} t^n\varphi,$$

$$\frac{\alpha}{\beta} s^n \leqslant (-1)^n g_n \leqslant \frac{\beta}{\alpha} t^n.$$

To turn this into an estimate of r_n let $\xi := \log(1+x)$ and write $r_n(x) =: \varrho_n(\xi)$. Then

$$\varrho_n'(\xi) = (1+x)r_n'(x),$$
$$\varrho_n''(\xi) = (1+x)\big((1+x)r_n'(x)\big)' = (1+x)g_n(x),$$

and by (10)

(11) $$\frac{\alpha}{\beta} s^n \leqslant (-1)^n \varrho_n''(\xi) \leqslant 2\frac{\beta}{\alpha} t^n.$$

From (7) we see that

$$\varrho_n(0) = \varrho_n(\log 2) = 0,$$

hence, by interpolation for $0 \leqslant \xi \leqslant \log 2$,

$$\varrho_n(\xi) = -\xi(\log 2 - \xi)\,\varrho_n''(\xi^*), \tag{12}$$

where $0 < \xi^* < \log 2$. The factor $\xi(\log 2 - \xi)$ as a function of x has simple zeros at 0 and 1 and is positive in between. Combining (11) and (12) we therefore obtain

$$c_2 s^n x(1-x) \leqslant (-1)^{n+1} r_n(x) \leqslant c_1 t^n x(1-x). \tag{13}$$

2.2. It remains to find a function φ such that (9) holds with reasonably good constants s, t. Since (8) appears too complicated to be handled directly we turn back to considering T. To minimize computation we wish to use functions ψ for which the right hand side of (4) can be put into some finite form. Such functions can be found by formally inverting T. Let χ be a given function and let us look for a function ψ such that $T\psi = \chi$. We assume that χ is defined for $0 \leqslant x < \infty$ (not only in $0 \leqslant x \leqslant 1$) and that (4) holds there. Then

$$\frac{\chi(x)}{1+x} - \frac{\chi(1+x)}{2+x} = \frac{1}{(1+x)(2+x)}\,\psi\left(\frac{1}{1+x}\right),$$

which gives

$$\psi(x) = \left(\frac{1}{x}+1\right)\chi\left(\frac{1}{x}-1\right) - \frac{1}{x}\chi\left(\frac{1}{x}\right) \quad \text{for} \quad 0 < x \leqslant 1.$$

If $\chi(x) = o(x)$ as $x \to \infty$ then the function ψ defined in this way indeed fulfils $T\psi = \chi$:

$$(T\psi)(x) = (1+x)\sum_{k=1}^{\infty}\left(\frac{1}{k+x}\chi(k-1+x) - \frac{1}{k+1+x}\chi(k+x)\right)$$
$$= (1+x)\left(\frac{\chi(x)}{1+x} - \lim_{k\to\infty}\frac{\chi(k+x)}{k+1+x}\right) = \chi(x).$$

The particular functions that we shall employ are

$$\psi_a(x) := \frac{1+x}{1+ax} - \frac{1}{1+(a+1)x}$$

which are obtained from

$$(T\psi_a)(x) = \frac{1}{1+a+x},$$

where a is some parameter.

We return to U via (5). If we set

$$\varphi_a(x) := \psi_a'(x) = \frac{1-a}{(1+ax)^2} + \frac{1+a}{(1+(1+a)x)^2}$$

then

$$(U\varphi_a)(x) = -(T\psi_a)'(x) = \frac{1}{(1+a+x)^2}.$$

Looking in particular at $\varphi = \varphi_0$ we find

$$\frac{\varphi}{U\varphi}(x) = (1+x)^2 + 1 \begin{cases} \leqslant 5 \\ \geqslant 2 \end{cases}$$

and therefore

$$\tfrac{1}{5}\varphi \leqslant U\varphi \leqslant \tfrac{1}{2}\varphi.$$

Thus (13) is proved with $s = \frac{1}{5}$, $t = \frac{1}{2}$. The calculation is equally easy for $a = 1$ but gives only $s = \frac{1}{6}$, $t = \frac{1}{2}$. Values of a in $0 < a < 1$ give better bounds but require more computation. The extremals of

$$\frac{\varphi_a}{U\varphi_a}(x)$$

can be determined explicitely and there is at most one of them in $0 \leqslant x \leqslant 1$. For $a = 0.31266$ very nearly

$$\frac{\varphi_a}{U\varphi_a}(0) = \frac{\varphi_a}{U\varphi_a}(1).$$

Computing these values and the minimum that the quotient takes in between yields

$$0.29\,\varphi_a \leqslant U\varphi_a \leqslant 0.31\,\varphi_a,$$

proving (13) with $s = 0.29$, $t = 0.31$.

Further improvements can be made using linear combinations of different φ_a. Thus

$$\varphi := 8\varphi_{a_1} - 7\varphi_{a_2}$$

with $a_1 = 0.6247$ and $a_2 = 0.7$ gives

$$s = 0.3020, \quad t = 0.3043.$$

2.3. The question naturally arises whether U has an eigenfunction with eigenvalue λ between s and t. I am not aware of any known theorem that would apply here. On the one hand, due to the discrete component

$$\sum \frac{1+x}{(k+x)^3(k+1+x)}\,g\left(\frac{1}{k+x}\right),$$

U is not compact. On the other hand Frobenius' theorem does not readily generalize, in fact there are operators of a quite similar appearance that do not possess eigenfunctions (see 3.3). The property of U that we shall be able to exploit is that it can be split,

$$U = F + U_1,$$

into two positive operators of which one, F, is actually a functional, which means that F takes every function into a constant. We shall prove this assertion now.

