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On a Diophantine inequality for forms of additive type

by

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Dedicated respectfully to Carl L. Siegel

§ 1. Let K be an algebraic number field of degree h over the field $\boldsymbol{Q}$ of rational numbers and let $\bar{K} = K \otimes_{\boldsymbol{Q}} \boldsymbol{R}$, the tensor product over $\boldsymbol{Q}$ of K with the field $\boldsymbol{R}$ of real numbers. Any element α in $\bar{K}$ can be represented as a diagonal matrix with diagonal elements $\alpha^{(1)}, \ldots, \alpha^{(h)}$ referred to as the 'conjugates' of α and assumed to be so ordered that $\alpha^{(i)} \in \boldsymbol{R}$ for $1 \leqslant i \leqslant r_1$ and $\alpha^{(k)} = \overline{\alpha^{(k+r_2)}}$ are complex for $r_1 < k \leqslant r_1 + r_2$ with $r_1 \geqslant 0$, $r_2 \geqslant 0$ and $r_1 + 2r_2 = h$. For $\alpha \in K$, $\alpha^{(1)}, \ldots, \alpha^{(h)}$ are just the conjugates of α over $\boldsymbol{Q}$. We define

$$\|\alpha\| = \max_{1 \leqslant k \leqslant h} |\alpha^{(k)}| \quad \text{for} \quad \alpha \in \bar{K}.$$

Let $m \geqslant 2$ be a natural number and let $f(x_1, \ldots, x_s) = \sum_{1 \leqslant r \leqslant s} a_r x_r^m$ be a *form of additive type over* $\bar{K}$, i.e. a polynomial in $x_1, \ldots, x_s$ of the above form with coefficients $a_1, \ldots, a_s$ which are invertible elements of $\bar{K}$. We call f *totally indefinite*, if, for every i with $1 \leqslant i \leqslant r_1$, there exist real numbers $p_{1i}, \ldots, p_{si}$ not all zero such that $\sum_{1 \leqslant r \leqslant s} a_r^{(i)} p_{ri}^m = 0$. Our object is to prove the following

Theorem. *Let $f(x_1, \ldots, x_s) = \sum_{1 \leqslant r \leqslant s} a_r x_r^m$ be a totally indefinite form of additive type over $K \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ which is not a scalar multiple of any polynomial in $x_1, \ldots, x_s$ with coefficients in K. If $s \geqslant 2^m + 1$, there exist, for any $\varepsilon > 0$, algebraic integers $\alpha_1, \ldots, \alpha_s$ not all zero in K such that $\|f(\alpha_1, \ldots, \alpha_s)\| < \varepsilon$.*

Remarks. This is an improved version of the Theorem stated in [3] and answers a question raised in [3], p. 300. The condition '$s \geqslant \max(2^m + 2, h2^{m-1}(m-1) + h^2 + h)$' of that Theorem is now replaced by the condition '$s \geqslant 2^m + 1$' which is clearly *independent of the degree h of K over $\boldsymbol{Q}$*. Further, the additional condition $mh \geqslant 4$ in [3] is no longer imposed here. We follow the same notation as in [3] and merely indicate the necessary modifications required to prove the Theorem stated above. For $m = 2$ this Theorem coincides with a well-known Theorem of Davenport and Heilbronn ([0], p. 158) for diagonal quadratic forms in 5 variables.

