fen $a \leqslant \Re s \leqslant \beta$ eine Schranke der Art $Ce^{-c\sigma(Y)}$ gefunden werden mit positiven Konstanten C und c, die nur von a, β, f, g abhängen. Mithin stellt (34), also auch (25) eine ganze Funktion dar. Die Invarianz von (34) bezüglich $s \to 2k - \frac{3}{2} - s$ folgt sofort aus der von (13) bezüglich $s \to \frac{3}{2} - s$.

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Explicit formulas for the eigenvalues of Hecke operators

by

YU. I. MANIN (Moscow)

To Professor C. L. Siegel on his 75th birthday

Introduction.

1. Put

$$H = \{z \in C \mid \operatorname{Im} z > 0\}, \quad \Gamma = \mathrm{SL}(2, \boldsymbol{Z})/(\pm 1),$$

$$\Gamma(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N\right\} \subset \Gamma.$$

Consider a cusp form $\Phi(z)$ of weight -2 for a congruence subgroup $\Gamma' \supset \Gamma(N)$. By definition, this means that $\Phi(z)\,dz$ is induced by a differential of the first kind on the compactification of $\Gamma' \backslash H'$. Let T_n be the nth Hecke operator, $(n, N) = 1$, and suppose that $\Phi \mid T_n = \lambda_n \Phi$, $\lambda_n \in \boldsymbol{C}$. In this note we give "explicit formulas" for the eigenvalues λ_n having a very simple arithmetic structure, and discuss some of the consequences. These formulas were first stated in [1], § 7, for the group

$$\Gamma' = \Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \bmod N\right\}$$

and for forms Φ satisfying the condition $\int_0^{i\infty} \Phi(z)\,dz \neq 0$. Here we show how one can get rid of these restrictions.

2. To state our main result we need some more definitions.

Let $d > 1$ be an integer. A solution $(\Delta, \Delta', \delta, \delta')$ of the equation $d = \Delta\Delta' + \delta\delta'$ is called *admissible* if it consists of integers satisfying the following supplementary conditions:

$$(\Delta, \delta) = (\Delta', \delta') = 1, \quad \Delta > \delta > 0$$

and

$$\text{either } \Delta' > \delta' > 0, \text{ or } \Delta = d,\ \Delta' = 1,\ 0 \leqslant \delta < d/2,\ \delta' = 0.$$

Let $P \subset \boldsymbol{Z} \times \boldsymbol{Z}$ be the set of all pairs of coprime integers. A function $y\colon P \to \boldsymbol{C}$ is called *locally constant* (implying adelic topology) if there exists an integer M such that $y(a, b)$ depends only on $(a \bmod M, b \bmod M)$.

Let $\varepsilon\colon \boldsymbol{Z}\to\boldsymbol{C}$ be a Dirichlet character. For $n>0$ put

$$\tau_\varepsilon(d)=\sum_{d|n}\varepsilon(d).$$

Finally, given a locally constant function $y\colon P\to C$ and a Dirichlet character ε, define a sequence of complex numbers

(1) $$\Lambda_n(y,\varepsilon)=\sum_{d|n}\left(\varepsilon\left(\frac{n}{d}\right)d-\tau_\varepsilon\left(\frac{n}{d}\right)\sum_{d=\Delta\Delta'+\delta\delta'}y(\Delta,\delta)\right)$$

where the inner sum is taken over all admissible solutions (and is zero for $d=1$).

3. Theorem. *Let Φ be a cusp form of weight -2 for a congruence subgroup $\Gamma'\subset\Gamma$. Suppose that Φ belongs to the Dirichlet character ε (for the definition see below, § 1) and that $\Phi|T_n=\lambda_n\Phi$ for all n coprime with a certain integer.*

Then there exist a primitive Dirichlet character χ (possibly principal), a locally constant function y and an integer M such that

(2) $$\lambda_n=\bar{\chi}(n)\Lambda_n(y,\varepsilon\chi^2)\quad\text{for all } n,\quad (n,M)=1.$$

In particular, if $\Gamma'=\Gamma_0(N)$ and $\int_0^{i\infty}\Phi(z)\,dz\neq0$, one may take $\chi=\varepsilon=1$, $M=2N$, and $y(\Delta,\delta)$ depends only on $(\Delta:\delta)\bmod N$, where we put $\Delta_1:\delta_1\equiv\Delta_2:\delta_2\bmod N$, if there exists an $a\in\boldsymbol{Z}$, $(a,N)=1$ such that $a\Delta_1\equiv\Delta_2\bmod N$, $a\delta_1\equiv\delta_2\bmod N$.