Let $\varphi \geqslant 0$. Then

$$(U\varphi)(x) \geqslant \sum_{k=1}^{\infty} \frac{k}{(k+1+x)^2} \int_{1/k+1+x}^{1/k+x} \varphi(y)\,dy = \int_0^1 K(x,y)\varphi(y)\,dy,$$

where

$$K(x,y) = \frac{\left[\frac{1}{y}-x\right]}{\left(\left[\frac{1}{y}-x\right]+x+1\right)^2}$$

and where [ ] denotes the integral part. Let $y \leqslant \frac{1}{3}$. Then $\left[\frac{1}{y}-x\right] \geqslant 2$ and since $t(t+x+1)^{-2}$ as a function of t decreases for $t \geqslant 2$ it follows that

$$K(x,y) \geqslant \frac{\frac{1}{y}-x}{\left(\frac{1}{y}+1\right)^2} \geqslant \frac{\frac{1}{y}-1}{\left(\frac{1}{y}+1\right)^2},$$

$$K(x,y) \geqslant \frac{y(1-y)}{(1+y)^2}.$$

If $\frac{1}{3} < y \leqslant \frac{1}{2}$ then $K = (2+x)^{-2}$ or $K = 2(3+x)^{-2}$, in any case $K \geqslant \frac{1}{9}$. Therefore

(14) $$U\varphi \geqslant F\varphi \quad \text{if} \quad \varphi \geqslant 0$$

where

(15) $$F\varphi := \int_0^{1/3} \frac{y(1-y)}{(1+y)^2}\varphi(y)\,dy + \frac{1}{9}\int_{1/3}^{1/2} \varphi(y)\,dy.$$

3.0. As before we write $U \geqslant 0$ for an operator on a function space if $(Uf)(x) \geqslant 0$ for every non-negative real-valued function f and every x. Accordingly $U_1 \geqslant U_2$ means $U_1 - U_2 \geqslant 0$.

An operator that takes each function into a constant is called a functional.

THEOREM. *Let B be the space of continuous complex valued functions over some compact set and let $\|\cdot\|$ denote the supremum-norm.*

Let U be a bounded linear operator of B into itself and F a linear functional on B such that

(16) $$U \geqslant F \geqslant 0.$$

Assume that there are a function $\varphi \in B$ and two constants s, t, $0 < s \leqslant t$ such that

(17) $$\varphi(x) \geqslant 0 \quad \text{for all } x,$$

(18) $$s\varphi \leqslant U\varphi \leqslant t\varphi,$$

and

(19) $$F\varphi > \left(1-\frac{s}{t}\right)\|U\varphi\|.$$

Then U has a real-valued eigenfunction Φ with

(20) $$\inf \Phi(x) > 0$$

with an eigenvalue

$$\lambda \in [s, t],$$

and for each $g \in B$, as $n \to \infty$, we have

(21) $$U^n g = \lambda^n \Phi G(g) + O\left(\left(\lambda - \frac{F\Phi}{\|\Phi\|}\right)^n \|g\|\right)$$

where G is a positive bounded linear functional. Moreover

(22) $$\frac{F\Phi}{\|\Phi\|} \geqslant t\frac{F\varphi}{\|U\varphi\|} - t + s$$

which by (19) *is positive.*

We can (essentially) restate (21) by saying that there is a projection $Pf := G(f)\Phi$ that commutes with U and that $U_1 := U(1-P)$ has spectral radius $\leqslant \lambda - F\Phi\|\Phi\|^{-1}$.

Remarks. I. If equality is allowed in (19) one can no longer prove the existence of an eigenfunction. We shall give a counterexample.

II. The estimate of the error term in (22) is best possible in the sense that there are examples for which $\lambda - F\Phi\|\Phi\|^{-1}$ cannot be replaced by a smaller value.

III. In applications the assertion of the theorem can help to establish the assumptions; for if the theorem applies although no φ, s, t fulfilling (18) and (19) are known then iterating U on an essentially arbitrary function g (with $G(g) \neq 0$) will produce an approximate eigenfunction which can be used as φ.

IV. Though we have formulated the theorem for spaces of continuous functions over compact sets the proof in fact makes use of the following properties only: B is a vector space of real or complex valued functions over the real or complex numbers respectively, B is complete with respect to the supremum-norm, B contains the constant functions and, in the complex case, $f \in B$ implies $\bar{f} \in B$.

V. Instead of the functional F one can more generally use an operator that takes each function into a multiple of some fixed function $f > 0$. Instead of (17) we would write

$$U \geqslant fF \geqslant 0 \tag{23}$$

where F again is a functional. The space B would have to be understood with respect to the norm

$$\|\varphi\|_f := \sup_x \frac{|\varphi(x)|}{f(x)}.$$

This generalization is easily obtained from the theorem itself by the isomorphism $\bar{\varphi} := f^{-1}\varphi$, $\bar{U} := f^{-1}Uf$, $\bar{F} := Ff$.

3.1. Construction of the eigenfunction. Set

$$\varphi_n := U^n \varphi$$

and choose s_n, t_n optimally such that

$$s_n\varphi_n \leqslant \varphi_{n+1} \leqslant t_n\varphi_n. \tag{24}$$

One can assume $s_0 = s$, $t_0 = t$. Since U is positive we may apply it to each term in (24) and find $s_n\varphi_{n+1} \leqslant \varphi_{n+2} \leqslant t_n\varphi_{n+1}$, therefore $s_{n+1} \geqslant s_n$ and $t_{n+1} \leqslant t_n$ are obvious. We have to improve upon this.