§ 2. Let $\{\omega_1, \ldots, \omega_h\}$ be a fixed basis of the ring $\mathfrak{o}$ of algebraic integers in K over the ring $\mathbf{Z}$ of rational integers. Let ϑ be the different and let $\{\varrho_1, \ldots, \varrho_h\}$ be the complementary $\mathbf{Z}$-basis of ϑ^{-1}. For $\alpha \in \bar{K}$, let $\sigma(\alpha)$ and $N\alpha$ denote the *trace* and the *norm* of α over $\mathbf{Q}$ respectively. For any ideal $\mathfrak{a}$, let $N\mathfrak{a}$ denote the norm of $\mathfrak{a}$. For $P > 0$, let $P\mathscr{B}_0 = \{\beta = \sum_{1\leqslant k\leqslant h} y_k\omega_k \in \bar{K} \mid -P \leqslant y_k < P$ for $1 \leqslant k \leqslant h\}$ and $\mathscr{R} = \{\alpha = \sum_{1\leqslant k\leqslant h} x_k\varrho_k \in \bar{K} \mid 0 \leqslant x_k < 1$ for $1 \leqslant k \leqslant h\}$. (We have identified here $\omega_k \otimes 1$ with ω_k and $\varrho_k \otimes 1$ with ϱ_k.) If f and g are two numbers or functions, we abbreviate "$|f| \leqslant \lambda\,|g|$ for an unspecified constant $\lambda > 0$ depending only on K" by "$f \ll g$". We also use the symbols O and o of Landau. For real $P > 0$, we write NP for P^h.

For $\alpha \in \bar{K}$ and a fixed number $P > 0$, we define the exponential sum

$$S(\alpha) = S(\alpha, P) = \sum_{x \in \mathfrak{o} \cap P\mathscr{B}_0} e^{2\pi i \sigma(\alpha x^m)}.$$

For $\beta \in \bar{K}$, we set

$$L(\beta) = \prod_{1\leqslant i\leqslant r_1} L_1(\beta^{(i)}) \prod_{r_1 < j \leqslant r_1+r_2} L_2(\beta^{(j)})$$

where $L_1(x) = (\sin\pi x/\pi x)^2$ or 1 according as the real number x is different from 0 or equal to 0 and $L_2(z) = \big(J_1(4\pi|z|)/\sqrt{2}\,|z|\big)^2$ or 1 according as the complex number z is different from or equal to 0 and J_1 denotes the usual Bessel function of order 1. For $\alpha \in \bar{K}$, let $\{d\alpha\}$ denote the volume element

$$\prod_{1\leqslant i\leqslant r_1} d\alpha^{(i)} \prod_{r_1<j\leqslant r_1+r_2} d(\operatorname{Re}\alpha^{(j)})\, d(\operatorname{Im}\alpha^{(j)}).$$

Then we have, for $\theta \in \bar{K}$,

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} L(y)\, e^{2\pi i\sigma(\theta y)}\{dy\} = M(\theta)$$

where

$$M(\theta) = \begin{cases} 0 & \text{if} \quad \|\theta\| > 1, \\ \prod_{1\leqslant i\leqslant r_1} (1-|\theta^{(i)}|) \prod_{r_1<j\leqslant r_1+r_2} \varphi(|\theta^{(j)}|) & \text{if} \quad \|\theta\| \leqslant 1 \end{cases}$$

with $\varphi(z) = 4\sin^{-1}(\sqrt{1-|z|^2}) - |z|\sqrt{1-|z|^2}$ or 0 according as $|z| \leqslant 1$ or $|z| > 1$.

To prove the theorem stated in § 1, it suffices to consider the case when $\varepsilon = 1$, since the case of general ε is deduced at once by taking $\varepsilon^{-1} f(x_1, \ldots, x_s)$ instead of $f(x_1, \ldots, x_s)$. We now assume that for every set of $x_1, \ldots, x_s$ not all 0 in $\mathfrak{o}$, we have

$$\|f(x_1, \ldots, x_s)\| \geqslant 1 \tag{1}$$

and derive a contradiction which will prove our Theorem.