The proof together with the explicit construction of the function $y=y_\Phi$ is given in §§ 1–3. Before proceeding to it, we reproduce two numerical examples from [1], § 8.

4. Examples. (a) $\Gamma'=\Gamma_0(11)$, $N=11$; here $\varepsilon=1$ and $\int_0^{i\infty}\Phi(z)\,dz\neq0$, where

$$\Phi(z)=\sum_{n=1}^{\infty}\lambda_n e^{2\pi inz}=e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi inz})^2(1-e^{22\pi inz})^2.$$

The function $y_{11}(\Delta,\delta)$ depends only on $\Delta:\delta\bmod 11$ and is given in the following table:

$\Delta:\delta \bmod 11$	0	∞	±1	±2	±3	±4	±5
y_{11}	2	-2	0	10	5	-5	-10

Formulas (1), (2) for a prime $n=p\neq2,11$ take the form:

(3a) $$1-\lambda_p+p=\sum_{p=\Delta\Delta'+\delta\delta'}y_{11}(\Delta,\delta).$$

The left-hand side of (3a) has also a very simple number-theoretic meaning, that is, it coincides with the number of solutions of the congruence $y^2+y\equiv x^3-x^2-10x-20\bmod p$ (including the solution "at infinity"). As this last number can be calculated in terms of the eigenvalues of a Frobenius, we can consider (3a) as a noncommutative reciprocity relation (cf. below, § 4).

(b) $\Gamma'=\Gamma_0(27)$, $N=27$, $\Phi(z)$ is the unique cusp form for this group. Again $\int_0^{i\infty}\Phi(z)\,dz\neq0$, $\varepsilon=1$. The function y_{27} depends only on $\Delta:\delta\bmod 27$ and is given in the following table:

$\Delta:\delta \bmod 27$	0	∞	±1	±2	±3	±4	±5	±6	±7	±8	±9	±10	±11	±12	±13
y_{27}	2	-2	0	6	3	3	3	1	-3	0	-1	0	-3	-3	-6

$\Delta:\delta \bmod 27$	1:3	2:3	4:3	5:3	7:3	8:3	1:9	2:9
y_{27}	-3	3	0	0	3	-3	1	-1

Putting $\Phi|T_p=\mu_p\Phi$ we again have

(3b) $$1-\mu_p+p=\sum_{p=\Delta\Delta'+\delta\delta'}y_{27}(\Delta,\delta),\qquad p\neq2,3.$$

The left-hand side of (3b) coincides with the number of solutions of $y^2\equiv4x^3+1\bmod p$. It can also be classically expressed by means of some exponential sums or of Hecke's Grössencharaktere. (The reason is that the elliptic curve $y^2=4x^3+1$ has complex multiplication, unlike the case $N=11$.)

§ 1. The main identity. Let us fix N and a cusp form $\Phi(z)$ for the group $\Gamma(N)$ as in n°1. Denote by $R_a\in\Gamma$ for $(a,N)=1$ a substitution represented by a matrix $\equiv\begin{pmatrix}a^{-1}&0\\0&a\end{pmatrix}\bmod N$. Then the coset $\Gamma(N)R_a$ and the form $\Phi|R_a$ are well defined. The form Φ is said to belong to the Dirichlet character ε, if $\Phi|R_a=\varepsilon(a)\Phi$ for all a, $(a,N)=1$. Put

$$\Gamma_1(N)=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\Big|\ \begin{pmatrix}a&b\\c&d\end{pmatrix}\equiv\begin{pmatrix}1&*\\0&1\end{pmatrix}\bmod N\right\}\subset\Gamma(N).$$

On the space of the cusp forms for the group $\Gamma_1(N)$ Hecke operators T_n (for $(n,N)=1$) are given by the formula

(4) $$T_n=\sum_{d|n}\sum_{b\bmod d}R_{n/d}\begin{pmatrix}n/d&b\\0&d\end{pmatrix}.$$

The following standard lemma shows that it suffices to consider eigenvalues of T_n only on forms for $\Gamma_1(N)$:

5. Lemma. *Let Φ be a cusp form for $\Gamma(N)$ belonging to the character ε and let $\Phi|T_n = \lambda_n\Phi$ for all n, $(n, N) = 1$. Put $\Psi(z) = \Phi(Nz)$. Then*

