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A note on zeta-functions of algebraic number fields

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It has first been shown by E. Artin [2] that if k is an algebraic number field [1] and K a normal extension field with icosahedral Galois group relative to k, then the quotient of the zeta-functions $\zeta_K(s)/\zeta_k(s)$ is an entire function of the complex variable s. Later, H. Aramata showed that Artin's method can be used to prove the same statement when K is a normal extension of the algebraic number field k with arbitrary Galois group; cf. Aramata [1], Brauer [4]. We shall give here further results of the same nature. It will be shown that if Ω_1 and Ω_2 are two algebraic number fields which are both normal over their intersection k, then

$$\zeta_K(s)\zeta_k(s)/\left(\zeta_{\Omega_1}(s)\zeta_{\Omega_2}(s)\right)$$

is an entire function. More generally, we shall prove the following

THEOREM. *Let $\Omega_1, \Omega_2, \ldots, \Omega_m$ be algebraic number fields which are normal over the field k and for which Ω_j intersects the compositum of $\Omega_{j+1}, \Omega_{j+2}, \ldots, \Omega_m$ in k; $(j = 1, 2, \ldots, m-1)$. For any non-empty subset T of the set $M = \{1, 2, \ldots, m\}$, let Ω_T denote the compositum of the fields Ω_j with $j \in T$. If $T = \emptyset$ is the empty set, we set $\Omega_\emptyset = k$. Set $\varepsilon(T)$ equal to 1 or -1 according as to whether $M-T$ contains an even or an odd number of elements. Then*

$$\xi(s) = \prod_{T \subseteq M} \left(\zeta_{\Omega(T)}(s)\right)^{\varepsilon(T)} \tag{1}$$

is an entire function of s.

Proof. We begin with some elementary field theoretic observations. We set $K = \Omega(M)$. If n_j denotes the degree of Ω_j over k $(j = 1, 2, \ldots, m)$,

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(1) All algebraic number fields considered are assumed to be of finite degree over the field of rational numbers.

we show easily by induction that for any subset T of M, the degree of $\Omega(T)$ over k is given by

(2) $$[\Omega(T):k] = \prod_{j\in T} n_j.$$

For two subsets T_1 and T_2, the compositum of $\Omega(T_1)$ and $\Omega(T_2)$ is $\Omega(T_1\cup T_2)$. Using (2), we see that the intersection of $\Omega(T_1)$ and $\Omega(T_2)$ is $\Omega(T_1\cap T_2)$.

Let $H(T)$ denote the Galois group of K relative to $\Omega(T)$. In particular, $H(\emptyset)$ is the Galois group G of K relative to k. If we set $H_j = H(M-\{j\})$, then H_j is a normal subgroup of G and we see that, for any subset T of M, we have

(3) $$H(T) = \prod_{j\notin T} H_j$$

where the product is direct. In particular,

(4) $$G = H_1\times H_2\times\ldots\times H_m.$$

We note that the order of H_j is n_j. Thus, the order $|H(T)|$ of $H(T)$ is given by

(5) $$|H(T)| = \prod_{j\notin T} n_j.$$

We can express the zeta-function $\zeta_{\Omega(T)}(s)$ as an Artin L-function, Artin [3]. If $1_{H(T)}$ denotes the principal character of $H(T)$, we have

$$\zeta_{\Omega(T)}(s) = L\big(s, 1_{H(T)}, K/\Omega(T)\big).$$

If φ is a character of a subgroup H of G, we denote by φ^G the induced character of G. Then

$$\zeta_{\Omega(T)}(s) = L\big(s, (1_{H(T)})^G, K/k\big).$$

It follows from (1) that

(6) $$\xi(s) = L(s, \psi, K/k)$$

where we set

$$\psi = \sum_{T\subseteq M} \varepsilon(T)(1_{H(T)})^G.$$

Consider an element α of G and set

(7) $$\alpha = \alpha_1\alpha_2\ldots\alpha_m$$

with $\alpha_j\in H_j$. The definition of induced characters shows that

(8) $$(1_{H(T)})^G(\alpha) = \big(1/|H(T)|\big)\sum_{\beta}{}' 1_{H(T)}(\beta\alpha\beta^{-1})$$

where β ranges over those elements of G for which $\beta^{-1}\alpha\beta\in H(T)$. If no such β exists, the expression (8) is zero. Now, (4) shows that no β exists if $\alpha_j\neq 1$ for some $j\in T$. On the other hand, if we have $\alpha_j = 1$ for all $j\in T$, then $\beta^{-1}\alpha\beta\in H(T)$ for all $\beta\in G$ and we find from (8) and (5) that

$$(1_{H(T)})^G(\alpha) = |G|/\prod_{j\notin T} n_j = \prod_{j\in T} n_j.$$

Let M_α denote the subset of M consisting of those indices j for which $\alpha_j = 1$ in (7). We have now shown that

$$(1_{H(T)})^G(\alpha) = \begin{cases} 0 & \text{for} \quad T\nsubseteq M_\alpha, \\ \prod_{j\in T} n_j & \text{for} \quad T\subseteq M_\alpha. \end{cases}$$

This implies that

$$\psi(\alpha) = \sum_{T\subseteq M_\alpha}\varepsilon(T)\prod_{j\in T} n_j = (-1)^m\sum_{T\subseteq M_\alpha}\prod_{j\in T}(-n_j) = (-1)^m\prod_{j\in M_\alpha}(1-n_j).$$

If ϱ_j denotes the regular character of H_j, i.e. if

$$\varrho_j = (1_1)^{H_j},$$

our result can be written in the form

(9) $$\psi(\alpha) = \big(\varrho_1(\alpha_1)-1\big)\big(\varrho_2(\alpha_2)-1\big)\ldots\big(\varrho_m(\alpha_m)-1\big).$$

As has been shown in [1] or [4], for any finite group H, the character $(1_1)^H - 1_H$ can be expressed as a linear combination of characters of H induced by non-principal characters of degree 1 of subgroups of H such that the coefficients are non-negative rational numbers. Then (9) shows that ψ is a linear combination with non-negative rational coefficients of characters of G induced by non-principal characters of degree 1 of subgroups of G. On account of Artin's results, this implies that $\xi(s)$ cannot have a pole for any finite s. Since the zeta-functions are meromorphic, it is now clear from (1) that $\xi(s)$ is an entire function as we wanted to show.

References

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