From (24) and (16) we deduce

$$\varphi_{n+1} - s_n\varphi_n \geqslant 0,$$

$$\begin{aligned} \varphi_{n+2} - s_n\varphi_{n+1} = U(\varphi_{n+1} - s_n\varphi_n) &\geqslant F(\varphi_{n+1} - s_n\varphi_n) \\ &\geqslant \varphi_{n+1}\frac{1}{\|\varphi_{n+1}\|}F(\varphi_{n+1} - s_n\varphi_n), \end{aligned}$$

$$s_{n+1} \geqslant s_n + \frac{1}{\|\varphi_{n+1}\|}F(\varphi_{n+1} - s_n\varphi_n). \tag{25}$$

Similarly

$$t_n\varphi_n - \varphi_{n+1} \geqslant 0,$$

$$\begin{aligned} t_n\varphi_{n+1} - \varphi_{n+2} = U(t_n\varphi_n - \varphi_{n+1}) &\geqslant F(t_n\varphi_n - \varphi_{n+1}) \\ &\geqslant \varphi_{n+1}\frac{1}{\|\varphi_{n+1}\|}F(t_n\varphi_n - \varphi_{n+1}), \end{aligned}$$

$$t_{n+1} \leqslant t_n - \frac{1}{\|\varphi_{n+1}\|}F(t_n\varphi_n - \varphi_{n+1}). \tag{26}$$

The combination of (25) and (26) gives

$$t_{n+1} - s_{n+1} \leqslant (t_n - s_n)\left(1 - \frac{F\varphi_n}{\|\varphi_{n+1}\|}\right),$$

or shorter

$$d_{n+1} \leqslant d_n(1 - q_n), \tag{27}$$

where

$$d_n := t_n - s_n, \qquad q_n := \frac{F\varphi_n}{\|\varphi_{n+1}\|}.$$

Now (24) implies

$$F\varphi_{n+1} \geqslant Fs_n\varphi_n = s_nF\varphi_n \quad \text{and} \quad \|\varphi_{n+2}\| \leqslant \|t_{n+1}\varphi_{n+1}\| = t_{n+1}\|\varphi_{n+1}\|.$$

Hence

$$q_{n+1} \geqslant \frac{s_n}{t_{n+1}}q_n. \tag{28}$$

Whether (28) will keep the q_n large enough to force the d_n by (27) towards zero is a question of the initial values. In fact our assumption (19), which we may rewrite as

$$q_0 - \frac{d_0}{t_0} > 0, \tag{29}$$

ensures exponential decrease of the d_n:

$$\begin{aligned} t_{n+1}q_{n+1} - d_{n+1} &\geqslant s_nq_n - d_n(1 - q_n) = t_nq_n - d_n, \\ t_nq_n - d_n &\geqslant t_0q_0 - d_0, \\ q_n &\geqslant \frac{1}{t_n}(t_0q_0 - d_0) \geqslant q_0 - \frac{d_0}{t_0}, \end{aligned} \tag{30}$$

$$d_n \leqslant d_0\left(1 - q_0 + \frac{d_0}{t_0}\right)^n, \qquad d_n \to 0. \tag{31}$$

The common limit of s_n and t_n as $n \to \infty$ shall be called λ. Next we define

$$\tilde{\varphi}_0 := \varphi_0 = \varphi, \qquad \tilde{\varphi}_n := \varphi_n(s_0s_1\ldots s_{n-1})^{-1}.$$

This turns (24) into

$$\tilde{\varphi}_n \leqslant \tilde{\varphi}_{n+1} \leqslant \frac{t_n}{s_n}\tilde{\varphi}_n = \left(1 + \frac{d_n}{s_n}\right)\tilde{\varphi}_n \leqslant \left(1 + \frac{d_n}{s_0}\right)\tilde{\varphi}_n.$$

Because of (29) and (31)

$$\gamma := \prod_{n=1}^{\infty}\left(1 + \frac{d_n}{s_0}\right) < \infty$$

hence

$$\tilde{\varphi}_n \leqslant \prod_{i=0}^{n-1}\left(1+\frac{d_i}{s_0}\right)\tilde{\varphi}_0 \leqslant \gamma\varphi.$$

Furthermore

$$0 \leqslant \tilde{\varphi}_{n+1}-\tilde{\varphi}_n \leqslant \frac{d_n}{s_0}\tilde{\varphi}_n \leqslant d_n\frac{\gamma}{s_0}\varphi$$

thus $\sum\limits_n \|\tilde{\varphi}_{n+1}-\tilde{\varphi}_n\|$ converges and by the completeness of our space B

$$\Phi := \lim_{n\to\infty}\tilde{\varphi}_n$$

exists. Letting n pass to ∞ in

$$s_n\tilde{\varphi}_n \leqslant U\tilde{\varphi}_n \leqslant t_n\tilde{\varphi}_n$$

we obtain

$$U\Phi = \lambda\Phi.$$

Because of $\tilde{\varphi}_{n+1} \geqslant \tilde{\varphi}_n \geqslant \ldots \geqslant \tilde{\varphi}_0 = \varphi$ we have

$$\Phi \geqslant \varphi \geqslant \frac{1}{t}U\varphi \geqslant \frac{1}{t}F\varphi > 0,$$

which proves (20).

