We can therefore select a subset $\mathscr{S}^*$ of $\mathscr{X}^*$, satisfying $|\mathscr{S}^*| = 2M^2$ and containing no elements of any of the sets

$$-c-a^{(\nu)}+\overline{\mathscr{N}} \quad (\nu = 1, 2, \ldots, |\mathscr{B}|).$$

This last property is equivalent to the desired condition $\mathscr{B}+\mathscr{S}^* \subset \mathscr{N}$.

References

[1] S. Chowla, *The Riemann zeta and allied functions*, Bull. American Math. Soc. 58 (1952), pp. 287–305.

[2] — *Some applications of a method of A. Selberg*, J. Reine Angew. Math. 217 (1965), pp. 128–132.

[3] Paul J. Cohen, *On a conjecture of Littlewood and indempotent measures*, American J. Math. 82 (1960), pp. 191–212.

[4] H. Davenport, *On a theorem of P. J. Cohen*, Mathematika 7 (1960), pp. 93–97.

[5] G. H. Hardy and J. E. Littlewood, *A new proof of a theorem on rearrangements*, J. London Math. Soc. 23 (1948), pp. 163–168.

[6] Uchiyama, née Katayama, Miyoko; Uchiyama, Saburô, *On the cosine problem*, Proc. Japan Acad. 36 (1960), pp. 475–479.

[7] Uchiyama, Saburô, *À propos d'un problème de M. J. E. Littlewood*, C. R. Acad. Sci. Paris 260 (1965), pp. 2675–2678.

[8] R. Salem, *On a problem of Littlewood*, American J. Math. 77 (1955), pp. 535–540.

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On gaps between numbers with a large prime factor

by

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In honour of Professor C. L. Siegel on his 75th birthday

§ 1. Introduction. Let k be a fixed natural number and $n_1, n_2, n_3, \ldots$ all the natural numbers (in the increasing order) which have at least one prime factor exceeding k. It is easy to see that

$$f(k) = \max_{i=1,2,\ldots} (n_{i+1}-n_i)$$

is finite. P. Erdös was the first to give the estimate $f(k) = O\left(\frac{k}{\log k}\right)$ ([3]) which goes beyond the trivial estimate $f(k) = O(k)$. (The problem of the O-constant here was also difficult and a non-trivial value for the O-constant was obtained by Sylvester and Schur independently. See [2].) His argument gives that the upper limit of $f(k)\left(\frac{k}{\log k}\right)^{-1}$ as k tends to infinity does not exceed 3. Even the lowering of this constant 3 seemed difficult and Erdös remarked in [3] that the proof of $f(k) < \pi(k)$ for all large k would prove to be considerably difficult (it is now known by Tijdeman's result to be mentioned immediately). The first author of the present paper reduced 3 to 1 in [5]. Further he made some partial progress in the direction of reducing this constant to $\frac{1}{2}$, in [6]. A part of the proof in [6] depended on an adaptation of the Roth–Halberstam method (see [3] of [6]), and an ingeneous adaptation of this method was made by Tijdeman [11] who reduced 1 to $\frac{1}{2}$. The other part of [6] depended on Baker's methods and in this paper we develop this method and use the results of [5] and [7] to prove the following

Main Theorem. *We have*

$$f(k) = O\left(\frac{k}{\log k}\left(\frac{\log\log\log\log k}{\log\log\log k}\right)^{1/2}\right)$$

where the O-constant is computable (but it is tedious to do so).

In our proof the approach of Roth–Halberstam–Ramachandra–Tijdeman is not necessary. In § 3 we can argue completely along Baker's lines [1] with suitable modifications but we use a result of Tijdeman [12] (which is an improvement of a result due to Mahler) since it will simplify our proof to some extent. We have also to use a choice of certain parameters $k_1, k_2, \ldots$ due to the first of us (see [7]). We also follow a simple version of the pattern of proof in [10].

