ACTA ARITHMETICA
XXIV (1973)

On Hirzebruch sums and a theorem of Schinzel

by

P. CHOWLA and S. CHOWLA (Princeton, N. J.)

Dedicated to Carl Ludwig Siegel on the occasion of his 75th birthday

Let N be a positive square-free integer. Expressing $\sqrt{N}$ as an infinite simple continued fraction we get

$$\sqrt{N} = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_s} + \ldots,$$

where the "period" starts with the second term, and consists of s terms. We can also define s as the least positive integer such that $a_s = 2a_0$. Recently (the second author learnt this in oral conversation with Hirzebruch a few months ago) Hirzebruch proved the following astonishing and surprising theorem:

If N is a prime $\equiv 3\,(4) > 3$, such that the class-number of $Q(\sqrt{+N})$ is 1, then

$$\Sigma_N = a_s - a_{s-1} + a_{s-2} - a_{s-3} + \ldots \pm a_1$$

has the property

$$\Sigma_N = 3h(-N).$$

Here $h(d)$ denotes the class-number of $Q(\sqrt{d})$.

This remarkable theorem led us to make the following conjectures.

Let N be a number $\equiv 3\,(4)$, $3 \nmid N$. Then

(i) $$\Sigma_N > 0;$$

(ii) $$3 \mid \Sigma_N.$$

Next let p be a prime $\equiv 3\,(4)$, then for $p > 3$,

(iii) $$\Sigma_p \text{ is an odd multiple of } 3.$$

(iv) If $N = 2p$, where N is a prime $\equiv 3(4)$ and $h(2p) = 1$, then

$$\Sigma_{2p} = 6h(-p).$$

Hence if p is a prime $\equiv 3(4)$ and if $h(p) = h(2p) = 1$ then ($p = 4519$ is an example)

$$\Sigma_{2p} = 2\Sigma_p.$$

Both (ii) and (iii) have been proved by Andrzej Schinzel, in fact in generalized form. His beautiful proof is very elementary.

PENNSYLVANIA STATE UNIVERSITY
INSTITUTE FOR ADVANCED STUDY

Received on 26. 9. 1972 (328)