3.2. Estimation of the remaining spectrum. To prove (21) it obviously suffices to consider real-valued functions $g \in B$. Set

$$g_n := U^n g.$$

Because of (20) we can determine u_n, v_n such that

$$(32) \qquad u_n\lambda^n\Phi \leqslant g_n \leqslant v_n\lambda^n\Phi$$

where u_n is maximal and v_n minimal. From the left hand inequality we deduce

$$g_{n+1}-u_n\lambda^{n+1}\Phi = U(g_n-u_n\lambda^n\Phi) \geqslant F(g_n-u_n\lambda^n\Phi) \geqslant \Phi\frac{1}{\|\Phi\|}F(g_n-u_n\lambda^n\Phi),$$

$$u_{n+1} \geqslant u_n+\frac{1}{\lambda^{n+1}\|\Phi\|}F(g_n-u_n\lambda^n\Phi),$$

in particular $u_{n+1} \geqslant u_n$. Similarly

$$v_{n+1} \leqslant v_n-\frac{1}{\lambda^{n+1}\|\Phi\|}F(v_n\lambda^n\Phi-g_n),$$

whence

$$v_{n+1}-u_{n+1} \leqslant (v_n-u_n)\left(1-\frac{F\Phi}{\lambda\|\Phi\|}\right).$$

Concerning the right hand factor we refer to (30): Since

$$q_n = \frac{F\varphi_n}{\|U\varphi_n\|} = \frac{F\tilde{\varphi}_n}{\|U\tilde{\varphi}_n\|}$$

we see that

$$\frac{F\Phi}{\|\Phi\|} = \lambda\frac{F\Phi}{\|U\Phi\|} = \lim_{n\to\infty} t_n q_n \geqslant t_0 q_0 - d_0, \qquad \frac{F\Phi}{\|\Phi\|} \geqslant t\frac{F\varphi}{\|U\varphi\|}-t+s,$$

which is (22). If now by $G(g)$ we denote the common limit of the u_n and v_n we have

$$u_n, v_n = G(g)+O\left((v_0-u_0)\left(1-\frac{F\Phi}{\lambda\|\Phi\|}\right)^n\right),$$

therefore by (32)

$$U^n g = \lambda^n\Phi G(g)+O\left((v_0-u_0)\left(\lambda-\frac{F\Phi}{\|\Phi\|}\right)^n\right).$$

From the definition of u_0, v_0 we see $|u_0|, |v_0| \leqslant \|g\|(\inf\Phi)^{-1}$. So v_0-u_0 in the O-term can be replaced by $\|g\|$, proving (21). For the same reason $G(g)/\|g\|$ is bounded. The linearity and positivity of G are obvious consequences of (21).

3.3. Proof of remarks I and II. Let B be the space $C[0,1]$ of continuous functions on $[0,1]$ and define the operator U by

$$(Ug)(x) = g(0)+xg(x),$$

the functional F by

$$Fg = g(0).$$

Apparently $U \geqslant F$. The only functions Φ that are continuous on $[0,1)$ and fulfil $U\Phi = \lambda\Phi$ with some λ are easily determined as the multiples of

$$\Phi(x) = \frac{1}{1-x},$$

where $\lambda = 1$. But $\Phi \notin C[0,1]$ and so U has no eigenfunction. However, putting

$$\varphi_\varepsilon(x) := \begin{cases} \dfrac{1}{1-x} & \text{if} \quad 0 \leqslant x \leqslant 1-\varepsilon, \\ \dfrac{1}{\varepsilon} & \text{if} \quad 1-\varepsilon \leqslant x \leqslant 1 \end{cases}$$

we have

$$(U\varphi_\varepsilon)(x) = \begin{cases} \dfrac{1}{1-x} & \text{if} \quad 0 \leqslant x \leqslant 1-\varepsilon, \\ 1+\dfrac{x}{\varepsilon} & \text{if} \quad 1-\varepsilon \leqslant x \leqslant 1. \end{cases}$$

Thus $s = 1$, $t = 1+\varepsilon$,

$$\frac{F\varphi_s}{\|U\varphi_s\|} = \frac{\varepsilon}{1+\varepsilon} = 1-\frac{s}{t}$$

which proves remark I. It is not accidental, by the way, that in this example there are functions φ_s for which $1-\frac{s}{t}$ becomes arbitrarily small; if equality holds in (19) one can still prove $d_n \to 0$, since otherwise $d_n \geqslant \delta > 0$ and by (30) and (27)

$$t_n q_n \geqslant d_n \geqslant \delta, \quad q_n \geqslant \frac{\delta}{t_n} \geqslant \frac{\delta}{t_0}, \quad d_n \leqslant d_0\left(1-\frac{\delta}{t_0}\right)^n \to 0.$$

It is the later steps of the proof therefore that break down in the present case.

Now we consider $B = C[0, a]$ where $0 < a < 1$ and define U and F as before. This time $\Phi(x) := \frac{1}{1-x}$ is indeed an eigenfunction (with $\lambda = 1$). Iteration of U is easy:

$$(U^n g)(x) = (1+x+\ldots+x^{n-1})g(0)+x^n g(x) = \frac{g(0)}{1-x}+x^n\left(g(x)-\frac{g(0)}{1-x}\right),$$

and we see that the remainder in (21), which is $O(a^n\|g\|)$, cannot be improved upon.