For $\alpha \in \bar{K}$, we set $T(\alpha) = \prod_{1\leqslant j\leqslant s} S(a_j\alpha)$. Then for $\alpha = \sum_{1\leqslant k\leqslant h} \alpha_k\varrho_k$, we have

$$d\alpha_1 \ldots d\alpha_h = c\{d\alpha\} \tag{2}$$

for a constant c depending only on K. By the definition of $L(\alpha)$, we have

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} T(\alpha)L(\alpha)\, d\alpha_1 \ldots d\alpha_h = c \sum_{x_1,\ldots,x_s} M\Big(\sum_{1\leqslant j\leqslant s} a_j x_j^m\Big)$$

where the summation on the right hand side is over all s-tuples $x_1, \ldots, x_s$ of elements of $\mathfrak{o} \cap P\mathscr{B}_0$ subject to the restriction that $\big\|\sum_{1\leqslant j\leqslant s} a_j x_j^m\big\| \leqslant 1$. In view of the hypothesis (1) on f, we have

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} T(\alpha)L(\alpha)\, d\alpha_1 \ldots d\alpha_h = c. \tag{3}$$

We shall obtain a contradiction to (3) for large P under the hypotheses of the Theorem, by splitting up the domain of integration in (3) suitably.

Since $f(x_1, \ldots, x_s)$ is not a scalar multiple of any form over K, we may suppose, without loss of generality that $a_1 a_2^{-1}$ is not in K. We divide the whole space $-\infty < \alpha_i < \infty$ $(1 \leqslant i \leqslant h)$ into three mutually non-overlapping subsets E_1, E_2, E_3 defined by

$$E_1 = \{\alpha \in \bar{K} \mid \|\alpha\| \leqslant c_1 P^{-m+\delta/h}\},$$
$$E_2 = \{\alpha \in \bar{K} \mid c_1 P^{-m+\delta/h} < \|\alpha\| \leqslant P^{\hat{\delta}/4h^2}\},$$
$$E_3 = \{\alpha \in \bar{K} \mid \|\alpha\| > P^{\hat{\delta}/4h^2}\}$$

where $\delta, \hat{\delta}$ are fixed real numbers such that $0 < \delta < 1$, $0 < \hat{\delta} < 1$ and c_1 is a positive constant so chosen that for $\alpha \in E_2$ and $a_i\alpha = \sum_{1\leqslant j\leqslant h} c_{ij}\varrho_j$, we have

$$\max_{1\leqslant j\leqslant h} |c_{ij}| > P^{-m+\delta/h} \quad \text{for} \quad i = 1, 2.$$

Remark. The change-over from $P^{1-m-\delta}$ in the definition of E_1 and E_2 in [3], p. 302, to the present $P^{-m+\delta/h}$ was suggested by arguments of K. Ramachandra in a paper *On the sums* $\sum_{j=1}^{K} \lambda_j f_j(p_j)$ (to appear).

Let

$$J_i = \int_{E_i} T(\alpha)L(\alpha)\, d\alpha_1 \ldots d\alpha_h \quad \text{for} \quad i = 1, 2, 3.$$

Then $J_1 + J_2 + J_3 = c$, from (3). We show that J_1 has an estimate from below involving P while J_2 and J_3 have upper estimates involving P which are of a strictly lower order than the lower estimate for J_1. The contradiction required for establishing the Theorem is obtained by letting P tend to infinity.

§ 3. We get here a lower estimate for J_1. For any $P>0$, let $Y(P)=\{x\in\bar{K} \mid \|x\|<P\}$ and for $\theta\in\bar{K}$, let $I(\theta)=\int\limits_{Y(P)}\dots\int e^{2\pi i\sigma(\theta x^m)}\{dx\}$. For any $\gamma\in K$, we may write $(\gamma)\vartheta=\mathfrak{b}\mathfrak{a}_\gamma^{-1}$ with coprime integral ideals $\mathfrak{a}_\gamma$ and $\mathfrak{b}$; we refer to $\mathfrak{a}_\gamma$ as the "denominator" of $(\gamma)\vartheta$ and for $\gamma=0$, we take $\mathfrak{a}_\gamma=\mathfrak{o}$. Let

$$G(\gamma)=\Big(\sum_{\mu \bmod \mathfrak{a}_\gamma} e^{2\pi i\sigma(\gamma\mu^m)}\Big)/N\mathfrak{a}_\gamma,$$

where μ runs over a complete set of representatives of residue classes of $\mathfrak{o}$ modulo $\mathfrak{a}_\gamma$. Then we have

