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Relations between the values of zeta and L-functions at integral arguments*

by

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Dedicated to Prof. C. L. Siegel on his 75th birthday

1. Introduction. The classical results of Euler, concerning the arithmetical nature of the sums

$$\sum_{n=1}^{\infty} n^{-2m} = \zeta(2m) \qquad (m = 1, 2, \ldots),$$

namely $\zeta(2m) = r_m \pi^{2m}$ with rational r_m, have been extended in many ways. Of special relevance in this respect are the contributions of C. L. Siegel, such as [19] and [20]. In this context one has to mention also the work of E. Hecke [8]; H. Klingen [9], [10]; C. Meyer [14]; H. Lang [11], and K. Barner [1] among others.

In contrast with the success of these investigations of the arithmetical character of numbers of the type $\zeta_K(2m)$ and $L(n,\chi)$ for $(-1)^n = \chi(-1)$ ($\zeta_K(s)$ = Dedekind's zeta function of the algebraic (usually totally real) number field K, $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ a Dedekind (or ray-class) L-function to the character $\chi(n)$) stands the dearth of results concerning the arithmetical nature of $\zeta(2m+1) = \sum_{n=1}^{\infty} n^{-(2m+1)}$, or, more generally, of $L(n,\chi)$, for $(-1)^n = -\chi(-1)$.

Some topics related to this problem were considered in [4], [5], and [6] and the present paper is a further contribution to it. Some early results, concerning particular instances are due to Ramanujan [17]. M. Mikolás [15], [16] found interesting representations for $\zeta(2m+1)$, while A. Guinand obtained already in [7] the principal result of [4]. An essentially equivalent formula for $\zeta(4m-1)$ had also been proved by H. F. Sandham [18]. Important recent contributions to this problem

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from a very different point of view are due to Lichtenbaum, Coates and Iwasawa among others (most results not yet published; see, however [3], [13]).

Following Section 2 with the notations is Section 3 which contains the main results of this paper (Theorems 1 and 2 and their corollaries). Explicit representations are found for $L(a, \overline{\chi})\zeta(a)$, where $\chi(n)$ is a primitive, even character to the modulus k and $a = 2m+1$ is an odd, natural integer. These explicit representations depend only on quantities whose arithmetic nature is known and on the values at $\tau = i$ of a certain function $H(\tau, a, \chi)$. The latter has an expansion in a rapidly convergent series and resembles similar functions encountered in [4], [5], and [6]. For fixed a and χ, the values of $H(\tau, a, \chi)$ at $\tau = i$ are denoted by β_j $(j = 0, 1, 2)$, where j depends on the parities of χ and of $(a-1)/2$. In case $k \equiv 1 \pmod 4$ is a fundamental discriminant and $\chi(n) = (k/n)$, the Kronecker symbol, then $\chi(n) = \overline{\chi}(n)$ is real and one has $L(s, \overline{\chi})\zeta(s) = \zeta_K(s)$, the Dedekind zeta function of the quadratic extention $K = Q(\sqrt{k})$ of the rational field Q. The expressions for $\zeta_K(2m+1)$ so obtained may then be compared with those, formally different ones, from [6]. It is also of interest to relate them to Lichtenbaum's conjectures concerning $\zeta_K(2m+1)$, but this will not be done here. The proofs follow in Section 4, the nature of the β^j is discussed in Section 5 and some numerical result in Section 6.

2. Notations. The symbols $\boldsymbol{Z}, \boldsymbol{Q}, \zeta(s), \zeta_K(s), L(s, \chi)$, etc. have their customary meaning. $\int\limits_{(c)}(\ldots)ds$ stands for $\lim\limits_{T\to\infty}\int\limits_{c-iT}^{c+iT}(\ldots)ds$. Following Leopoldt [12] B_χ^m stands for the mth Bernoulli number corresponding to the character $\chi(n)$. If $\chi(n) = 1$ for all $n \in \boldsymbol{Z}$, we suppress the subscript and B^m are the ordinary Bernoulli numbers in the "even superscript" notation (except that $B^1 = \frac{1}{2}$) as in [12]. All characters that occur are assumed to be primitive, non-principal characters, except for specific mention to the contrary. $\overline{\chi}(n)$ denotes the character conjugate to $\chi(n)$. $\chi(n)$ is said to be even if $\chi(-1) = 1$; otherwise $\chi(-1) = -1$ and $\chi(n)$ is called odd. $\tau(\chi)$ stands for the normalized Gaussian sum $\sum\limits_{m \bmod k}\chi(m)e^{2\pi i m/k}$. For natural n, real r and character $\chi(n)$ we set

$$\sigma_r(n) = \sum_{d|n} d^r \quad \text{and} \quad \sigma_r(n, \chi) = \sum_{d|n}\chi^2(d)d^r.$$

Further notations will be introduced as needed.