4.0. To apply the theorem to our problem we could use the function φ_a as considered in 2.2 with $a = 0.31266$, for which we have $s = 0.29$, $t = 0.31$. Because of its slightly simpler expression, however, we use $U\varphi_a$ instead, for which the same values of s and t apply. That is, we have

$$\varphi(x) := \frac{1}{(1+a+x)^2}.$$

The functional F is the one given by (15) which after elementary calculation becomes

$$F\varphi = \frac{3a+4}{a^3}\log\frac{4(a+1)}{3a+4}+\frac{1}{8a}-\frac{1}{a^2}-\frac{2}{9(2a+3)}-\frac{1}{24(3a+4)}.$$

Because of $U\varphi \leqslant t\varphi$ we have

$$\frac{tF\varphi}{\|U\varphi\|} \geqslant \frac{F\varphi}{\|\varphi\|} = (1+a)^2 F\varphi > 0.033.$$

Since this is greater than $t-s = 0.02$ the theorem applies: U has a continuous positive valued eigenfunction Φ with an eigenvalue $\lambda \in [0.29, 0.31]$ and for $g_n = (-U)^n g_0$, $n \to \infty$

$$g_n = (-\lambda)^n G(g_0)\Phi + O(\mu^n)$$

with

$$\mu \leqslant \lambda - 0.013. \tag{33}$$

By a new normalization of Φ and G we can make

$$G(g_0) = 1.$$

Now we define the function Ψ by

$$\big((1+x)\Psi'(x)\big)' = \Phi(x), \quad \Psi(0) = \Psi(1) = 0. \tag{34}$$

By the relations between S, T and U we have

$$\big((1+x)(S\Psi)'(x)\big)' = -(U\Phi)(x) = -\lambda\Phi(x),$$

which together with $(S\Psi)(0) = 0$, $(S\Psi)(1) = \Psi(1) = 0$ implies

$$S\Psi = -\lambda\Psi.$$

The procedure of 2.1 now yields

$$m_n(x) = \frac{\log(1+x)}{\log 2}+(-\lambda)^n\Psi(x)+O\big(x(1-x)\mu^n\big),$$

which is assertion (1) of the introduction.

Working similarly on the function $\varphi = U(8\varphi_{a_1}-7\varphi_{a_2})$ with the values of a_1, a_2 specified at the end of 2.2 we can improve (33) to

$$\mu \leqslant \lambda - 0.031. \tag{35}$$

4.1. The theorem also provides the means for computing λ to any given accuracy. Handier than U for actual computation is the operator T. The influence of the constant eigenfunction can be eliminated by renormalizing after each step of the iteration. That is, if we consider a sequence $f_0, f_1, \ldots$

$$f_n := Tf_{n-1}^*, \quad f_n^*(x) := \frac{f_n(x)-f_n(0)}{f_n(1)-f_n(0)}$$

then from the theorem and relation (5) it is easily seen that

$$\prod_{\nu=0}^{n}\big(f_\nu(1)-f_\nu(0)\big) = (-\lambda)^n G(f_0')\int_0^1 \Phi(x)\,dx + O(\mu^n)$$

and therefore

$$f_n(0)-f_n(1) = \lambda+O\left(\frac{\mu^n}{\lambda^n}\right).$$

The value of λ given in the introduction was determined this way. The functions $f_n(x)$ were calculated for x in steps of 0.01 from -0.02 to 1.05. The additional points outside the interval $[0,1]$ allow the use of centered fifth-order interpolation in computing $f_n\left(\frac{1}{k+x}\right)$ for all these x and all k.

The first hundred terms of the series (4) were computed individually while for the remainder the Euler–Mac Laurin sum formula was applied, replacing f_n by its interpolation polynomial. The program was checked by starting from $f_0(x) := 1+x-(1+x)^{-1}$ (the function φ_0 of 2.2). The table of f_1^* did represent the function $2x(1+x)^{-1}$ within the computing accuracy of 10 digits.

The computation was carried further by R. Steinbach who found

$$\lambda = 0.3036630028987326586\ldots$$

5. In this paragraph we show the eigenfunctions Φ and Ψ of U and S respectively to be analytic. Their analytic continuations are holomorphic in C^*, the complex plane with a cut along the negative real axis from ∞ to -1, which constitutes the natural boundary of these functions. Because of (34) it suffices to deal with any one of them.

5.1. Analyticity of Φ in some rectangle. Let $\varepsilon > 0$ and

$$D_\varepsilon := \{x+iy;\ 0<x<1,\ |y|<\varepsilon\}.$$

If $z \in D_\varepsilon$ then $(z+k)^{-1} \in D_\varepsilon$. We consider the functions

$$\varphi_0(z) = 1, \quad \varphi_n = U^n\varphi_0$$

on D_ε, where U is defined by formula (8) with z instead of x and the integrals being taken along any paths inside D_ε. All φ_n are holomorphic in D_ε. We know that

$$\Phi(x) = \lim_{n\to\infty}\frac{\varphi_n(x)}{\lambda^n} \quad \text{for} \quad 0<x<1.$$

We shall show that the right hand side converges uniformly in D_ε if ε is small enough and thereby provides the analytic continuation of Φ.

Let

$$\delta_n := \frac{\varphi_n}{\lambda^n}-\frac{\varphi_{n+1}}{\lambda^{n+1}}.$$

By our theorem

$$(36)\qquad |\delta_n(x)| \leqslant c_1\left(\frac{\mu}{\lambda}\right)^n \quad \text{for} \quad 0<x<1,\ n=0,1,2,\ldots$$

with some constant c_1 and $0<\mu<\lambda$. Assuming, as we may, $\mu \geqslant \frac{1}{4}$ we shall prove

$$(\delta_n)\qquad |\delta_n(z)| \leqslant 2c_1\left(\frac{\mu}{\lambda}\right)^n \quad \text{for} \quad z\in D_\varepsilon,$$

if ε is sufficiently small. This will of course establish uniform convergence and therefore holomorphy of $\lim\lambda^{-n}\varphi_n$. Formula (δ_n) is proved by induction together with

$$(\delta_n')\qquad |\delta_n'(z)| \leqslant \frac{K}{|1+z|^2}\left(\frac{\mu}{\lambda}\right)^n \quad \text{for} \quad z\in D_\varepsilon$$

with suitable K.