Lemma 1 (Siegel). *For $\alpha\in\bar{K},\gamma\in K$ with $N\mathfrak{a}_\gamma\leqslant P^{\delta/2h}$ and $\|\alpha-\gamma\|\leqslant c_1P^{-m+\delta/h}/N\mathfrak{a}_\gamma$, we have*

$$S(\alpha)=c^{-1}G(\gamma)I(\alpha-\gamma)+O(P^{h-1+\delta/h}) \tag{4}$$

where c is the constant defined by (2). Moreover,

$$I(\alpha-\gamma)=O\big(P^hN(\min(1,P^{-1}|\alpha-\gamma|^{-1/m}))\big). \tag{5}$$

The proof is exactly the same as in Siegel [5], p. 128.

Lemma 2.

$$\Big|\int\limits_{E_1}\dots\int\sum_{1\leqslant j\leqslant s} I(a_j\alpha)L(\alpha)\{d\alpha\}\Big|\gg NP^{s-m},\quad \textit{for}\quad s>2m.$$

Proof. To prove the lemma, it is enough to show that for some $\varrho>0$, we have

$$\int\limits_{\|\alpha\|\geqslant c_1P^{-m+\delta/h}}\dots\int\prod_{1\leqslant j\leqslant s} I(a_j\alpha)L(\alpha)\{d\alpha\}\ll NP^{s-m-\varrho} \tag{6}$$

and further

$$\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\prod_{1\leqslant j\leqslant s} I(a_j\alpha)L(\alpha)\{d\alpha\}\gg NP^{s-m}. \tag{7}$$

Taking $\gamma=0$ in (5), we get for the left hand side of (6), the upper bound

$$NP^s\int\limits_{\|\alpha\|\geqslant P^{-m+\delta/h}}\dots\int\prod_{1\leqslant j\leqslant s}N(\min(1,P^{-1}|a_j\alpha|^{-1/m}))L(\alpha)\{d\alpha\}$$

$$\ll NP^s\int\limits_{\|\alpha\|\geqslant P^{-m+\delta/h}}\dots\int N(\min(1,|P^m\alpha|^{-s/m}))L(\alpha)\{d\alpha\}$$

$$\ll NP^{s(1-1/h)}\int_{P^{-m+\delta/h}}^{\infty} t^{-s/m}t^{\lambda}P^{-m(h-\lambda-1)}\,dt$$

with $\lambda=0$ or 1, in view of the fact that

$$1+P^{(\lambda+1)m-s}P^{(\lambda+1-s/m)(-m+\delta/h)}=1+P^{-\delta(s-m\lambda-m)/(mh)}\leqslant 2$$

for $s\geqslant 2m, P\geqslant 1$ and $\lambda=0$ or 1. We are thus led to the estimation (6) with $\varrho=(s-2m)\delta/mh^2>0$ for $s>2m$. The estimate (7) is obtained exactly as in Lemma 2 of [3].

Lemma 3. *For $s\geqslant m^2+1$ and $\delta\leqslant 1/4$, we have $J_1\gg NP^{s-m}$.*

Proof. From Lemma 1, we have

$$T(\alpha)=\prod_{1\leqslant j\leqslant s}S(a_j\alpha)=\big(c^{-1}G(\gamma)\big)^s\prod_{1\leqslant j\leqslant s} I\big(a_j(\alpha-\gamma)\big)+$$
$$+O(NP^{s-1/h+\delta/h^2})N\big(\min(1,|P^m(\alpha-\gamma)|^{-(s-1)/m})\big)+O(NP^{s-s/h+s\delta/h^2}).$$

On the other hand, we know from Siegel ([4], p. 335) that for $a>m$,

$$\int_0^1\dots\int_0^1 N\Big(\min\Big(1,\Big|P^m\sum_{1\leqslant k\leqslant h}x_k\omega_k\Big|^{-a/m}\Big)\Big)dx_1\dots dx_h=O(NP^{-m}).$$