3. Main results. Let $\chi(n)$ be a primitive, non-principal, even character modulo k and let a be an odd natural integer. Set

$$\varphi(s) = \zeta(s)\zeta(s+a)L(s, \chi)L(s+a, \overline{\chi})$$

and

$$\Phi_0(s) = (4\pi^2/k)^{-s}\varphi(s)\Gamma^2(s).$$

In case $\chi(n)$ is even, $\Phi_0(s)$ has double poles at $s = -1, -3, \ldots, -a+2$, and in case $\chi(n)$ is odd, $\Phi_0(s)$ has double poles at $s = 0$ and $s = -a+1$. We therefore introduce the functions

$$\Phi_1(s) = p_1(s)\Phi_0(s) \quad \text{and} \quad \Phi_2(s) = p_2(s)\Phi_0(s)$$

with

$$p_1(s) = (s+1)(s+3)\ldots(s+a-2) \quad \text{and} \quad p_2(s) = s(s+a-1).$$

$\Phi_1(s)$ and $\Phi_2(s)$ have only simple poles at $s = 1, 0, -1, -2, \ldots, -a$. For $c > 1$ we define

$$F_j(\tau) = \frac{1}{2\pi i}\int\limits_{(c)}\Phi_j(s)(\tau/i)^{-s}ds, \quad j = 1, 2.$$

In particular,

$$F_j(i) = \frac{1}{2\pi i}\int\limits_{(c)}\Phi_j(s)ds, \quad j = 1, 2.$$

Clearly, $F_j(\tau)$ depends also on a and on χ, but for simplicity, this dependence will not be emphasized by the notation.

Let

$$u(a) = \tfrac{1}{4}\{(1+(-1)^{(a-1)/2})a + 3(-1)^{(a+1)/2}+1\},$$

$$v(a) = -\frac{i}{2}i(1+(-1)^{(a-1)/2})$$

and define

$$H = H(\tau, a, \chi)$$
$$= (1+\chi(-1))\big(u(a)F_1(\tau) + v(a)F_1'(\tau)\big) + (1-\chi(-1))F_2(\tau),$$

i.e.

$$H(\tau, a, \chi) = \begin{cases}(a-1)F_1(\tau) - 2iF_1'(\tau) & \text{if } \chi \text{ is even and } a \equiv 1 \pmod 4,\\ 2F_1(\tau) & \text{if } \chi \text{ is even and } a \equiv 3 \pmod 4,\\ 2F_2(\tau) & \text{if } \chi \text{ is odd}.\end{cases}$$

We shall be interested in the values $H(i, a, \chi) = \beta_j$ $(j = 0, 1, 2)$, where β depends on the parities of χ and of $(a-1)/2$. Specifically, let

$$\beta_0 = (a-1)F_1(i) - 2iF_1'(i), \quad \beta_1 = 2F_1(i), \quad \beta_2 = 2F_2(i).$$

These functions generalize the sums $\sum\limits_{m=1}^{\infty} m^{-a}(e^{2\pi m}-1)^{-1}$ and similar ones that occur in [4] and [5].

With these notations, the following statements hold:

THEOREM 1. *If $\chi(n)$ is even and $a \equiv 1 \pmod 4$, then*

$$(1)\quad 2^{-(a+3)/2}(a-1)(a-1)!\left\{\left(\frac{a-1}{2}\right)!\right\}^{-1}\tau(\chi)L(1,\bar\chi)L(a,\bar\chi)\zeta(a)$$

$$= (4\pi^2/k)^a 2^{(a-7)/2}\tau(\bar\chi)\left\{2\left(\frac{a-1}{2}\right)!\{a!\,(a+1)!\}^{-1}L(1,\chi)B_\chi^{a+1}B^{a+1} -\right.$$

$$\left.\sum_{m=1}^{(a-1)/4}(-1)^m\frac{(m-1)!\left(\frac{a-1}{2}-m\right)!(a+1-4m)}{\{(2m)!(a+1-2m)!\}^2}B_\chi^{2m}B^{2m}B_\chi^{a+1-2m}B^{a+1-2m}\right\}-\beta_0.$$