§ 2. Some preparations. Let $P(u, k)$ denote the maximum prime factor of the product $(u+1)(u+2)\ldots(u+k)$.

LEMMA 1. *Let $k \leqslant u \leqslant k^{3/2}$. Then*

$$P(u, k) \geqslant u+1.$$

Proof. Apply Hoheisel–Ingham–Montgomery–Huxley theorem [4] that

$$p_{n+1} - p_n \ll p_n^{\frac{1}{2}+\frac{1}{12}+\varepsilon}$$

where p_n denotes the nth prime and $\varepsilon > 0$ is an arbitrary constant.

LEMMA 2. *Let $k^{3/2} \leqslant u \leqslant k^{\log\log k}$. Then*

$$P(u, k) > k^{1+2^{\lambda}} \quad \textit{where} \quad \lambda = \lambda(u, k) = -\left(8 + \frac{\log u}{\log k}\right).$$

Proof. This is the main theorem in [5].

LEMMA 3. *If $u > k^{\log\log k}$ then*

$$P(u, k) \gg \min\left\{k^2, k\log k\left(\frac{\log\log u}{\log\log k + \log\log\log u}\right)^{1/2}\right\},$$

where the implied constant is absolute.

Proof. See the introduction in [7].

LEMMA 4. *If $k \geqslant 2$ then $P(u, k) \gg \log\log u$, where the implied constant is absolute.*

Proof. See the introduction in [7]. This follows by Baker's work on the diophantine equation $y^2 = ax^3 + bx^2 + cx + d$.

Let now $k \geqslant k_0$, a large absolute constant,

$$k^{\log\log k} \leqslant u \leqslant e^{(\log k)^L} \quad \text{where} \quad 2 \leqslant L \leqslant (\log k)^{1/4}$$

and we shall specify L later. (If $u \geqslant e^{(\log k)^L}$ Lemmas 3 and 4 give $P(u, k) \gg L^{1/2} k\log k$ where the implied constant is absolute.) In this range we know by the results of [5] that $P(u, k) \geqslant \frac{3}{4} k\log k$. We shall suppose that

$$P(u, k) \leqslant \tfrac{1}{2} Mk\log k \quad \text{where} \quad \tfrac{3}{2} \leqslant M \leqslant (\log k)^{1/4}$$

(we shall specify L and M later and arrive at a contradiction). Let us write $n = m'm''$ where $u < n \leqslant u+k$ and m' is the product of all powers of primes not exceeding k which divide n and m'' consists of primes exceeding k. Let $\omega(m'')$ denote the number of distinct prime factors of m''. Then it is easy to see that

$$\sum_n \omega(m'') \leqslant \tfrac{3}{5} Mk.$$

Hence the number of integers n with $\omega(m'') \geqslant M$ does not exceed $\frac{3}{5}k$. Thus there exist at least $[\frac{2}{5}k]$ natural numbers n with $\omega(m'') \leqslant M$. For each prime $q \leqslant k$ we omit amongst these n, one n for which q divides n to a maximal power. If star denotes the omission of these numbers n, then it follows by an argument of Erdös that

$$\prod_n{}^{*} m' \leqslant k^k.$$

The number of n's counted in this product is $\geqslant [\frac{2}{5}k] - \pi(k) \geqslant \frac{1}{3}k$ and so the number of numbers n (counted in this product) for which $m' \geqslant k^{100}$ is at most $k/100$. Hence we have the following

LEMMA 5. *Let*

$$k \geqslant k_0, \quad k^{\log\log k} \leqslant u \leqslant e^{(\log k)^L} \quad \textit{where} \quad 2 \leqslant L \leqslant (\log k)^{1/4}$$

and

$$P(u, k) \leqslant \tfrac{1}{2} Mk\log k \quad \textit{where} \quad \tfrac{3}{2} \leqslant M \leqslant (\log k)^{1/4}.$$

Then there exist amongst the natural numbers $u+1, u+2, \ldots, u+k$ at least $[\frac{1}{4}k]$ distinct natural numbers n for which

$$n = m'm''$$

where m' has all its prime factors $\leqslant k$, $m' \leqslant k^{100}$ and the distinct prime factors of m'' are all $> k$ and their number $\omega(m'')$ is at most $M_0 = [M]$.