Using these two estimates with $\gamma=0$, we obtain for $s\geqslant m^2+1$ that

$$\Big|J_1-\int\limits_{E_1}\dots\int\prod_{1\leqslant j\leqslant s} I(a_j\alpha)L(\alpha)\,d\alpha_1\dots d\alpha_h\Big| \tag{8}$$
$$\ll NP^{s-1/h+\delta/h^2}\int\limits_{E_1}\dots\int N\big(\min(1,|P^m\alpha|^{-(s-1)/m})\big)L(\alpha)\,d\alpha_1\dots d\alpha_h+$$
$$+NP^{s-s/h+s\delta/h^2}\int\limits_{E_1}\dots\int L(\alpha)\,d\alpha_1\dots d\alpha_h.$$

Since $L(\beta)\ll 1$ for $\beta\in\bar{K}$ and $L(\gamma)\ll|N\gamma|^{-2}$ for invertible γ in $\bar{K}$, we have for the right hand side of (8), the estimate

$$NP^{s-m-1/h+\delta/h^2}+NP^{s-m-s/h+(s+h)\delta/h^2}\ll NP^{s-m-\varrho'}$$

for a $\varrho'>0$ and $\delta\leqslant 1/4$.

§ 4. An upper estimate for J_3 is provided by

Lemma 4. *For $s\geqslant 2^m$ and large P, we have $J_3\ll NP^{s-m-\delta/8h^3}$.*

The proof is on the same lines as in Lemma 4 in [3].

For the estimation of J_2, we have to slightly modify the definition of major arcs used in [3]. For the given P, δ, the **Z**-basis $\{\varrho_1,\dots,\varrho_h\}$ of ϑ^{-1} and $\gamma=\sum_{1\leqslant k\leqslant h}\gamma_k\varrho_k$ in K with $N\mathfrak{a}_\gamma\leqslant P^{\delta/2h}$, we define the *major arc* $\hat{B}_\gamma$ as the set of $x=\sum_{1\leqslant k\leqslant h}x_k\varrho_k\in\bar{K}$ for which $|x_k-\gamma_k|\leqslant P^{-m+\delta/h}/N\mathfrak{a}_\gamma$ for $1\leqslant k\leqslant h$. Let $\mathfrak{m}$ denote the complement in $\mathscr{R}$ of the union of all $\hat{B}_\gamma$ for $\gamma\in K$ with $N\mathfrak{a}_\gamma\leqslant P^{\delta/(2h)}$. For $0<\theta<1$, let $\mathfrak{m}_0=\{\alpha\in\mathfrak{m}\mid$ there do not exist $\lambda\neq 0$ in $\mathfrak{o}\cap P^{(m-1)\theta+\delta/(4h^2)}\mathscr{B}_0$ and μ in ϑ^{-1} such that for $\lambda\alpha-\mu=\sum_{1\leqslant k\leqslant h}\varepsilon_k\varrho_k$ we have $|\varepsilon_k|<P^{-m+(m-1)\theta+\delta/(2h)}$ for $1\leqslant k\leqslant h\}$. For small enough θ and large P, we claim that $\mathfrak{m}_0=\mathfrak{m}$. If possible, let otherwise, there exist α

in $\mathfrak{m}$ and not in $\mathfrak{m}_0$. Then there exist suitable λ, μ as referred to above. Setting $\gamma = \mu/\lambda$, we have $\lambda \in \mathfrak{a}_\gamma$ and further, if $(m-1)\theta < \delta/(4h^2)$, then

$$N\mathfrak{a}_\gamma \leqslant |N\lambda| \ll NP^{(m-1)\theta+\delta/(4h^2)} \leqslant P^{\delta/(2h)}.$$