(a) *Ψ is a cusp form for $\Gamma_1(N^2)$, belonging to ε.*

(b) *$\Psi|T_n = \lambda_n\Psi$ for all n, $(n, N) = 1$.*

Proof. First of all,

$$\Psi\Big|\begin{pmatrix} a & b \\ c & d\end{pmatrix}(z) = \Phi\Big|\begin{pmatrix} a & Nb \\ N^{-1}c & d\end{pmatrix}(Nz).$$

If

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_1(N^2),$$

then

$$\begin{pmatrix} a & Nb \\ N^{-1}c & d\end{pmatrix}\in\Gamma(N)$$

so that $\Psi(z)$ is a cusp form for $\Gamma_1(N^2)$. Moreover, if

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\equiv\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\bmod N^2,$$

then

$$\begin{pmatrix} a & Nb \\ N^{-1}c & d\end{pmatrix}\equiv\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\bmod N.$$

Hence Ψ belongs to the same character ε as Φ does. To check the last statement one can make a direct calculation or apply a Hecke theorem (Ogg [2], Theorem 1.2, p. IV–22) to the effect that Φ (and Ψ) are eigenfunctions for T_p and R_p ($p\nmid N$ prime) if and only if their Mellin transforms have an Euler product for p of a definite type. This last property of Ψ easily follows from the corresponding property of Φ. Q.E.D.

6. Lemma. *Let Φ be a cusp form for $\Gamma_1(N)$ belonging to ε and let $\Phi|T_n = \lambda_n\Phi$ for some n, $(n, N) = 1$. Then*

$$(5)\qquad \Big(\sum_{d|n}\varepsilon(n/d)d-\lambda_n\Big)\int_0^{i\infty}\Phi\,dz = \sum_{d|n}\tau_\varepsilon(n/d)\sum_{\substack{0<b<d/2\\(b,d)=1}}\Big(\int_0^{b/d}+\int_0^{-b/d}\Big)\Phi\,dz.$$

Proof. Integrate the identity $\lambda_n\Phi(z)dz = \Phi|T_n dz$ along the imaginary half axis. Using (4) and $\Phi|R_{n/d} = \varepsilon(n/d)\Phi$ we get:

$$\lambda_n\int_0^{i\infty}\Phi\,dz = \sum_{d|n}\varepsilon(n/d)\sum_{b\bmod d}\int_0^{i\infty}\Phi\Big(\frac{n}{d^2}z+\frac{b}{d}\Big)d\Big(\frac{n}{d^2}z+\frac{b}{d}\Big)$$

$$= \sum_{d|n}\varepsilon(n/d)\sum_{b\bmod d}\Big(\int_{b/d}^{i\infty}+\int_{i\infty}^{0}+\int_0^{i\infty}\Big)\Phi(z)\,dz.$$

Transfer all the last integrals to the left-hand side and change the signs:

$$(6)\qquad \Big(\sum_{d|n}\varepsilon(n/d)d-\lambda_n\Big)\int_0^{i\infty}\Phi\,dz = \sum_{d|n}\varepsilon(n/d)\int_0^{b/d}\Phi(z)\,dz.$$

Each fraction b/d with $(b, d) = 1$ enters the right-hand side of (6) (as an upper limit of integration) $\tau_1(n/d)$ times as $b\delta/d\delta$ for $\delta|n/d$. The coefficient of the corresponding integral is $\varepsilon(n/d\delta)$ and $\sum_{\delta|n/d}\varepsilon(n/d\delta) = \tau_\varepsilon(n/d)$. Hence

$$(7)\qquad \Big(\sum_{d|n}\varepsilon(n/d)d-\lambda_n\Big)\int_0^{i\infty}\Phi\,dz = \sum_{d|n}\tau_\varepsilon(n/d)\sum_{\substack{b\bmod d\\(b,d)=1}}\int_0^{b/d}\Phi\,dz.$$

To deduce (5) it suffices to remark that $\int_0^{(d-b)/d}\Phi\,dz = \int_0^{-b/d}\Phi\,dz$ because $\Phi(z+1) = \Phi(z)$. This concludes the proof.

§ 2. The case $\int_0^{i\infty}\neq 0$. In this section we prove Theorem 3 in the case when $\int_0^{i\infty}\Phi(z)\,dz\neq 0$.

