On a Diophantine inequality for forms of additive type

by

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Dedicated respectfully to Carl L. Siegel

§ 1. Let $K$ be an algebraic number field of degree $h$ over the field $Q$ of rational numbers and let $\bar{K} = K \otimes Q R$, the tensor product over $Q$ of $K$ with the field $R$ of real numbers. Any element $a$ in $\bar{K}$ can be represented as a diagonal matrix with diagonal elements $a^{(0)}, \ldots, a^{(h)}$ referred to as the 'conjugates' of $a$ and assumed to be so ordered that $a^{(0)} \in R$ for $1 \leq i \leq r_1$ and $a^{(h)} = a^{(h+1)}$ are complex for $r_1 < h = r_1 + r_2$ with $r_1 \geq 0$, $r_2 \geq 0$ and $r_1 + 2r_2 = h$. For $a \in K$, $a^{(0)}, \ldots, a^{(h)}$ are just the conjugates of $a$ over $Q$. We define

$$||a|| = \max_{1 \leq i \leq h} |a^{(i)}| \quad \text{for} \quad a \in \bar{K}.$$

Let $m \geq 2$ be a natural number and let $f(x_1, \ldots, x_m) = \sum_{1 \leq i \leq h} a_i x_i^m$ be a form of additive type over $\bar{K}$, i.e., a polynomial in $x_1, \ldots, x_m$ of the above form with coefficients $a_1, \ldots, a_h$ which are invertible elements of $\bar{K}$. We call $f$ totally indefinite, if, for every $i$ with $1 \leq i \leq r_1$, there exist real numbers $p_{i1}, \ldots, p_{ih}$ not all zero such that $\sum_{1 \leq i \leq h} a_i p_i^m = 0$. Our object is to prove the following

**Theorem.** Let $f(x_1, \ldots, x_m) = \sum_{1 \leq i \leq h} a_i x_i^m$ be a totally indefinite form of additive type over $K \otimes Q R$ which is not a scalar multiple of any polynomial in $x_1, \ldots, x_m$ with coefficients in $K$. If $s \geq 2^m + 1$, there exist, for any $\varepsilon > 0$, algebraic integers $a_1, \ldots, a_s$ not all zero in $K$ such that $||f(a_1, \ldots, a_s)|| < \varepsilon$.

**Remarks.** This is an improved version of the Theorem stated in [3] and answers a question raised in [3], p. 300. The condition $s \geq \max(2^m + 1, h^2h^{-1}(m-1) + h^2 + h)$ of that Theorem is now replaced by the condition $s \geq 2^m + 1$, which is clearly independent of the degree $h$ of $K$ over $Q$. Further, the additional condition $mh \geq 4$ in [3] is no longer imposed here. We follow the same notation as in [3] and merely indicate the necessary modifications required to prove the Theorem stated above. For $m = 2$ this Theorem coincides with a well-known Theorem of Davenport and Helbig ([9]), p. 168 for diagonal quadratic forms in 5 variables.

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§ 2. Let \( \{\omega_1, \ldots, \omega_h\} \) be a basis of the ring \( O \) of algebraic integers in \( K \) over the ring \( Z \) of integers. Let \( \phi \) be the different of \( K \) and \( \{\Omega_1, \ldots, \Omega_h\} \) be the complementary \( Z \)-basis of \( \phi^{-1} \). For \( \alpha \in K \), let \( J_0 \) and \( J_0^0 \) denote the trace and the norm of \( \alpha \) over \( Q \) respectively. For any ideal \( \mathfrak{a} \), let \( \mathfrak{a} \mathfrak{A} \) denote the norm of \( \mathfrak{a} \). For \( P > 0 \), let \( P \mathfrak{A} = \{ \beta = \sum_{1 \leq k < h} y_k \omega_k \in \mathfrak{A} | -P \leq y_k < P \text{ for } 1 \leq k \leq h \} \) and \( \# \mathfrak{A} = \sum_{1 \leq k \leq A} x_k \omega_k \in \mathfrak{A} | 0 
less x_k < 1 \text{ for } 1 \leq k \leq h \}. \) (We have identified here \( \omega_0 \bigotimes_{1} 1 \) with \( \omega_k \bigotimes_{k} 1 \) and \( \omega_k \bigotimes_{1} 1 \) with \( \omega_k \).) If \( f \) and \( g \) are two numbers or functions, we abbreviate \( "|f| < \lambda |g|" \) for an unspecified constant \( \lambda > 0 \) depending only on \( K \) by \( "f \ll g" \). We also use the symbols \( O \) and \( o \) of Landau. For real \( P > 0 \), we write \( NP \) for \( P^2 \).