COROLLARY 1.1. *Under the conditions of Theorem 1,*

$$\pi^{-2a}L(a,\bar\chi)\zeta(a) = \frac{\tau(\bar\chi)L(1,\chi)}{\tau(\chi)L(1,\bar\chi)}V_1(a)+\frac{\tau(\bar\chi)}{\tau(\chi)L(1,\bar\chi)}V_2(a)+\frac{\beta_0}{\pi^{2a}\tau(\chi)L(1,\bar\chi)}V_3(a)$$

with algebraic $V_j(a)$ $(j = 1, 2, 3)$.

COROLLARY 1.2. *If under the conditions of Theorem 1, $\chi(n)$ is a real character, then $\chi(n) = \bar\chi(n)$ and*

$$\pi^{-2a}L(a,\chi)\zeta(a) = R_1(a)+R_2(a)/L(1,\chi)+\big(\pi^{-2a}\beta_0/\tau(\chi)L(1,\chi)\big)R_3(a)$$

with rational $R_j(a)$ $(j = 1, 2, 3)$.

COROLLARY 1.3. *If $\chi(n) = (k/n)$, $k \equiv 1 \pmod 4$ and $\zeta_K(s)$ is the Dedekind zeta function of the quadratic field $K = Q(\sqrt{k})$, then $\tau(\chi) = k^{1/2}$, $L(1,\chi) = 2hk^{-1/2}\log\varepsilon$* ($h$ = class number of K, ε = fundamental unit of K) *and*

$$(2)\quad \pi^{-2a}\zeta_K(a) = R_1(a)+R_2(a)k^{1/2}/2h\log\varepsilon+R_3(a)\beta_0\pi^{-2a}/2h\log\varepsilon$$

with rational $R_j(a)$ $(j = 1, 2, 3)$.

Remark. Formula (2) should be compared on the one hand with Corollary 1.1 of [6] according to which, for every totally real field K of degree n,

$$\pi^{-na}\zeta_K(a) = R_1'(a)+R_2'(a)S'(1)\pi^{-n(a+1/2)}/R$$

with $S'(1)$ a quantity analogous to β_0, rational $R_1'(a)$, $R_2'(a)$, and R the regulator of K; and on the other hand with Lichtenbaum's conjecture which (in the particular case of a quadratic field) predicts a simple arithmetic interpretation for the quantity $\pi^{2-2a}\zeta_K(a)$, rather than $\pi^{-2a}\zeta_K(a)$.

THEOREM 2. *If $\chi(n)$ is even and $a \equiv 3 \pmod 4$, then*

$$(3)\quad 2^{-(a+1)/2}\frac{(a-1)!}{\left(\frac{a-1}{2}\right)!}\tau(\chi)L(1,\bar\chi)L(a,\chi)\zeta(a)$$

$$= (2\pi)^{2a}2^{(a-3)/2}k^{-a}\tau(\bar\chi)\left(\frac{a-1}{2}\right)!\{(a+1)!\}^{-2}L(1,\chi)B_\chi^{a+1}B^{a+1} -$$

$$-(2\pi)^{2a}2^{(a-7)/2}k^{-a}\tau(\bar\chi)\sum_{m=1}^{(a-1)/2}(-1)^m(m-1)!\left(\frac{a-1}{2}-m\right)!\times$$

$$\times\{(2m)!(a+1-2m)!\}^{-2}B_\chi^{2m}B^{2m}B_\chi^{a+1-2m}B^{a+1-2m}-\beta_1.$$

COROLLARY 2.1. *Under the conditions of Theorem 2,*

$$\pi^{-2a}L(a,\chi)\zeta(a) = \frac{\tau(\bar\chi)L(1,\chi)}{\tau(\chi)L(1,\bar\chi)}v_1(a)+\frac{\tau(\bar\chi)}{\tau(\chi)L(1,\bar\chi)}v_2(a)+\frac{\beta_1}{\pi^{2a}\tau(\chi)L(1,\bar\chi)}v_3(a)$$

with algebraic $v_j(a)$ $(j = 1, 2, 3)$.