7. Lemma. *Let Φ be a cusp form for $\Gamma_1(N)$ and let $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\mathrm{SL}(2, \mathbf{Z})$. Put*

$$(8)\qquad Y(c, d) = \Big(\int_{b/d}^{a/c}+\int_{-b/d}^{-a/c}\Big)\Phi(z)\,dz.$$

Then $Y(c, d) = Y(-c, -d)$ depends (Φ being given) *only on* ($c \bmod N$, $d \bmod N$), *hence Y is a locally constant function in the sense of* n° 2.

Proof. Let $X(\mathbf{C})$ be the Riemann surface, which is a standard compactification of $\Gamma_1(N)\backslash H$. For $\alpha, \beta\in H\cup\mathbf{Q}\cup(i\infty)$ denote by $\{\alpha, \beta\}\in H_1(X(\mathbf{C}), \mathbf{R})$ the homology class on $X(\mathbf{C})$ of the image of a path from α to β in H. If this image is not closed, its homology class is defined by integrating the differentials of the first kind as is explained in [1].

It suffices to check that $\{b/d, a/c\}$ depends only on ($c \bmod N$, $d \bmod N$), because

$$Y(c, d) = \int_{\{\frac{b}{d},\frac{a}{c}\}+\{-\frac{b}{d},-\frac{a}{c}\}}\varphi$$

where φ is the differential on $X(\mathbf{C})$ induced by $\Phi(z)dz$. Now

$$\Big\{\frac{b}{d}, \frac{a}{c}\Big\} = \Big\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}(0), \begin{pmatrix} a & b \\ c & d\end{pmatrix}(i\infty)\Big\}.$$

Moreover $\{g(\alpha), g(\beta)\} = \{\alpha, \beta\}$ for each $g \in \Gamma_1(N)$. Hence $\{b/d, a/c\}$ depends only on the coset $\Gamma_1(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which in turn is easily seen to depend only on $(c \bmod N, d \bmod N)$.Q.E.D.

8. Lemma. *Under the assumptions of Lemma 8 for all d, $(d, 2N) = 1$, we have*

$$\sum_{\substack{0<b<d/2 \\ (b,d)=1}} \left(\int_0^{b/d} + \int_0^{-b/d}\right) \Phi\, dz = \sum_{d=\Delta\Delta'+\delta\delta'} Y(\Delta, \delta) \tag{9}$$

where the right-hand sum is taken over all admissible solutions.

Proof. We reproduce the reasoning of [1]. Let

$$\frac{b}{d} = \frac{b_n}{d_n}, \quad \frac{b_{n-1}}{d_{n-1}}, \ \ldots, \ \frac{b_0}{d_0} = \frac{0}{1}$$

be the consecutive convergents of b/d. Then $b_{i-1}d_i - b_i d_{i-1} = (-1)^i$, that is

$$\begin{pmatrix} b_i & (-1)^{i+1} b_{i-1} \\ d_i & (-1)^{i+1} d_{i-1} \end{pmatrix} \in \Gamma.$$

Moreover

$$\int_0^{b/d} \Phi(z)\, dz = \sum_{i=1}^{n} \int_{b_{i-1}/d_{i-1}}^{b_i/d_i} \Phi(z)\, dz, \qquad \int_0^{-b/d} \Phi\, dz = \sum_{i=1}^{n} \int_{-b_{i-1}/d_{i-1}}^{-b_i/d_i} \Phi\, dz.$$

Hence by (8)

$$\left(\int_0^{b/d} + \int_0^{-b/d}\right) \Phi\, dz = \sum_{i=1}^{n} Y(d_i, d_{i-1}).$$

Now sum over b, $0 < b < d/2$, $(b, d) = 1$ and apply a Heilbronn lemma ([1], Lemma 7.7). It states that the family of pairs of denominators (d_i, d_{i-1}) of the consecutive convergents of all our b/d coincides with the family of pairs (Δ, δ) taken from all admissible solutions of $d = \Delta\Delta' + \delta\delta'$. This concludes the proof.