For \( \alpha \in K \) and a fixed number \( P > 0 \), we define the exponential sum \( S(\alpha, P) = \sum_{x \in \mathfrak{A} \mathfrak{A}^0} e^{2\pi i x \alpha} \).

For \( \beta \in K \), we set \( L(\beta) = \prod_{1 \leq k < l} L_k(\beta|0) \prod_{r_1 \leq r_2} L_{r_1}(\beta|0) \prod_{r_2} L_{r_2}(\beta|0) \)

where \( L_k(\beta|0) = (\sin \pi \beta \omega_k^n)^2 \) if \( \pm 0 \) according to the real number \( \alpha \) is different from \( 0 \) or equal to \( 0 \) and \( L_r(\beta|0) = \frac{1}{J_r} |J_r(\beta|0)| \) or \( 1 \) according to the complex number \( \alpha \) is different from \( 0 \) or equal to \( 0 \) and \( J_r \) denotes the usual Bessel function of order \( 1 \). For \( \alpha \in K \), let \( \{\alpha\} \) denote the volume element \( \prod_{1 \leq k < l} d\alpha(\beta|0) \prod_{r_1 \leq r_2} d(\Re \alpha(\beta|0)) \).

Then we have, for \( \alpha \in K \),

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{2\pi i x \alpha} \chi(\beta|0) d\beta \chi(y|0) d\beta \chi(z|0) d\beta = M(\theta)
\]

where

\[
M(\theta) = \begin{cases} 0 & \text{if } |\theta| > 1, \\ \prod_{1 \leq k < l} (1 - \theta|0|) \prod_{r_1 \leq r_2} \phi(|\theta|) & \text{if } |\theta| \leq 1 \end{cases}
\]

with \( \phi(x) = 4\sin^{-1}(x) - |x| \sqrt{1 - |x|^2} \) or \( 0 \) according as \( |x| < 1 \) or \( |x| > 1 \).

To prove the theorem stated in § 1, it suffices to consider the case when \( e = 1 \), since the case of general \( e \) is deduced at once by taking \( e^{-1}f(x_1, \ldots, x_h) \) instead of \( f(x_1, \ldots, x_h) \). We now assume that for every set of \( x_1, \ldots, x_h \) not all \( 0 \) in \( \alpha \), we have

(1) \[ \|
\]

and derive a contradiction which will prove our Theorem.

For \( \alpha \in K \), we set \( T(\alpha) = \prod_{i \leq h} S(\alpha, \omega_i) \). Then for \( \alpha = \sum_{i \leq h} a_i \omega_i \), we have

(2) \[ \alpha = \cdots \alpha = c(\{\alpha\}) \]

for a constant \( c \) depending only on \( K \). By the definition of \( L(\alpha) \), we have

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} T(\alpha) L(\alpha) d\alpha_1 \cdots d\alpha_h = c \sum_{a_1, \ldots, a_h} M(\sum_{i \leq h} a_i \omega_i^n)
\]

where the summation on the right hand side is over all \( s \)-tuples \( x_1, \ldots, x_h \) of elements of \( \alpha \cap \mathfrak{A}^0 \) subject to the restriction that \( \|
\]

In view of the hypothesis (1) on \( f \), we have

(3) \[ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} T(\alpha) L(\alpha) d\alpha_1 \cdots d\alpha_h = c. \]