COROLLARY 2.2. *If, under the conditions of Theorem 2, $\chi(n)$ is a real character, then $\chi(n) = \bar\chi(n)$ and*

$$\pi^{-2a}L(a,\chi)\zeta(a) = r_1(a)+r_2(a)/L(1,\chi)+\beta_1 r_3(a)\pi^{-2a}/\tau(\chi)L(1,\chi)$$

with rational $r_j(a)$ $(j = 1, 2, 3)$.

COROLLARY 2.3. *If $\chi(n) = (k/n)$, $k \equiv 1 \pmod 4$ and $\zeta_K(s)$ is the Dedekind zeta function of the quadratic field $K = Q(\sqrt{k})$, then, with the notations of Corollary* 1.3,

$$\pi^{-2a}\zeta_K(a) = r_1(a)+r_2(a)k^{1/2}/2h\log\varepsilon+\beta_1 r_3(a)\pi^{-2a}/2h\log\varepsilon.$$

Remark. If $\chi(n)$ is an odd character modulo k a similar statement holds, specifically,

$$(4)\quad -\frac{a-1}{a!}\left(\frac{2\pi}{k}\right)^{a-1}B_\chi^a L(1,\chi)\zeta(a)+$$

$$(2\pi)^a\frac{\tau(\chi)}{i}\sum_{m=0}^{(a+1)/2}\frac{(2m-1)(a-2m)}{4(2m)!(a+1-2m)!}B^{2m}B^{a+1-2m}L(2m,\bar\chi)L(a+1-2m,\bar\chi)$$

$$+\left(\frac{2\pi}{k}\right)^a\frac{\tau(\bar\chi)}{i}\sum_{m=1}^{(a-3)/2}\frac{(2m)(a-2m-1)}{4(2m+1)!(a-2m)!}B_\chi^{2m+1}B_\chi^{a-2m}\zeta(a-2m)\zeta(2m+1)$$

$$= (-1)^{(a-1)/2}\beta_2.$$

The simplest instance of this formula corresponds to $k = a = 3$, $\chi(n) = \left(\frac{-3}{n}\right) = 0, \pm 1$, with $\chi\left(\frac{-3}{n}\right) \equiv n \pmod 3$, and reads

$$-\tfrac{1}{10}L(4, \chi)+\tfrac{16}{81}\zeta(3)-\tfrac{1}{8}L^2(2, \bar{\chi}) = 3\sqrt{3}\pi^{-3}\beta_2.$$

By restricting the character to be real (4) can be simplified somewhat, $L(1, \chi)$ may be replaced by $-\pi k^{-3/2} r$ ($r = \sum_{0<m<k} \chi(m) m$ an integer), etc. However, contrary to the case of even $\chi(n)$, each term here contains quantities, of unknown arithmetic nature, so that the usefulness of (4) is doubtful and we do not pursue the matter further.

4. Proofs. The method of proof is well known (see, e.g., [4] or [5]); therefore, it will be sufficient to sketch the argument only briefly.

On account of the functional equations of the Γ, ζ, and L-functions (the latter used very conveniently in the specific form given in [12]) and of the classical equation $\tau(\chi)\tau(\bar{\chi}) = \chi(-1)k$ (a neat new proof of this relation is due to B. C. Berndt, see [2]) we have

$$\Phi_1(1-s-a) = (-1)^{(a-1)/2}\Phi_1(s), \qquad \Phi_2(1-s-a) = -\Phi_2(s).$$

First consider the case of even $\chi(n)$. Then for $-\varepsilon - a \leqslant \sigma$ $(=\operatorname{Re} s) \leqslant 1+\varepsilon$, $0<\varepsilon<1$, $\Phi_1(s)$ has only the simple poles $s = -a, 1-a, \ldots, -1, 0, 1$ and it follows with $\sigma_2 = 1+\varepsilon$ and $\sigma_1 = 1-a-\sigma_2$ that

$$F_1(\tau) = \frac{1}{2\pi i}\int_{(\sigma_2)} \Phi_1(s)(\tau/i)^{-s}ds = \frac{1}{2\pi i}\int_{(\sigma_1)} \Phi_1(1-s-a)(\tau/i)^{s+a-1}ds$$
$$= (-1)^{(a-1)/2}\frac{1}{2\pi i}\int_{(\sigma_1)} \Phi_1(s)(\tau/i)^{s+a-1}ds.$$