9. Theorem. *Let Φ be a cusp form of weight -2 for $\Gamma_1(N)$ belonging to ε and let $\Phi | T_n = \lambda_n \Phi$ for all n, $(n, 2N) = 1$.*

(a) *If $\int_0^{i\infty} \Phi\, dz = 0$, then*

$$\sum_{d|n} \tau_\varepsilon(n/d) \sum_{d=\Delta\Delta'+\delta\delta'} Y(\Delta, \delta) = 0 \quad \textit{for all } n, \quad (n, 2N) = 1. \tag{10}$$

(b) *If $\int_0^{i\infty} \Phi\, dz \neq 0$, then putting*

$$y(\Delta, \delta) = Y(\Delta, \delta) \Big/ \int_0^{i\infty} \Phi\, dz \tag{11}$$

we have

$$\sum_{d|n} \varepsilon(n/d) d - \lambda_n = \sum_{d|n} \tau_\varepsilon(n/d) \sum_{d=\Delta\Delta'+\delta\delta'} y(\Delta, \delta) \quad \textit{for all } n, \quad (n, 2N) = 1. \tag{12}$$

The last sum in (10) *and* (12) *is taken over all admissible solutions. Functions $Y(\Delta, \delta)$ and $y(\Delta, \delta)$ depend only on $(\Delta, \delta) \bmod N$.*

Proof. To prove (10) substitute (9) in (5). Then divide by $\int_0^{i\infty} \Phi\, dz$ to get (12). Note that (12) is equivalent to (2) for $\chi \equiv 1$.

10. Particular cases. Let $n = p$ be a prime. Identities (10) and (12) become respectively

$$\sum_{p=\Delta\Delta'+\delta\delta'} Y(\Delta, \delta) = 0 \quad \text{for all } p \nmid 2N, \tag{13}$$

$$\varepsilon(p) - \lambda_p + p = \sum_{p=\Delta\Delta'+\delta\delta'} y(\Delta, \delta) \quad \text{for all } p \nmid 2N. \tag{14}$$

§ 3. Reduction to the case $\int_0^{i\infty} \neq 0$. Let Φ be a cusp form fulfilling the assumptions of Theorem 9. Let $\Phi(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ and, for a Dirichlet character $\chi \bmod m$, $(m, N) = 1$, put $\Phi_\chi(z) = \sum_{n=1}^{\infty} \chi(n) a_n e^{2\pi i n z}$.

11. Lemma. *$\Phi_\chi(z)$ is a cusp form for $\Gamma_1(m^2 N)$ belonging to the character $\varepsilon\chi^2$. Moreover, $\Phi_\chi | T_n = \chi(n) \lambda_n \Phi_\chi$ for all n, $(n, mN) = 1$.*

To prove the lemma, combine Theorems 14 and 12 of Ogg's book [2].

12. Lemma. *Suppose that $\int_0^{i\infty} \Phi\, dz = 0$. Then there exist an infinity of primes l and primitive characters $\chi \bmod l$ such that $\int_0^{i\infty} \Phi_\chi dz \neq 0$.*

Proof. (a) Let $X(C)$ and $\{\alpha, \beta\} \in H_1(X(C), \boldsymbol{R})$ be the same as in the proof of Lemma 7. First we show that the classes $\{0, b/l\}$ for all $l > C$ (any constant) and all $b \bmod l$ generate the whole group $H_1(X(C), \boldsymbol{Z})$. In fact, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$. We have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ * & cx+d \end{pmatrix}.$$

By the Dirichlet theorem we can make $cx + d$ a prime $> C$ choosing x appropriately. Hence $\Gamma_1(N)$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and matrices whose lower right coefficient is a big prime. But there is the canonical surjective map $\Gamma_1(N)\to H_1(X(C), \boldsymbol{Z})$: $g\mapsto\{0, g(0)\}$ (cf. [1], Proposition 1.4). This shows that the set of classes $\{0, b/l\}$, $l > C$ coincides in fact with the whole homology group.

(b) Suppose now that $\int_0^{i\infty} \Phi_\chi dz = 0$ for all $\chi \bmod l$, $l > C$. We get a contradiction.

For $\chi \neq 1$ we have:

$$\Phi_\chi(z) = \frac{g(\chi)}{l} \sum_{b \bmod l} \bar{\chi}(-b)\Phi(z+b/l)$$

where

$$g(\chi) = \sum_{b \bmod l} \chi(b) e^{2\pi i \frac{b}{l}} \neq 0$$

is a Gauss sum.