We shall obtain a contradiction to (3) for large \( P \) under the hypotheses of the Theorem, by splitting up the domain of integration in (3) suitably. Since \( f(x_1, \ldots, x_h) \) is not a scalar multiple of any form over \( K \), we may suppose, without loss of generality that \( a_1 \alpha_1^{-1} \Gamma_1 \) is not in \( K \). We divide the whole space \( -\infty < a_i < \infty (1 \leq i \leq h) \) into three mutually non-overlapping subsets \( E_1, E_2, E_3 \) defined by

\[
E_1 = \{ \alpha \in K \| \alpha \| \leq c_1 \Gamma_1^{m-\delta} \},
E_2 = \{ \alpha \in K \| \alpha \| \leq c_2 \Gamma_1^{m+\delta} \},
E_3 = \{ \alpha \in K \| \alpha \| > \Gamma_1^{m+\delta} \}
\]

where \( \delta, \delta \) are fixed real numbers such that \( 0 < \delta < 1, 0 < \delta < 1 \) and \( \delta \) is a positive constant so chosen that for \( \alpha \in E_i \) and \( a_\alpha = \sum_{i \leq h} a_i \omega_i \), we have

\[
\max_{1 \leq i \leq h} |a_i| > \Gamma_1^{m+\delta} \text{ for } i = 1, 2.
\]

Remark. The change-over from \( P^{m-\delta} \Phi \) in the definition of \( E_1 \) and \( E_2 \) in [3], p. 302, to the present \( P^{m+\delta} \Phi \) was suggested by arguments of K. Ramachandra in a paper On the sums \( \sum_{f \in \mathcal{F}} K \) (to appear).

Let

\[
J_i = \sum_{\beta \in \mathcal{F}} J(\alpha) L(\alpha) d\alpha_1 \cdots d\alpha_h
\]

Then \( J_1 + J_2 + J_3 = 0 \) from (3). We show that \( J_1 \) has an estimate from below involving \( P \) while \( J_2 \) and \( J_3 \) have upper estimates involving \( P \) which are of lower order than the lower estimate for \( J_1 \). The contradiction required for establishing the Theorem is obtained by letting \( P \) tend to infinity.
§ 3. We get here a lower estimate for $J_1$. For any $P > 0$, let $Y(P) = \{ a \in K : |a| < P \}$ and for $\theta \in \mathbb{R}$, let $I(\theta) = \int \int f_{\text{hk}}(a)(d\alpha)$. For any $\gamma \in K$, we may write $(\gamma) = b_{\alpha}a_{\alpha}^n$ with coprime integral ideals $a_{\alpha}$ and $b_{\alpha}$; we refer to $a_{\alpha}$ as the "denominator" of $(\gamma)$. For $\gamma = 0$, we take $a_{\alpha} = 0$. Let
\[ G(\gamma) = \left( \sum_{v \mod a_{\alpha}} e_{hk}(v^{m^n}) / N a_{\alpha} \right), \]
where $\mu$ runs over a complete set of representatives of residue classes of $a$ modulo $a_{\alpha}$. Then we have

**Lemma 1** (Siegel). For $a \in K$, $\gamma \in K$ with $N a_{\alpha} \leq P^{1/n}$ and $|a - \gamma| \leq P^{1/n}/N a_{\alpha}$, we have
\[ S(a) = o^{-1} G(\gamma) I(a - \gamma) + O(1/1 + 4/\alpha) \]
where $\alpha$ is the constant defined by (2). Moreover,
\[ I(a - \gamma) = O(P^{1/n} N \min(1, P^{1/m} |a - \gamma|^{-1/m})). \]

The proof is exactly the same as in Siegel [5], p. 128.

**Lemma 2.**
\[ \left| \prod_{\alpha \in \mathbb{R}} I(\alpha) \right| \geq N P^{2m-2}, \quad \text{for } s > 2m. \]

**Proof.** To prove the lemma, it is enough to show that for some $\varepsilon > 0$, we have
\[ \prod_{\alpha \in \mathbb{R}} I(\alpha) \geq N P^{2m-2 - \varepsilon} \]
and further
\[ \int_{-\infty}^{\infty} \left| \prod_{\alpha \in \mathbb{R}} I(\alpha) \right| \geq N P^{2m-2}. \]

Taking $\gamma = 0$ in (5), we get for the left hand side of (6), the upper bound
\[ N P^{2m} \int_{|a| \leq P^{1/m + 4/\alpha}} \prod_{\alpha \in \mathbb{R}} I(\alpha) \geq N P^{2m - 2}. \]

For $s > 2m$ and large $P$, we have $J_2 \leq N P^{2m - 2} / \alpha^n$. The proof is on the same lines as in Lemma 4 in [3].