Taking K sufficiently large we satisfy (δ_0') for all $\varepsilon \leqslant 1$.

By (36), (δ_n') implies (δ_n):

$$\delta_n(x+iy) = \delta_n(x)+\int_0^y \delta_n'(x+it)\,i\,dt,$$

$$|\delta_n(x+iy)| \leqslant (c_1+|y|K)\left(\frac{\mu}{\lambda}\right)^n \leqslant (c_1+\varepsilon K)\left(\frac{\mu}{\lambda}\right)^n \leqslant 2c_1\left(\frac{\mu}{\lambda}\right)^n,$$

provided

$$(37)\qquad \varepsilon K \leqslant c_1.$$

From (δ_n) and (δ_n') we prove (δ_{n+1}'). Differentiation of $\delta_{n+1} = \frac{1}{\lambda}U\delta_n$ produces

$$(38)\qquad \delta_{n+1}' = \frac{1}{\lambda}(V\delta_n+W\delta_n'),$$

where

$$(W\delta')(z) = -\sum_{k=1}^{\infty}\frac{1+z}{(k+z)^5(k+1+z)}\,\delta'\left(\frac{1}{k+z}\right)$$

and the operator V is bounded, say

$$(39)\qquad (V\delta)(w) \leqslant c_2\sup_z|\delta(z)| \quad \text{for} \quad w\in D_\varepsilon,\ \varepsilon\leqslant 1.$$

From (δ_n') it follows that

$$|(W\delta_n')(z)| \leqslant \frac{K}{|1+z|^2}\left(\frac{\mu}{\lambda}\right)^n\sum_{k=1}^{\infty}\left|\frac{1+z}{(k+z)(k+1+z)}\right|^3.$$

Since

$$\frac{1+z}{(k+z)(k+1+z)} = \left(2k+z+\frac{k(k-1)}{z+1}\right)^{-1}$$

and

$$\operatorname{Re}\left(z+\frac{k(k-1)}{z+1}\right)\geqslant 0 \quad \text{if} \quad \operatorname{Re} z\geqslant 0$$

we have

$$\sum_{1}^{\infty}\left|\frac{1+z}{(k+z)(k+1+z)}\right|^3\leqslant\sum\frac{1}{(2k)^3}=\frac{1}{8}\zeta(3)<\frac{1}{6},$$

$$|(W\delta_n')(z)|\leqslant\frac{K}{6|1+z|^2}\left(\frac{\mu}{\lambda}\right)^n. \tag{40}$$

Since furthermore $|1+z|^2\leqslant 5$ in D_ε if $\varepsilon\leqslant 1$ we finally obtain from (38), (39), (δ_n) and (40)

$$|\delta_{n+1}'(z)|=\frac{1}{\lambda}\left(2c_1c_2+\frac{K}{6|1+z|^2}\right)\left(\frac{\mu}{\lambda}\right)^n$$
$$\leqslant\frac{1}{|1+z|^2}\left(10c_1c_2+\frac{K}{6}\right)\frac{\mu^n}{\lambda^{n+1}}\leqslant\frac{K}{|1+z|^2}\cdot\frac{1}{4}\cdot\frac{\mu^n}{\lambda^{n+1}},$$

provided

$$10c_1c_2\leqslant\tfrac{1}{12}K. \tag{41}$$

Because of our assumption $\mu\geqslant\frac{1}{4}$ this proves (δ_{n+1}').

As (37) and (41) can be arranged by taking K big and ε small all (δ_n) follow now by induction and our proposition is proved; that is $\Phi(x)$ has a holomorphic extension $\Phi(z)$ into D_ε. Furthermore Φ is eigenfunction of U on D_ε:

$$U\Phi=U\lim_n\lambda^{-n}\varphi_n=\lim_n\lambda^{-n}U\varphi_n=\lim_n\lambda^{-n}\varphi_{n+1}=\lambda\lim_n\lambda^{-n-1}\varphi_{n+1}=\lambda\Phi.$$

5.2. Continuation into the slit plane. Since Φ is holomorphic in D_ε so is the function Ψ defined by (34). As in the real case we conclude that $S\Psi=-\lambda\Psi$, that is

$$\Psi(z)=\frac{1}{\lambda}\sum_{k=1}^{\infty}\left(\Psi\left(\frac{1}{k+z}\right)-\Psi\left(\frac{1}{k}\right)\right) \tag{42}$$

and

$$\Psi'(z)=-\frac{1}{\lambda}\sum_{k=1}^{\infty}\frac{1}{(k+z)^2}\Psi'\left(\frac{1}{k+z}\right)$$

for $z\in D_\varepsilon$. The boundedness of Φ on D_ε implies boundedness of Ψ' and therefore convergence of these series. Iterating n times leads to

$$\Psi'(z)=\left(\frac{-1}{\lambda}\right)^n\sum_{k_1,\ldots,k_n=1}^{\infty}\frac{1}{(q_n+zq_{n-1})^2}\Psi'(T_n(z)), \tag{43}$$

where

$$T_n(z):=\frac{p_n+p_{n-1}z}{q_n+q_{n-1}z}:=[0;k_1,\ldots,k_{n-1},k_n+z],$$

and the $p_\nu=p_\nu(k_1,\ldots,k_\nu)$, $q_\nu=q_\nu(k_1,\ldots,k_\nu)$ are determined by the well-known algorithm

$$p_{\nu+1}=k_{\nu+1}p_\nu+p_{\nu-1},\quad p_{-1}=1,\ p_0=0,$$
$$q_{\nu+1}=k_{\nu+1}q_\nu+q_{\nu-1},\quad q_{-1}=0,\ q_0=1.$$

So far (43) has been proved for $z\in D_\varepsilon$ only. We shall see, however, that the right hand side makes sense and defines an analytic function for z ranging through a much wider region.