Hence

$$\int_0^{i\infty} \Phi_\chi dz = \frac{g(\chi)}{l} \sum_{b \bmod l} \bar{\chi}(-b) \int_{b/l}^{i\infty} \Phi(z)\,dz = \frac{g(\chi)}{m} \sum_{b \bmod l} \bar{\chi}(-b) \int_{\{b/l, 0\}} \varphi$$

where φ is the differential of the first kind on $X(C)$ induced by Φdz. (To change the upper limit $i\infty$ to 0 in the last integral we use $\sum_{b \bmod l} \bar{\chi}(-b) = 0$.)

So if $\int_0^{i\infty} \Phi_\chi dz = 0$ for all $\chi \bmod l$, then $\int_{\{b/l, 0\}} \varphi = 0$ for all $b \bmod l$. But this cannot be true for all primes $l > C$ because in that case the first part of the proof shows that φ has zero periods. This concludes the proof of the lemma.

Combining Lemmas 11 and 12 with Theorem 9 we finally get the missing part of Theorem 3 (except the statement for $\Gamma_0(N)$, which is proved in [1]).

13. Corollary. *Given the assumptions of Theorem 9, suppose that* $\int_0^{i\infty} \Phi_\chi dz \neq 0$ *for a fixed* $\chi \bmod l$ *(such an* χ *exists). For all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \boldsymbol{Z})$ *put*

$$y(c, d) = \Big(\int_{b/d}^{a/c} + \int_{-b/d}^{-a/c}\Big) \Phi_\chi(z)\,dz \Big/ \Big(\int_0^{i\infty} \Phi_\chi(z)\,dz\Big). \tag{15}$$

This function depends only on $(C \bmod l^2N, d \bmod l^2N)$. *Moreover, for all* n, $(n, l^2N) = 1$,

$$\sum_{d|n} \varepsilon\chi^2(n/d)\,d - \chi(n)\lambda_n = \sum_{d|n} \tau_{\varepsilon\chi^2}(n/d) \sum_{d=\Delta\Delta'+\delta\delta'} y(\Delta, \delta) \tag{16}$$

which is equivalent to (2).

§ 4. Corollaries and remarks.

14. The identity (2) means that the two number-theoretical functions defined by completely different means in fact coincide. So we can try to get some information on the eigenvalues λ_n by looking at admissible solutions or *vice versa* to interpret the known properties of λ_n's as indicating some features of the distribution of admissible solutions. We state below some results of the second type because the first possibility has not been explored at all. We would like just to mention once more a particularly baffling question concerning the statistics of admissible solutions to which one can reduce the Sato-Tate conjecture [3] on the distribution of arguments of the Frobenius automorphisms of elliptic curves uniformized à la Weil.

15. The distribution of admissible solutions in residue classes. The total number of admissible solutions is asymptotically (as $d\to\infty$)

$$\sum_{\Delta\Delta'+\delta\delta'=d} 1 = \frac{6\ln 2}{\pi^2}\varphi(d)\Big[\ln d + \sum_{p|d} \frac{\ln p}{p-1}\Big] + O\Big(\sum_{\delta|d} \delta\Big) \tag{17}$$

where $\varphi(d)$ is the Euler function. In particular, for $d = p$ (prime):

$$\sum_{\Delta\Delta'+\delta\delta'=p} 1 = \frac{6\ln 2}{\pi^2} p\ln p + O(p). \tag{18}$$

The first term in (17) is given by Heilbronn, the second one and the error term by T. Tonkov; their methods are elementary. The error term is unusually bad and possibly conceals something interesting. The coefficient $6\ln 2/\pi^2$ arises naturally in the statistics of continued fractions, irrational ones (distribution "almost everywhere", Kuzmin, Khintchin, Lévy) and rational ones (with a fixed denominator, Heilbronn).

Our identities (3a), (3b), (14) may be interpreted as displaying some properties of uniform distribution of admissible solutions in residue classes (modulo the period of y). For example, we get from (3a) (using Hasse's and Eichler's inequality $|\lambda_p| < 2\sqrt{p}$):

$$\Big(2\sum_{\Delta/\delta\equiv\pm 2} 1 + \sum_{\Delta/\delta\equiv\pm 3} 1\Big) - \Big(2\sum_{\Delta/\delta\equiv\pm 5} 1 + \sum_{\Delta/\delta\equiv\pm 4} 1\Big) = \frac{p+1}{5} - \frac{\lambda_p}{5} = \frac{p}{5} + O(\sqrt{p}). \tag{19}$$