**§ 4.** An upper estimate for $J_2$ is provided by

**Lemma 4.** For $s > 2m$ and large $P$, we have $J_3 \leq N P^{2m - 2} / \alpha^n$.

For the estimation of $J_3$, we have to slightly modify the definition of major arcs used in [3]. For the given $P$, $\delta$, the $\mathbb{Z}$-basis $\{ a_1, \ldots, a_k \}$ of $\delta^{-1}$ and $\gamma = \sum_{\gamma \in \mathbb{R}} \chi_{\gamma} a_\gamma$ in $K$ with $N a_{\gamma} \leq P^{1/\alpha}$, we define the major arc $\hat{B}_\gamma$ as the set of $x = \sum_{\alpha \in \mathbb{R}} \chi_{\gamma} a_\gamma$ for which $|x_\gamma - \gamma_\gamma| \leq P^{1/m + 4/\alpha}/N a_{\gamma}$ for $1 \leq k \leq k$. Let $m$ denote the complement in $\mathbb{R}$ of the union of all $\hat{B}_\gamma$ for $\gamma \in K$ with $N a_{\gamma} \leq P^{1/\alpha}$. For $0 < \theta < 1$, let $m_\theta = \{ a \in \mathbb{Z} : a \neq 0 \}$, there do not exist $\lambda = 0$ in $\mathbb{R}$ such that $\lambda = \sum_{\gamma \in \mathbb{R}} \chi_{\gamma} a_\gamma$ and $\mu$ in $\delta^{-1}$ such that for $k \mu = \sum_{\gamma \in \mathbb{R}} \chi_{\gamma} a_\gamma$ we have $|\gamma_\gamma| \leq P^{m - 1/(m + 4/\alpha)}$. For small enough $\theta$ and large $P$, we claim that $m_\theta = m$. If possible, let otherwise, there exist $a$ for $s > 2m$, $P > 1$, and $\lambda = 0$ or 1. We are thus led to the estimation (6) with $c = (s - 2m) / \alpha^n$ for $s > 2m$. The estimate (7) is obtained exactly as in Lemma 2 of [3].
in $m$ and not in $m_{0}$. Then there exist suitable $\lambda, \mu$ as referred to above. Setting $k = \mu/\lambda$, we have $\lambda \in \Lambda$, and further, if $(m-1)\theta < \delta(4h^{3})$, then

$$N_{\lambda} < \lbrack N_{\Lambda} \rbrack < NP^{(m-1)/2}(4h^{3}) \leq P^{m/2h}.$$  

Now $a - \gamma = \lambda^{-1} \sum_{1 \leq i \leq \lambda} a_{i} \delta_{i} = \sum_{1 \leq i \leq \lambda} a_{i} \delta_{i}$, with $a_{i} = \sum_{1 \leq i \leq \lambda} a_{i} \delta_{i}$, $(\alpha_{q}, \beta_{j})$ being the regular representation matrix of $\Lambda$ with respect to the basis $\{\delta_{1}, \ldots, \delta_{\lambda}\}$ of $K$ over $\mathbb{Q}$. Now

$$|a_{j}| \leq P^{(m-1)/2}(m-1)2^{m} / \lbrack N_{\Lambda} \rbrack \leq P^{(m-1)/2}(\delta_{h}) / \lbrack N_{\Lambda} \rbrack$$

and

$$|\beta_{j}| \leq P^{-\mu}(m-1)^{2} \leq h^{2} / \lbrack N_{\Lambda} \rbrack$$  

for $1 \leq i, j \leq \lambda$, contradicting the fact that $\alpha \in m$. This establishes our claim above.

**Lemma 5.** For $\theta < \delta(4h^{3}(m-1))$ and large $P$, $m_{0} = m$ and hence, for all $\mu$,

$$|S(\alpha_{1})| \leq NP^{(m-1)/2}(m-1)\theta.$$  

The proof is the same as that of Lemma 11 of [3].

**Lemma 6.** Let $\alpha_{1}, \alpha_{2}$ be invertible elements of $K$ such that $\alpha_{1}^{-1} \in K$. Then, for $\alpha \in E_{2}$ and large $P$,

$$\min(|S(\alpha_{1}a_{1})|, |S(\alpha_{2}a_{1})|) \leq NP^{(m-1)/2}(\delta_{h})$$

provided that $\delta = \delta(4h^{3}(m-1))$.