The last integral is evaluated by moving the line of integration back to σ_2, and taking into account the sum of the residues of the poles of the integrand with $-a \leqslant \sigma \leqslant 1$, which we denote be $S_1(\tau, a)$:

$$(5)\quad F_1(\tau) = (-1)^{(a-1)/2}\left\{\frac{1}{2\pi i}\int_{(\sigma_2)} \Phi_1(s)(\tau/i)^{s+a-1}ds - S_1(\tau, a)\right\}$$
$$= (-1)^{(a-1)/2}\left\{(\tau/i)^{a-1}\,\frac{1}{2\pi i}\int_{(\sigma_2)} \Phi_1(s)\left(\frac{-1/\tau}{i}\right)^{-s} - S_1(\tau, a)\right\}$$
$$= (-1)^{(a-1)/2}\{(\tau/i)^{a-1}F_1(-1/\tau) - S_1(\tau, a)\}.$$

The sum of the residues is computed routinely. If $a \equiv 1 \pmod 4$ then (5) becomes

$$F_1(\tau) - (\tau/i)^{a-1}F_1(-1/\tau) = -S_1(\tau, a).$$

We now set $\tau = it$ $(t>0)$, divide both sides by $t-1$ and let $t \to 1$, and the result is equation (1).

If $a \equiv 3 \pmod 4$, (5) becomes

$$F_1(\tau) + (\tau/i)^{a-1}F_1(-1/\tau) = S_1(\tau, a).$$

For $\tau = i$ this yields equation (3).

The Corollaries follow almost trivially from the respective theorems, by recalling that Leopoldt's generalized Bernoulli numbers B_χ^m (see [12]) are algebraic and belong to the cyclotomic field generated over the rationals by the values of the character $\chi(n)$.

Next consider the case of $\chi(n)$ an odd character. Then, for $-a-\varepsilon \leqslant \sigma \leqslant 1+\varepsilon$ $(0<\varepsilon<1)$, $\Phi_2(s)$ has only the simple poles at $s = -a, 1-a, \ldots, -1, 0, 1$, and, proceeding as before, we obtain successively

$$F_2(\tau) = \frac{1}{2\pi i}\int_{(\sigma_2)} \Phi_2(s)(\tau/i)^{-s}ds = \frac{1}{2\pi i}\int_{(\sigma_1)} \Phi_2(1-s-a)(\tau/i)^{s+a-1}ds$$
$$= -\frac{1}{2\pi i}\int_{(\sigma_1)} \Phi_2(s)(\tau/i)^{s+a-1}ds$$
$$= -\left\{\frac{1}{2\pi i}\int_{(\sigma_2)} \Phi_2(s)(\tau/i)^{s+a-1}ds - S_2(\tau, a)\right\}$$
$$= -\{(\tau/i)^{a-1}F_2(-1/\tau) - S_2(\tau, a)\},$$

or

$$F_2(\tau) + (\tau/i)^{a-1}F_2(-1/\tau) = S_2(\tau, a),$$

where $S_2(\tau, a)$ is the sum of the residues of the integrand in the strip $-a-\varepsilon \leqslant \sigma \leqslant 1+\varepsilon$. The content of the Remark now follows by setting $\tau = i$.

5. The values β_j. In the theorems and corollaries occur the quantities β_j $(j = 0, 1, 2)$. The arithmetical nature of the β_j's is not clear, but some things are known. The β_j's may be represented by series similar to Lambert series, the role of the exponential e^x (which is the inverse Mellin transform of $\Gamma(s)$) being played essentially by a function $g(x)$, which is the inverse Mellin transform of $\Gamma^2(s)$, and by the derivatives of $g(x)$.