The sums are taken over admissible solutions of $\Delta\Delta'+\delta\delta' = p$ and the congruences are mod 11. Comparing this with (18) we see that the number of solutions with $\Delta/\delta \equiv \pm 2, \pm 3$, although asymptotically the same as the number of solutions with $\Delta/\delta \equiv \pm 4, \pm 5$ (counted with weights), is still noticeably greater: the main term of the difference is only (const. $\ln p$) times less than the total number of solutions. The identity (3b) can be interpreted similarly. But a suitable distribution modulo $3^3 \cdot 11$ is much more regular:

$$\sum (y_{11} - y_{27})(\Delta, \delta) = \mu_p - \lambda_p = O(\sqrt{p}). \tag{20}$$

The formulas (13) occurring for $\int_0^{i\infty} \Phi(z)\,dz = 0$ are probably even more surprising:

$$\sum_{p=\Delta\Delta'+\delta\delta'} Y(\Delta, \delta) = 0 \quad \text{for all } p > C.$$

To show the existence of nontrivial identities of this kind, that is of forms Φ with $Y(\Delta, \delta) \not\equiv 0$, one can proceed as follows. Take a cusp form Φ for the group $\Gamma_0(N)$ ($\varepsilon \equiv 1$) with rational coefficients corresponding to the differential of the first kind on an elliptic curve E over Q (cf. [1], § 5). The condition $\int_0^{i\infty} \Phi\,dz = 0$ means that the L-function of E has a zero in the centre of its critical strip. This last condition in the range of tables (and conjecturally always) means that the rank $E(Q)$ is greater than 0.

All these conditions are fulfilled, for example, for the curve E: $y^2 + y = x^3 - x$ with $N = 37$. For the form Φ of this type we have $Y(\Delta, \delta) \not\equiv 0$, because otherwise all periods of Φ would be imaginary, which is impossible. It would be interesting to look at some such Y. They can be calculated by means of an algorithm described in [1], § 8 and applied there to the cases $N = 11, 17, 19, 27$.

16. The density of some sets of primes. V. Shokurov made the following remark: the density of those primes p for which

$$\sum_{p=\Delta\Delta'+\delta\delta'} y_{11}(\Delta, \delta) \equiv 0 \bmod l \tag{21}$$

($l \neq 5$ a fixed prime) is equal to $1/(l-1)^2$.

The proof is a combination of the following facts. The left-hand side of (21) coincides with $\mathrm{card}(\bar{E}(F_p))$, where $\bar{E} = E \bmod p$, E is the elliptic curve over Q, given in n° 3. Let G_l be the Galois group of the extension of Q by the points of the order l on E. Serre ([4], p. 309) showed that $G_l \cong \mathrm{GL}(2, F_l)$ if $l \neq 5$. On the other hand, let $H_l = \left\{\begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix}\right\} \subset G_l$ and let K_l' be equal to the subfield of K_l, corresponding to H_l for $l \neq 5$. Then Shimura [5] showed that the congruence $\mathrm{Card}(\bar{E}(F_p)) \equiv 0 \bmod l$ holds if and only if p has a prime divisor of degree 1 in K_l'. The density of such primes is easily calculated by means of Tchebotarev's theorem (cf. [3]).

17. Some non-linear identities. We have $\lambda_m \lambda_n = \lambda_{mn}$ if $(m, n) = 1$, and $\lambda_{p^a} = \lambda_p \lambda_{p^{a-1}} - p\lambda_{p^{a-2}}$, p prime. Applying these identities to the functions $\Lambda_n(y, \varepsilon)$ we get some non-linear relations between them which

can hardly be seen directly. For example the identity $\lambda_{p^2} = \lambda_p^2 - p$ applied to y_{11} from (3a) gives

$$\sum_{p^2=\Delta\Delta'+\delta\delta'} y_{11}(\Delta, \delta) = 2p \sum_{p=\Delta\Delta'+\delta\delta'} y_{11}(\Delta, \delta) - \Big(\sum_{p=\Delta\Delta'+\delta\delta'} y_{11}(\Delta, \delta)\Big)^2.$$

18. Possible generalizations. The methods of this note generalize to cusp forms of higher weight. This involves the vector forms of Eichler–Shimura, and will be discussed in a later publication. The case of the Hilbert modular group can probably also be treated similarly but looks more difficult. Finally, the functional $\Phi \mapsto \int_0^{i\infty} \Phi\,dz$ has interesting properties from the point of view of representation theory. This work can be considered as a preliminary attempt to get an insight into these properties.

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