Now $\alpha - \gamma = \lambda^{-1} \sum_{1\leqslant i\leqslant h} \varepsilon_i \varrho_i = \sum_{1\leqslant j\leqslant h} \beta_j \varrho_j$ with $\beta_j = \sum_{1\leqslant i\leqslant h} a_{ij}\varepsilon_i$, (a_{ij}) being the regular representation matrix of $1/\lambda$ with respect to the basis $\{\varrho_1, \ldots, \varrho_h\}$ of K over Q. Now

$$|a_{ij}| \ll P^{(h-1)((m-1)\theta+\delta/(4h^2))}/|N\lambda| \ll P^{(h-1)\delta/(2h^2)}/|N\lambda|$$

and

$$|\beta_j| \ll P^{-m+(m-1)\theta+\delta/(2h)+(h-1)\delta/(2h^2)}/N\mathfrak{a}_\gamma \leqslant P^{-m+\delta/h}/N\mathfrak{a}_\gamma$$

for $1 \leqslant i, j \leqslant h$, contradicting the fact that $\alpha \in \mathfrak{m}$. This establishes our claim above.

LEMMA 5. *For $\theta < \delta/(4h^2(m-1))$ and large P, $\mathfrak{m}_0 = \mathfrak{m}$ and hence, for $\alpha \in \mathfrak{m}$,*

$$|S(\alpha)| \ll NP^{1-\delta/(2^{m+1}h^2(m-1))}.$$

The proof is the same as that of Lemma 10 of [3].

LEMMA 6. *Let a_1, a_2 be invertible elements of $\bar{K}$ such that $a_1 a_2^{-1} \notin K$. Then, for $\alpha \in E_2$ and large P,*

$$\min(|S(a_1\alpha)|, |S(a_2\alpha)|) \ll NP^{1-\hat{\delta}/2h} \tag{9}$$

provided that $\hat{\delta} = \delta/(2^m h(m-1))$.

Proof. Let $a_i\alpha = \sum_{1\leqslant j\leqslant h} c_{ij}\varrho_j$ with $c_{ij} \in \boldsymbol{R}$ and $\mu_i = \sum_{1\leqslant j\leqslant h} b_{ij}\varrho_j$ with $b_{ij} \in \boldsymbol{R}$ such that $c_{ij} - b_{ij} \in \boldsymbol{Z}$ and $0 \leqslant b_{ij} < 1$ for $i = 1, 2$ and $1 \leqslant j \leqslant h$. Then $\mu_i \in \mathscr{R}$ and $S(a_i\alpha) = S(\mu_i)$ for $i = 1, 2$. Suppose that $\mu_i \in \hat{B}_\gamma$ with $N\mathfrak{a}_\gamma \leqslant P^{\delta/2h}$. Then, by Lemma 1, we have

$$|S(a_i\alpha)| \ll NP/(N\mathfrak{a}_\gamma)^{1/m} + O(NP^{1-1/h+\delta/h^2}).$$

If $\mu_i \notin \hat{B}_\gamma$ for any $\gamma \in K$ with $N\mathfrak{a}_\gamma \leqslant P^{\delta/2h}$, then $\mu_i \in \mathfrak{m}$ ($= \mathfrak{m}_0$ for some θ and large P) so that $|S(a_i\alpha)| = |S(\mu_i)| \ll NP^{1-\hat{\delta}/2h}$ in view of Lemma 5. Thus, if either μ_1 or μ_2 is in $\mathfrak{m}$ or belongs to some $\hat{B}_\gamma$ with $P^{m\delta/(2^{m+1}h(m-1))} \leqslant N\mathfrak{a}_\gamma \leqslant P^{\delta/2h}$, then $|S(a_i\alpha)| \ll NP^{1-\hat{\delta}/2h}$. In order to prove the Lemma completely, it suffices to show that *the case*

$$\mu_1 \in \hat{B}_{\gamma_1},\ \mu_2 \in \hat{B}_{\gamma_2} \quad \text{with} \quad N\mathfrak{a}_{\gamma_i} \leqslant P^{m\hat{\delta}/2h}, \quad i = 1, 2,$$