Let

$$E_\delta:=\{z;\ -\pi+\delta<\arg z<\pi-\delta \text{ or } \operatorname{Re} z>-1+\delta\}.$$

The half plane $\operatorname{Re} z>-1$ is mapped by T_n onto the interior H_1 of the circle that intersects the real axis orthogonally at

$$T_n(\infty)=\frac{p_{n-1}}{q_{n-1}}\quad\text{and}\quad T_n(-1)=\frac{p_n-p_{n-1}}{q_n-q_{n-1}}=\frac{(k_n-1)p_{n-1}+p_{n-2}}{(k_n-1)q_{n-1}+q_{n-2}}.$$

If $n\geqslant 4$, both points are situated in $(0,1)$. Since the latter one lies between $\frac{p_{n-2}}{q_{n-2}}$ and $\frac{p_{n-1}}{q_{n-1}}$ and since $q_\nu\geqslant 2^{\nu/2}$, whatever $k_1,k_2,\ldots,k_n$, the diameter d_1 of the disk H_1 is

$$d_1\leqslant\left|\frac{p_{n-1}}{q_{n-1}}-\frac{p_{n-2}}{q_{n-2}}\right|=\frac{1}{q_{n-1}q_{n-2}}\leqslant 2\varepsilon$$

and therefore

$$H_1\subset D_\varepsilon \tag{44}$$

if only n is sufficiently large.

Similarly T_n maps the domain $-\pi+\delta<\arg z<\pi-\delta$ onto the union of the two disks H_2, $\overline{H}_2$ the boundaries of which meet the real axis at $T_n(0)=\frac{p_n}{q_n}$ and $T_n(\infty)=\frac{p_{n-1}}{q_{n-1}}$ at an angle δ. The diameter d_2 of these disks is

$$d_2=\frac{1}{\sin\delta}\left|\frac{p_n}{q_n}-\frac{p_{n-1}}{q_{n-1}}\right|=\frac{1}{q_nq_{n-1}\sin\delta}<\frac{1}{q_n}<\varepsilon$$

for large n. Since the points of intersection have at least distance $1/q_n$ from the end-points of the interval $(0,1)$ we see that

$$H_2\cup\overline{H}_2\subset D_\varepsilon. \tag{45}$$

By (44) and (45) all terms $\Psi'(T_n(z))$ are defined and uniformly bounded for $z \in E_\delta$ if $n \geqslant n_0(\varepsilon, \delta)$. To prove uniform convergence of (43) we note that if $z \in E_\delta$ then either $-\pi+\delta < \arg z < \pi-\delta$, and then

$$|q_n + zq_{n-1}| \geqslant q_n \sin\delta,$$

or $\operatorname{Re} z > -1+\delta$, and therefore

$$|q_n + zq_{n-1}| \geqslant q_n + q_{n-1}\operatorname{Re} z \geqslant q_n - q_{n-1}(1-\delta) \geqslant q_n - q_n(1-\delta) = q_n\delta,$$

hence in any case

$$\frac{1}{(q_n+zq_{n-1})^2}\,\Psi(T_n(z)) \ll \frac{1}{q_n^2} \ll \frac{1}{q_n(q_n+q_{n-1})}.$$

Finally

$$\sum_{k_1,\ldots,k_n} \frac{1}{q_n(q_n+q_{n-1})} = 1,$$

since

$$\frac{1}{q_n(q_n+q_{n-1})} = |[0; k_1, \ldots, k_{n-1}, k_n+1] - [0; k_1, \ldots, k_n]|.$$

Thus for $n \geqslant n_0(\varepsilon, \delta)$ formula (43) provides the analytic continuation of Ψ' into E_δ. As $\delta \to 0$ the regions E_δ exhaust the slit plane C^*. Therefore Ψ', Ψ and Φ are holomorphic in C^*.

Analytic continuation also validifies (42) for all $z \in C^*$. A functional equation is easily deduced from it:

$$\Psi(z) - \Psi(1+z) = \frac{1}{\lambda}\sum_k \left(\Psi\left(\frac{1}{k+z}\right) - \Psi\left(\frac{1}{k+1+z}\right)\right) = \frac{1}{\lambda}\left(\Psi\left(\frac{1}{1+z}\right) - \Psi(0)\right),$$

$$\Psi(z) - \Psi(1+z) = \frac{1}{\lambda}\Psi\left(\frac{1}{1+z}\right). \tag{46}$$