**Proof.** Let $a_{1} = \sum a_{i} \delta_{i}$ with $a_{i} \in R$ and $\mu_{1} = \sum b_{i} \delta_{i}$ with $b_{i} \in R$ such that $a_{i} = b_{i} + \varepsilon_{i}$ and $0 \leq \varepsilon_{i} < 1$ for $1 \leq i \leq \lambda$ and $1 \leq j \leq \lambda$. Then $\mu_{1} \in R$ and $S(\alpha_{1}a_{1}) = S(\mu_{1})$ for $1 \leq i \leq 2$. Suppose that $\mu_{0} \in E_{2}$, then, by Lemma 1, we have

$$|S(\alpha_{1}a_{1})| \leq NP^{(m-1)/2} + O(\delta h^{3}P^{m/2})$$

If $\mu_{1} \in E_{2}$ for any $\gamma \in K$ with $N_{\lambda} \leq P^{m/2h}$, then $\mu_{0} \in m_{0} = m_{0}$ for some $\theta$ and large $P$ so that $|S(\alpha_{1}a_{1})| = |S(\mu_{1})| \leq NP^{(m-1)/2}$ in view of Lemma 5. Thus, if either $\mu_{1}$ or $\mu_{0}$ is in $m_{0}$ or belongs to some $E_{2}$ with $P^{m/2h}$ as above, $N_{\lambda} \leq P^{m/2h}$, then $|S(\alpha_{1}a_{1})| \leq NP^{(m-1)/2}$. In order to prove the Lemma completely, it suffices to show that the case

$$\mu_{1} \in E_{2}, \mu_{0} \in E_{2}, \quad \mu_{0} \in m_{0} \leq P^{m/2h}, \quad i, 1, 2,$$

does not arise at all. Writing $\gamma_{1} = \sum a_{i} \delta_{i}$ and $\gamma_{1} = \mu_{1} + a_{i} \delta_{i}$ for $1 \leq i \leq \lambda$, we see that $(\gamma_{1})^{\theta}$ has the same denominator as $(\gamma_{1})^{\theta}$. Now

$$|a_{j} - g_{j}| \leq P^{-m/2}(\delta_{h}) / \lbrack N_{\lambda} \rbrack$$  

for $1 \leq j \leq \lambda$ and $i = 1, 2,$

Observe that, for fixed $i$, not all the numbers $g_{j}$ ($1 \leq j \leq \lambda$) can vanish, since, otherwise, this would contradict a being in $E_{2}$. For $i = 1, 2$, let us write $\gamma_{1} = t_{1}u_{1}$ with $u_{1}, t_{1} \in K$ having their greatest common ideal-divisor belonging to a fixed finite set of integral ideals in $\mathbb{Q}$. Then $N_{u_{1}}$ is the same as $|N_{\lambda}|$ except for a positive constant depending only on $K$. From above, we have

$$a_{i} = t_{1}u_{1}^{-1}(1 + O(P^{-m/2}))$$

and hence,

$$a_{i}a_{i}^{-1} = (t_{1}u_{1})^{-1}(1 + O(P^{-m/2})).$$

Further,

$$|N_{u_{1}}| \leq |N(\alpha_{i}a_{i})| |N_{u_{1}}| \leq P^{m/2h}$$

and

$$|N_{u_{1}}| = O(P^{(m+1)/2}h^{3}).$$

By Lemma 11 of [3], there exist infinitely many $a_{0}d_{0}^{-1}K$ with $d_{0}$ of the same order of magnitude as all its conjugates, $N(\alpha_{0}, d_{0})$ and $N(\alpha_{0})$ tending to infinity, such that