For each n occurring in connection with this lemma we can write

$$n = m' p_1^{l_1} \ldots p_j^{l_j}$$

(where $j = M_0$ and $p_1, \ldots, p_j$ are distinct and exceed k) where all the prime factors of m' are $\leqslant k$, $m' \leqslant k^{100}$ and $l_1, \ldots, l_j$ are non-negative integers not exceeding $(\log k)^L$. For each n we fix a definite ordering of the prime factors $p_1, p_2, \ldots, p_j$. The total number of j-tuples $(l_1, l_2, \ldots, l_j)$ for various n does not exceed $(\log k)^{ML}$. Hence for at least $K = [\frac{1}{4}k(\log k)^{-3ML}]$ natural numbers n, we have the same j-tuple $(l_1, l_2, \ldots, l_j)$. We now order the numbers m' (occurring in various n) in the non-decreasing order, say

$$1 \leqslant m_1 \leqslant m_2 \leqslant \ldots \leqslant m_K \leqslant k^{100}.$$

Let J be an integer $\leqslant K(\log\log k)^{-1}$, to be chosen later, and x_0 a positive real number with

$$\frac{m_{J+a}}{m_a} \geqslant (1+x_0), \qquad a = 1, 2, \ldots, K-J.$$

This would lead to a contradiction if $k^{200J} < (1+x_0)^{K-J}$ (so it suffices to set $x_0 = 400JK^{-1}\log k$ to get a contradiction).

Hence we have

$$0 < \log\frac{m_{J+a}}{m_a} \leqslant 400JK^{-1}\log k$$

for at least one a in $1 \leqslant a \leqslant K-J$. Next we order the first primes p_1 for those n's for which the numbers m' satisfy $m_a \leqslant m' \leqslant m_{J+a}$, say

$$2 \leqslant p_1^{(1)} < p_1^{(2)} < \ldots < p_1^{(J)} \leqslant kM\log k.$$

Arguing as above we end up with a natural number $J_1 \leqslant J(\log\log k)^{-1}$ such that

$$0 < \log\frac{p_1^{(J_1+a_1)}}{p^{(a_1)}} \leqslant x_1$$

where x_1 satisfies $(Mk\log k)^{J_1} < (1+x_1)^{J-J_1}$ say $x_1 = (2J_1\log k)/J$. Proceeding similarly we get

$$0 < \log\frac{p_r^{(J_r+a_r)}}{p_r^{(a_r)}} \leqslant x_r$$

where $x_r = \dfrac{2J_r\log k}{J_{r-1}}$, $J_r \leqslant \dfrac{J_{r-1}}{\log\log k}$ and $r = 2, 3, \ldots, j = M_0 = [M]$.

Let $a = \dfrac{1}{2(M+1)}$, $J = K^{a(j+1)}$, $J_1 = K^{ja}, \ldots, J_j = K^a$ so that $K^a \geqslant 4$ and so there exist two integers n_λ and n_μ satisfying $u < n_\lambda < n_\mu \leqslant u+k$ for which the corresponding numbers m' and the primes $p_1, \ldots, p_j$ satisfy

$$\left|\log\frac{m'_\lambda}{m'_\mu}\right| \leqslant \frac{(\log k)^2}{K^a}, \qquad \left|\log\frac{p_r^{(\lambda)}}{p_r^{(\mu)}}\right| \leqslant \frac{(\log k)^2}{K^a} \qquad (r = 1, 2, \ldots, j).$$

Also since $1 < \dfrac{n_\mu}{n_\lambda} \leqslant \dfrac{u+k}{u}$ we have $0 < \log\dfrac{n_\mu}{n_\lambda} \leqslant \dfrac{k}{u}$ and we arrive at the following