5.3. The singularities. In what follows it is necessary to know that the zero of Ψ at 0 is simple. Indeed by (46) we have

$$\Psi'(0) = \left(1 - \frac{1}{\lambda}\right)\Psi'(1),$$

by (34) on the other hand

$$2\Psi'(1) - \Psi'(0) = \int_0^1 \Phi(x)\,dx$$

together

$$-\frac{1+\lambda}{1-\lambda}\Psi'(0) = \int_0^1 \Phi(x)\,dx > 0.$$

It is possible, therefore, to normalize in such a way, changing Ψ into Ψ_0, that

$$\Psi_0'(0) = -\lambda. \tag{47}$$

With this normalization for $\delta > 0$, $\delta < \pi/2$, we show

$$\Psi_0(z) = \log z + O(1) \quad \text{as} \quad z \to \infty,\ |\arg z| < \pi - \delta, \tag{48}$$

$$\Psi_0(-1+w) = \frac{-1}{\lambda}\log w + O(1), \quad \text{as} \quad w \to 0,\ |\arg w| < \pi - \delta, \tag{49}$$

and for every rational $r > 1$, $r = [a_0; a_1, \ldots, a_n]$, say, where $a_0, \ldots, a_n$ are natural numbers, $a_n \geqslant 2$,

$$\Psi_0(-r+w) = -\left(1+\frac{1}{\lambda}\right)\frac{1}{\lambda^{n+1}}\log w + O(1), \tag{50}$$

$$\text{as } w \to 0,\ \delta < |\arg w| < \pi - \delta.$$

By (50) the cut from -1 to ∞ is natural boundary of Ψ.

Proof. If $|\arg z| < \pi - \delta$ then $|k+z| \geqslant k\sin\delta$. Thus, from (42) and (47) it follows that

$$\Psi(z) = \sum_k \left(\frac{1}{k} - \frac{1}{k+z} + O\left(\frac{1}{k^2}\right)\right) = \sum_k \left(\frac{1}{k} - \frac{1}{k+z}\right) + O(1),$$

which by a well-known theorem on $\dfrac{\Gamma'}{\Gamma}$ implies (48). From this (46) leads to (49):

$$\Psi(-1+w) = \Psi(w) + \frac{1}{\lambda}\Psi\left(\frac{1}{w}\right) = O(1) + \frac{1}{\lambda}\log\frac{1}{w} \quad (w \to 0).$$

Next follows (50) with $r = 2$ (that is $n = 0$, $a_0 = 2$):

$$\Psi(-2+w) = \Psi(-1+w) + \frac{1}{\lambda}\Psi\left(-1 - \frac{w}{1-w}\right)$$

$$= \frac{-1}{\lambda}\log w + \frac{-1}{\lambda^2}\log(-w) + O(1).$$

Finally the remaining cases of (50) are obtained by induction with respect to n and a_0: If $r > 1$, $w \to 0$, then

$$\Psi\left(-\left(1+\frac{1}{r}\right)+w\right) = \Psi\left(-\frac{1}{r}+w\right) + \frac{1}{\lambda}\Psi\left(-r - \frac{r^2w}{1-rw}\right)$$

$$= O(1) + \frac{1}{\lambda}\Psi\left(-r - \frac{r^2w}{1-rw}\right)$$

and

$$\Psi(-(r+1)+w) = \Psi(-r+w) + \frac{1}{\lambda}\Psi\left(\frac{1}{-r+w}\right) = \Psi(-r+w) + O(1).$$

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Notes on small class numbers

by

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1. Introduction. In this paper we study the problem of obtaining lower bounds for the class number $h = h(-d)$ of an imaginary quadratic field $Q(\sqrt{-d})$, $d > 0$. We recall that Siegel [13] has shown that $h > d^{\frac{1}{2}-\varepsilon}$ for $d > d_0$, and that his argument does not permit one to determine all fields with given class number. Recently the problem of obtaining effective lower bounds for h has received considerable attention (see, for example, Baker [1], [3], and Stark [16]).

From the Deuring–Heilbronn formulae it may be shown that if $h < d^{\frac{1}{2}-\delta}$ then all non-trivial zeros of certain L-functions are on the critical line, at least up to a height depending on d, on δ, and on the L-function. If the class number is somewhat smaller, $h < d^{\frac{1}{4}-\delta}$, then the imaginary parts of these zeros can also be described; it is found that the zeros are quite evenly spaced, so that two zeros of the same L-function cannot be very close together. To state this more precisely, let $\varrho = \frac{1}{2}+i\gamma$ and $\varrho' = \frac{1}{2}+i\gamma'$ be consecutive zeros on the critical line of an L-function $L(s, \chi)$, where χ is a primitive character (mod k). Put

$$\lambda(K) = \min \frac{1}{2\pi}|\gamma-\gamma'|\log K,$$

where the minimum is over all $k \leqslant K$, all χ (mod k), and all $\varrho = \frac{1}{2}+i\gamma$ of $L(s, \chi)$ with $|\gamma| \leqslant 1$. In this range the average of $|\gamma-\gamma'|$ is $2\pi/\log k$, so trivially $\overline{\lim}\,\lambda(K) \leqslant 1$. Presumably $\lambda(K)$ tends to 0 as K increases; if this could be shown effectively then the effective lower bound $h > d^{\frac{1}{2}-\varepsilon}$ would follow. In fact the weak inequality $\lambda(K) < \frac{1}{4}-\delta$ for $K > K_0$ implies that $h > d^{4\delta-\varepsilon}$ for $d > C(K_0, \varepsilon)$; the function $C(K_0, \varepsilon)$ can be made explicit. Even $\lambda(K) < \frac{1}{2}-\delta$ has striking consequences.

The initial remark of the previous paragraph makes it clear that in bounding $\lambda(K)$ one may assume that all the zeros of the L-functions under consideration are on the critical line. In this situation the techniques