$$|a_{0}a_{0}^{-1} - d_{0}^{-1}d_{0}| \leq |d_{0}|^{-1/2}(m-1).$$

We now assume that $P$ is a natural number of the same magnitude as $\delta^{m}(m-1)^{3/2}$, the ideals $(\alpha_{0}d_{0}^{-1})$ and $(t_{1}u_{1}/t_{1}u_{1})$ are distinct, since, otherwise, as in the proof of Lemma 12 of [3], we can show that $P^{(m+1)/2}$ which will give a contradiction for large $P$ and $\theta < 1$. In particular, $\tau = c_{0}d_{0}^{-1} - t_{1}u_{1} / t_{1}u_{1} \neq 0$. Writing $\tau = c_{0}d_{0}^{-1} - d_{0}^{-1}d_{0} + a_{0}a_{0}^{-1} + c_{0}d_{0}^{-1} - t_{1}u_{1} / t_{1}u_{1}$, we have

$$0 < |\tau| \leq P^{(m-1)/2} + P^{-m/2h} \leq P^{-1} \leq |d_{0}|^{-(m-1)/2h.}$$

This implies that:

$$|d_{0}|^{-1/2} \leq N_{u_{1}} < |N(\gamma_{1})| \lessgtr |N(\alpha_{0}u_{1}) - d_{0}^{-1}d_{0}| / |N(\alpha_{0})| \lessgtr 1 / |d_{0}|^{3} |M_{u_{1}}| / |N_{u_{1}}|.$$  

From (10) and above, we get

$$|d_{0}| \leq |N_{u_{1}}| |N_{u_{1}}| \leq P^{m/2h} \leq |d_{0}| ^{m/2h} |d_{0}|^{3/2},$$

which for large $|d_{0}|$ gives a contradiction. Lemma 6 is thus proved.

**Lemma 7.** For $\delta$ defined as above and under the given hypotheses of the Theorem,

$$J_{z} = O(NP^{m-1/3}h^{3}).$$
Proof. By Lemma 6, we have

\[ J_2 \ll N^{1-\varepsilon} \int \prod_{2 \leq j \leq \varepsilon} |S(a_j \alpha)| L(a_j) \, da_j + \int \prod_{2 \leq j \leq \varepsilon} |S(a_j \alpha)| L(a_j) \, da_j. \]

Applying Hölder's inequality and Körner's theorem ([2], Satz 6), we obtain, as in [3], for \( s - 1 \geq 2^m \) that \( J_2 \ll N^{1-\varepsilon} \sum_{N^{1-\delta/2} \leq n \leq N^{1-\varepsilon}} N^{1/m} \), provided that \( P \) is large enough (to ensure that \( 2^{1/m} \) exceeds a certain power of \( \log P \)) and Lemma 7 is proved.

As mentioned on p. 501, Lemmas 3, 4 and 7 together with (3) for \( \delta \leq 1/4 \) and large \( P \), prove our Theorem.

Remark. It seems reasonable to expect that the condition \( s \geq 2^m + 1 \) in the Theorem may be improved to \( s \gg e^{m \log m} \) (for large \( m \)) as in Davenport–Roth [1].

References


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On the theorem of Gauss-Kusmin-Lévy
and a Frobenius-type theorem for function spaces

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I. Introduction. If one wants to investigate the distribution of values of \( a_n \) in the regular continued-fraction expansion

\[ a = [0; a_1, a_2, \ldots] := \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}, \]

where \( a \) varies randomly through the interval \((0, 1)\), one is readily led to considering the (Lebesgue-) measure \( m_n(x) \) of the set

\[ \{a; [0, a_n+1, a_{n+2}, \ldots] < x\}, \]

where \( 0 \leq x \leq 1 \) (see for instance Khintchine [3]). Gauss [2], in a letter to Laplace, stated that

\[ m_n(x) \sim \frac{\log(1 + x)}{\log 2} \quad \text{as} \quad n \to \infty. \]

The first one to publish a proof of this theorem was Kusmin [4] in 1928. Actually he proved that if we put

\[ m_n(x) = \frac{\log(1 + x)}{\log 2} + r_n(x) \]

then \( r_n(x) = O(q^n) \) as \( n \to \infty \), where \( q \) is some constant, \( 0 < q < 1 \). Lévy [5] independently proved

\[ r_n(x) = O(q^n) \]

by a different method (using probabilistic notions). As Szűsz [6] has shown this same result can also be obtained by Kusmin's approach. Szűsz' proof is easier than the two earlier ones and appears to give a smaller value (\( q = 0.486 \)) than Lévy's \( q = 0.7 \) if one accepts the trouble of some calculation. He does not give all details though.