LEMMA 6. *Suppose that the assumptions of Lemma 5 are satisfied. Then there exists a solution of the inequality*

$$0 < \left|\sum_{i=1}^{j_0}\beta_i\log a_i\right| \leqslant \frac{k}{u} \qquad (j_0 \leqslant M)$$

where $a_1, a_2, \ldots, a_{j_0}$ are multiplicatively independent positive rational numbers with height not exceeding k^{100}, $\beta_1, \ldots, \beta_{j_0}$ are rational integers not all zero with absolute values not exceeding $\dfrac{\log(2u)}{\log k} \leqslant (\log k)^L$, *and finally $|\log a_i|$ for $i = 1, 2, \ldots, j_0$ does not exceed*

$$\left(\frac{k}{8}(\log k)^{-3ML}\right)^{-1/(2(M+1))}(\log k)^2 \leqslant k^{-3/(10M)}(\log k)^{5L} \leqslant k^{-1/4M}$$

since

$$LM \leqslant \frac{1}{20}\,\frac{\log k}{\log\log k}.$$

We shall now fix $L = (\log k)^{1/4}$ and observe that the heights of a_i and β_i do not exceed k^{100}. We apply the theorem of the next section with $n = j_0$, $A = 400j_0$, $S_1 = k^{100}$. (If $j_0 = 1$ we have $|\beta_1\log a_1| \geqslant k^{-100}$ which contradicts $u > k^{\log\log k}$ whereas if $j_0 \geqslant 2$ we can write

$$\left|\sum_{i=1}^{j_0}\beta_i\log a_i\right| \geqslant \left|\sum_{i=1}^{j_0-1}\beta'_i\log a_i - \log a_{j_0}\right|$$

where the heights of β'_i (β'_i rational) do not exceed k^{100}.) Thus we have

$$\frac{k}{u} > \exp\{-(400j_0^2)^{Cj_0^2}\log k^{100}\},$$

i.e.,

$$u < k\exp\{100(20M)^{2CM^2}\log k\}.$$

This leads to a contradiction if we set

$$M = d\left(\frac{\log\log\log k}{\log\log\log\log k}\right)^{1/2}$$

(d, a small positive absolute constant), since we have already imposed $u > k^{\log\log k}$. Thus we have proved

LEMMA 7. *If $k^{\log\log k} < u < \exp(\log k)^{(\log k)^{1/4}}$ then*

$$P(u, k) \gg k\log k\left(\frac{\log\log\log k}{\log\log\log\log k}\right)^{1/2}$$

where the implied constant is absolute.

From Lemmas 1, 2, 3, 4 and 7 the main theorem easily follows. However, the proof of Lemma 7 depended on the results of the next section and so the results of the next section will complete the proof of the Main Theorem stated in the introduction.

§ 3. Some results relating to Baker's theory. The main object of this section is to prove the following theorem. (The notation in this section is completely independent of the rest of the paper and should not cause any confusion.) The proof of the following theorem can be simplified slightly but we retain the following proof since it admits some generalizations.

THEOREM. *Let $n > 1$ be an integer. Let $a_1, \ldots, a_n$ be rational numbers satisfying*

(i) $a_1 > 0, \ldots, a_n > 0$ *are multiplicatively independent.*

(ii) $|\log a_i| \leqslant \exp\left(-\frac{1}{A}\log S_1\right)$, $1 \leqslant i \leqslant n$ *and* $A > 1$.

(iii) *The size of* $a_1, \ldots, a_n$ *do not exceed* S_1.

(*The size of a rational number* $\frac{a}{b}$, $(a, b) = 1$ *is defined as* $|b| + \left|\frac{a}{b}\right|$.)

If $\beta_1, \ldots, \beta_{n-1}$ *are rational numbers of size not exceeding* S_1, *then*

$$|\beta_1 \log a_1 + \ldots + \beta_{n-1}\log a_{n-1} - \log a_n| > \exp\left(-(nA)^{cn^2}\log S_1\right)$$

where c is an effectively computable positive constant, which is independent of n, A and S_1.