does not arise at all. Writing $\gamma_i = \sum_{1\leqslant j\leqslant h} e_{ij}\varrho_j$ and $\gamma_i' = \gamma_i - \mu_i + a_i\alpha = \sum_{1\leqslant j\leqslant h} g_{ij}\varrho_j$ for $i = 1, 2$, we see that $(\gamma_i')\vartheta$ has the same denominator as $(\gamma_i)\vartheta$. Now

$$|e_{ij} - g_{ij}| \leqslant P^{-m+\delta/h}/N\mathfrak{a}_{\gamma_i} \quad \text{for } 1 \leqslant j \leqslant h \text{ and } i = 1, 2.$$

Observe that, for fixed i, not all the numbers g_{ij} $(1 \leqslant j \leqslant h)$ can vanish, since, otherwise, this would contradict α being in E_2. For $i = 1, 2$, let us write $\gamma_i' = t_i/u_i$ with $u_i, t_i \in \mathfrak{o}$ having their greatest common ideal-divisor belonging to a fixed finite set of integral ideals in K. Then $N\mathfrak{a}_{\gamma_i}$ is the same as $|Nu_i|$ except for a positive constant depending only on K. From above, we have

$$a_i\alpha = t_i u_i^{-1}(1 + O(P^{-m+\delta/h}))$$

and hence,

$$a_1 a_2^{-1} = (t_1 u_2/t_2 u_1)(1 + O(P^{-m+\delta/h})).$$

Further,

$$|Nt_i| \leqslant |N(a_i\alpha)|\,|Nu_i| \ll P^{(2m+1)\hat{\delta}/4h}$$

and

$$N(t_2 u_1) = O(P^{(4m+1)\hat{\delta}/4h}). \tag{10}$$

By Lemma 11 of [3], there exist infinitely many $c_0 d_0^{-1} \in K$ with d_0 of the same order of magnitude as all its conjugates, $N(c_0, d_0)$ bounded and $|N(d_0)| = O(|d_0|^h)$ tending to infinity, such that

$$\|a_1 a_2^{-1} - c_0^{-1} d_0\| \ll \|d_0\|^{-(1+1/h)}.$$

We now assume that P is a natural number of the same magnitude as $\|d_0\|^{(h+1)/h}$. The ideals $(c_0 d_0^{-1})$ and $(t_1 u_2/t_2 u_1)$ are distinct, since, otherwise, as in the proof of Lemma 12 of [3], we can show that $P^{(4m+1)\hat{\delta}/4h} \gg N(t_2 u_1) \gg P^{h^2/h+1}$ which will give a contradiction for large P and $\delta < 1$. In particular $\tau = c_0 d_0^{-1} - t_1 u_2/t_2 u_1 \neq 0$. Writing $\tau = c_0 d_0^{-1} - a_1 a_2^{-1} + a_1 a_2^{-1} - t_1 u_2/t_2 u_1$, we have

$$0 < |\tau| \ll \|d_0\|^{-(h+1)/h} + P^{-m+\delta/h} \ll P^{-1} \ll \|d_0\|^{-(h+1)/h}.$$

This implies that

$$\|d_0\|^{-h-1} \gg |N\tau| \gg |N(c_0 u_1 t_2 - d_0 u_2 t_1)|/(|Nd_0|\,|Nt_2|\,|Nu_1|)$$
$$\gg 1/(\|d_0\|^h\,|Nt_2|\,|Nu_1|).$$

From (10) and above, we get

$$\|d_0\| \ll |Nt_2|\,|Nu_1| \ll P^{(4m+1)\hat{\delta}/4h} \ll \|d_0\|^{(4m+1)\hat{\delta}/(4h+4)}$$

which for large $\|d_0\|$ gives a contradiction. Lemma 6 is thus proved.