The function

$$\varphi(s) = \zeta(s)\zeta(s+a)L(s, \chi)L(s+a, \bar{\chi})$$

is represented for $\sigma>1$ by the Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{with} \quad a_n = \sum_{d|n} \chi(d)\sigma_{-a}(n/d)\sigma_{-a}(n, \chi),$$

and, in case $\chi(n)$ is a real character,

$$a_n = \sum_{d|n} \chi(d)\sigma_{-a}(n/d)\sigma_{-a}(d).$$

In any case

$$|a_n| \leqslant \sum_{d|n} \sigma_{-a}(n/d)\sigma_{-a}(d) \leqslant \sigma_0(n)\max_{d|n}\sigma_{-a}(n/d)\sigma_{-a}(d) = O(n^{\varepsilon})$$

for any $\varepsilon > 0$ and $n\to\infty$.

With $p_1(s)$ as defined in Section 3 we can write, for any $c > 1$,

$$F_1(\tau) = \frac{1}{2\pi i}\int_{(c)} \Big\{\sum_{n=1}^{\infty} a_n n^{-s}\Big\} \Gamma^2(s)(4\pi^2/k)^{-s} p_1(s)(\tau/i)^{-s}\,ds$$
$$= \sum_{n=1}^{\infty} a_n \frac{1}{2\pi i}\int_{(c)} p_1(s)\Gamma^2(s)(4\pi^2 n\tau/ki)^{-s}\,ds.$$

Here the interchange of summation and integration can be justified without difficulty for $c > 1$, by using Stirling's formula for $\Gamma(s)$ and previous estimate $|a_n| = O(n^{\varepsilon})$.

In order to study this sum, let $g(x)$ be the inverse Mellin transform of $\Gamma^2(s)$, i.e.

$$g(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma^2(s)x^{-s}\,ds, \qquad \Gamma^2(s) = \int_0^{\infty} g(x)x^{s-1}\,dx.$$

Set $h(x) = x^{-1/2}g(x^{1/2})$ and observe that if $a = 2m+1$, then

$$\psi(x) = x^{a/2}h^{(m)}(x) = (-\tfrac{1}{2})^m \frac{1}{2\pi i}\int_{(c)} p_1(s)x^{-s/2}\Gamma^2(s)\,ds.$$

Consequently,

$$\beta_1 = 2F_1(i) = (-1)^m 2^{m+1}\sum_{m=1}^{\infty} a_n\psi(16\pi^4 n^2/k^2).$$

Similarly, if we set $f(x) = (a-1)\psi(x) - 4\psi'(x)$, then

$$\beta_0 = (a-1)F_1(i) - 2iF_1'(i)$$
$$= \sum_{n=1}^{\infty} a_n \frac{1}{2\pi i}\int_{(c)} \Gamma^2(s)p_1(s)\,(2s+a-1)(4\pi^2 n/k)^{-s}\,ds$$
$$= (-1)^m 2^m \sum_{n=1}^{\infty} a_n f(16\pi^4 n^2/k^2).$$

Finally, let

$$u(x) = x^{1-a}g(x),\ v(x) = x^a u'(x), \quad \text{and} \quad w(x) = v'(x);$$

then

$$\beta_2 = 2F_2(i) = 2\sum_{=1}^{\infty} a_n w(4\pi^2 n/k).$$

From

$$\int_0^{\infty} g(x)x^m\,dx = (m!)^2 \quad \text{for} \quad m\in Z^+$$

and

$$\int_0^{\infty} e^{-u^{\gamma}}u^m\,du = \frac{1}{\gamma}\Gamma\Big(\frac{m+1}{\gamma}\Big) < (m!)^2 < \frac{1}{\delta}\Gamma\Big(\frac{m+1}{\delta}\Big) = \int_0^{\infty} e^{-u^{\delta}}u^m\,du,$$

valid for any $\delta \leqslant \frac{1}{2} < \gamma$ and sufficiently large m, it follows that, for $u\to\infty$, $g(u)$ decreases essentially like $e^{-u^{1/2}}$. It follows that $\psi(16\pi^4 n^2/k^2)$ is comparable to $(-\frac{1}{4})^m(2\pi(n/k)^{1/2})^m e^{-2\pi(n/k)^{1/2}}$, $f(16\pi^4 n^2/k^2)$ is comparable to $2m(-\frac{1}{4})^m(2\pi(n/k)^{1/2})^m e^{-2\pi(n/k)^{1/2}}$ and $w(4\pi^2 n/k)$ to $4^{-1}e^{-2\pi(n/k)^{1/2}}$. The analogous terms of the sums in [4] are comparable to $n^{1-a}e^{-2\pi n}$ and $n^{-a}e^{-2\pi n}$, respectively. This finishes the proof of an earlier statement that $H(\tau)$ has an expansion in a rapidly convergent series and that it generalizes the functions $2F(\tau)$ and $(a-1)F(\tau)-2iF'(\tau)$ of [4], or $F_a(\tau)$ and $H_a(\tau)$ of [5], respectively.