The proof of the theorem depends on the following lemmas.

LEMMA 1. *Suppose that the coefficients of the p-linear forms* $y_k = a_{k,1}x_1 + \ldots + a_{k,q}x_q$ $(k = 1, \ldots, p; p < q)$ *are integers in an algebraic number field of degree h and let* $\overline{|a_{k,l}|} \leqslant A$ ($\overline{|a_{k,l}|}$ *denotes the maximum of the absolute values of the conjugates of* $a_{k,l}$). *Then there exist rational integers* $x_1, \ldots, x_q$, *not all zero, satisfying* $y_1 = 0, \ldots, y_p = 0$ *and such that*

$$|x_k| < 1 + (2qA)^{\frac{ph(h+1)}{2q - ph(h+1)}}, \quad k = 1, \ldots, q$$

provided $2q > ph(h+1)$ *and* $A \geqslant 1$.

This is a generalization of a lemma due to Siegel. See Ramachandra [9], p. 16.

LEMMA 2. *Let m, s and t be positive integers and set $r = st$. Let* $\alpha_0, \ldots, \alpha_{m-1}$ *and* $\beta_0, \ldots, \beta_{s-1}$ *be m and s distinct complex numbers, respectively, and let*

$$a = \max_{0 \leqslant \nu < m}(|\alpha_\nu|, 1), \qquad b = \max_{0 \leqslant \sigma < s}(|\beta_\sigma|, 1),$$

$$a_0 = \min_{\substack{0 \leqslant \nu < m \\ 0 \leqslant \mu < m \\ \nu \neq \mu}}(|\alpha_\mu - \alpha_\nu|, 1), \qquad b_0 = \min_{\substack{0 \leqslant \sigma < s \\ 0 \leqslant \varrho < s \\ \varrho \neq \sigma}}(|\beta_\varrho - \beta_\sigma|, 1).$$

Put for arbitrary complex numbers A_ν

$$E(z) = \sum_{\nu=0}^{m-1} A_\nu e^{\alpha_\nu z}$$

and

$$A = \max_{0 \leqslant \nu < m}|A_\nu|, \qquad E = \max_{\substack{0 \leqslant \varrho < t \\ 0 \leqslant \sigma < s}}|E^{(\varrho)}(\beta_\sigma)|.$$

Assume that

(1) $$r \geqslant 2m + 13ab.$$

Then

$$A \leqslant s\sqrt{m!}\,e^{7ab}\left(\frac{1}{2a_0 b}\right)^{m-1}\left(\frac{72b}{b_0\sqrt{s}}\right)^r E.$$

This is due to Tijdeman [12].

Proof of the Theorem. Without loss of generality, assume that S_1 (and also S to be introduced) exceeds certain absolute positive constant. Define

$$\beta = |\beta_1 \log a_1 + \ldots + \beta_{n-1}\log a_{n-1} - \log a_n|$$

and assume that $\beta < \frac{1}{2}$. Assume that the size of $\beta_1, \ldots, \beta_{n-1} \leqslant S$. Consider the following auxiliary function

$$\Phi(z_1, \ldots, z_{n-1}) = \sum_{\lambda_1=0}^{L} \ldots \sum_{\lambda_n=0}^{L} p(\lambda_1, \ldots, \lambda_n)\, a_1^{\gamma_1 z_1} \ldots a_{n-1}^{\gamma_{n-1} z_{n-1}}$$

where $\gamma_r = \lambda_r + \lambda_n \beta_r$, $1 \leqslant r < n$, and $p(\lambda_1, \ldots, \lambda_n)$ are rational integers, not all zero, to be determined under the condition

(2) $$q(l, m_1, \ldots, m_{n-1}) = 0$$

for $1 \leqslant l \leqslant h$ and for all non-negative integers $m_1, \ldots, m_{n-1}$ such that $0 \leqslant m_1 + \ldots + m_{n-1} \leqslant k$, where

$$q(l, m_1, \ldots, m_{n-1}) = \sum_{\lambda_1=0}^{L} \ldots \sum_{\lambda_n=0}^{L} p(\lambda_1, \ldots, \lambda_n)\, a_1^{\lambda_1 l} \ldots a_n^{\lambda_n l} \gamma_1^{m_1} \ldots \gamma_{n-1}^{m_{n-1}}$$

(L, h, k are large integers to be suitably chosen).