LEMMA 7. *For $\hat{\delta}$ defined as above and under the given hypotheses of the Theorem,*

$$J_2 = O(NP^{s-m-\hat{\delta}/8h}).$$

Proof. By Lemma 6, we have

$$J_2 \ll NP^{1-\delta/2h}\Big[\underset{\substack{E_2\\ |S(a_1\alpha)|\leqslant|S(a_2\alpha)|}}{\int\ldots\int} \prod_{j=2}^{s} |S(a_j\alpha)|L(\alpha)\{d\alpha\} + $$
$$+ \underset{\substack{E_2\\ |S(a_2\alpha)|\leqslant|S(a_1\alpha)|}}{\int\ldots\int} \prod_{j\neq 2} |S(a_j\alpha)|L(\alpha)\{d\alpha\}\Big].$$

Applying Hölder's inequality and Körner's theorem ([2], Satz 5), we obtain, as in [3], for $s-1 \geqslant 2^m$ that $J_2 \ll NP^{1-\delta/2h}NP^{s-1-m+\delta/8h^3}NP^{\delta/4h}$, provided that P is large enough (to ensure that $P^{\delta/8h^2}$ exceeds a certain power of $\log P$) and Lemma 7 is proved.

As mentioned on p. 501 Lemmas 3, 4 and 7 together with (3) for $\delta \leqslant 1/4$ and large P, prove our Theorem.

Remark. It seems reasonable to expect that the condition $s \geqslant 2^m+1$ in the Theorem may be improved to $s \geqslant c'm\log m$ (for large m) as in Davenport–Roth [1].

References

[0] H. Davenport, *Analytic Methods for Diophantine Equations and Diophantine Inequalities*, Ann Arbor 1962.
[1] — and K. F. Roth, *The solubility of certain Diophantine inequalities*, Mathematika 2 (1955), pp. 81–96.
[2] O. Körner, *Über Mittelwerte trigonometrischen Summen und ihre Anwendungen in algebraischen Zahlkörpern*, Math. Ann. 147 (1962), pp. 205–239.
[3] S. Raghavan and K. G. Ramanathan, *Solvability of a Diophantine inequality in algebraic number fields*, Acta Arith. 20 (1972), pp. 299–315.
[3] C. L. Siegel, *Sums of m-th powers of algebraic integers*, Ann. Math. 46 (1945), pp. 313–339; Gesam. Abhandlungen, 1966, vol. 3, pp. 12–38.
[5] — *Generalization of Waring's problem to algebraic number fields*, Amer. J. Math. 66 (1944), pp. 122–136; Gesam. Abhandlungen, 1966, vol. 2, pp. 405–420.

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On the theorem of Gauss-Kusmin-Lévy and a Frobenius-type theorem for function spaces

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1. Introduction. If one wants to investigate the distribution of values of a_n in the regular continued-fraction expansion

$$a = [0;\, a_1, a_2, \ldots] := \cfrac{1}{a_1+\cfrac{1}{a_2+\ldots}},$$

where a varies randomly through the interval $(0, 1)$, one is readily led to considering the (Lebesgue-) measure $m_n(x)$ of the set

$$\{a;\, [0, a_{n+1}, a_{n+2}, \ldots] < x\},$$

where $0 \leqslant x \leqslant 1$ (see for instance Khintchine [3]). Gauss [2], in a letter to Laplace, stated that

$$m_n(x) \to \frac{\log(1+x)}{\log 2} \quad \text{as} \quad n\to\infty.$$

The first one to publish a proof of this theorem was Kusmin [4] in 1928. Actually he proved that if we put

$$m_n(x) = \frac{\log(1+x)}{\log 2} + r_n(x)$$

then $r_n(x) = O(q^{\sqrt{n}})$ as $n\to\infty$, where q is some constant, $0<q<1$. Lévy [5] independently proved

$$r_n(x) = O(q^n)$$

by a different method (using probabilistic notions). As Szüsz [6] has shown this same result can also be obtained by Kusmin's approach. Szüsz' proof is easier than the two earlier ones and appears to give a smaller value ($q = 0.485$) than Lévy's $q = 0.7$ if one accepts the trouble of some calculation. He does not give all details though.