6. A numerical example. Theorem 2 (see also Corollary 2.3) yields for $a = 3$, $k \equiv 1 (\mathrm{mod}\, 4)$ (hence, $\chi(n) = \left(\frac{k}{n}\right)$ is even) that

$$(6) \qquad \pi^{-6}\zeta_k(3) = -\tfrac{1}{135}B_\chi^4 k^{-3} + \tfrac{1}{36}(B_\chi^2)^2 k^{-5/2}/h\log\varepsilon - \pi^{-6}\beta_1/h\log\varepsilon.$$

On the other hand, from [6] it follows that

$$\pi^{-6}\zeta_K(3) = -\tfrac{4}{135}B_\chi^4 k^{-3} + \pi^{-7}S'(1)/h\log\varepsilon.$$

It is somewhat surprising that the same algebraic term, $B_\chi^4 k^{-3}/135$ appears in both formulae, but with a different coefficient.

It easily follows that $\pi^{-6}\beta_1/\log\varepsilon$ and $\pi^{-7}S'(1)/\log\varepsilon$ cannot both be algebraic, as this would imply that the middle term on the right of (6) is algebraic, which is false. Similarly, $\pi^{-6}\beta_1$ and $\pi^{-7}S'(1)$ cannot both be algebraic; indeed, if we eliminate $\pi^{-6}\zeta_K(3)$, we obtain a relation of the form

$$\pi^{-6}\beta_1 + \pi^{-7}S'(1) = \alpha - 3\beta\log\varepsilon$$

with α and β algebraic, while $\log\varepsilon$ is not algebraic.

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Sur la représentation de zéro par une somme de carrés dans un corps algébrique

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§ 1. Soient donnés le corps algébrique K de degré n et le nombre naturel $m \geqslant 3$. Dans plusieurs mémoires j'ai étudié la résolubilité des équations diophantiennes du type

$$x_1^2 + x_2^2 + \ldots + x_m^2 = 0 \tag{1}$$

en nombres $x_1, x_2, \ldots, x_m$ (le cas $x_1 = x_2 = \ldots = x_m = 0$ étant exclu) appartenant au corps K; voir Nagell [3], [4] et [5]. Il faut évidemment que le corps K soit *totalement imaginaire*, c'est-à-dire que tous les corps conjugués soient imaginaires, et que n soit pair $= 2\nu$. Dans la suite nous considérons seulement les corps algébriques totalement imaginaires.

Si $x_1, x_2, \ldots, x_m$ satisfont à (1) nous dirons que $[x_1, x_2, \ldots, x_m]$ est une solution de cette équation. Cette solution est appelée *réductible*, s'il y a dans (1) une somme partielle des carrés x_i^2 qui s'annule. Dans le cas contraire la solution sera appelée *irréductible*.

Sans restreindre à la généralité nous pouvons supposer que $x_1 \neq 0$. Soit $[x_1, x_2, \ldots, x_m]$ une solution de (1) dans K. Désignons par K^* le corps engendré par les $m-1$ nombres $x_2/x_1, x_3/x_1, \ldots, x_m/x_1$. Ce corps est un sous-corps de K. Si K^* est identique à K nous dirons que la solution est *effective* dans K. Si K^* est un sous-corps véritable de K il doit être totalement imaginaire. La solution est alors effective dans K^*.

Pour reconnaitre si l'équation (1) est résoluble ou non dans le corps totalement imaginaire K nous avons le critère suivant (voir Nagell [4]):

Pour que l'équation (1) soit résoluble dans K il faut et il suffit que la congruence

$$x_1^2 + x_2^2 + \ldots + x_m^2 \equiv 0 \pmod 8 \tag{2}$$

soit résoluble dans K, de façon que $(x_1, x_2, \ldots, x_m, 2) = 1$.

Cependant, il faut noter que la démonstration de ce critère n'est pas constructive et qu'il s'agit seulement d'un théorème d'existence.