Condition (2) is a set of at most $h(k+1)^{n-1}$ linear equations in $(L+1)^n$ variables $p(\lambda_1, \ldots, \lambda_n)$. Assume that

(3) $$(L+1)^n > 2h(k+1)^{n-1}.$$

The coefficients of $p(\lambda_1, \ldots, \lambda_n)$ in equations (2), in absolute value, do not exceed $S_1^{2nLh}(2SL)^{2k}$. Multiply each of the equations (2) by a natural number $\leqslant S_1^{nLh}(2SL)^{2k}$ so that the coefficients of $p(\lambda_1, \ldots, \lambda_n)$ are rational integers. Hence by Lemma 1, there exist rational integers $p(\lambda_1, \ldots, \lambda_n)$, not all zero, satisfying (2) and

$$|p(\lambda_1, \ldots, \lambda_n)| \leqslant S_1^{4nLh}(2SL)^{5k}.$$

Suppose (2) is true for $1 \leqslant l \leqslant h_1$ $(\geqslant h)$, $m_i \geqslant 0$ and $0 \leqslant m_1 + \ldots + m_{n-1} \leqslant k_1$ $(\leqslant k)$. Then we shall choose β quite small and claim that for suitable h_2, k_2 (integers)

$$q(l, m_1, \ldots, m_{n-1}) = 0 \quad \text{for} \quad 1 \leqslant l \leqslant h_2 \quad (h_2 > h_1)$$

and for all non-negative integers $m_1, \ldots, m_{n-1}$ such that $m_1 + \ldots + m_{n-1} \leqslant k_2$ $(k_2 < k_1)$. If not, then there exists $(l, m_1, \ldots, m_{n-1})$ with $h_1 < l \leqslant h_2$ and $0 \leqslant m_1 + \ldots + m_{n-1} \leqslant k_2$, $m_i \geqslant 0$, such that

$$q(l, m_1, \ldots, m_{n-1}) \neq 0.$$

Consider the following function

$$f(z) = \Phi_{m_1, \ldots, m_{n-1}}(z, \ldots, z).$$

($\Phi_{m_1, \ldots, m_{n-1}}(z, \ldots, z)$ is the value of $\left(\frac{\partial}{\partial z_1}\right)^{m_1} \ldots \left(\frac{\partial}{\partial z_{n-1}}\right)^{m_{n-1}} \Phi(z_1, \ldots, z_{n-1})$ at the point $z_r = z$, $1 \leqslant r < n$.) Notice that

$$(4) \quad |(\log \alpha_1)^{-m_1} \ldots (\log \alpha_{n-1})^{-m_{n-1}} f(l) - q(l, m_1, \ldots, m_{n-1})| \leqslant (L+1)^n S_1^{4nLh} (2SL)^{5k} S_1^{2nLh_2} (2SL)^{2k} \beta e^{2Lh_2} \leqslant \beta S_1^{7nLh_2} (2SL)^{7k}.$$

Notice that $q(l, m_1, \ldots, m_{n-1})$ is a non-zero rational number whose denominator does not exceed $S_1^{nLh_2}(2SL)^{2k}$. Hence

$$(5) \quad |q(l, m_1, \ldots, m_{n-1})| \geqslant S_1^{-nLh_2}(2SL)^{-2k}.$$

Combining (4) and (5), we get

$$(6) \quad |(\log \alpha_1)^{-m_1} \ldots (\log \alpha_{n-1})^{-m_{n-1}} f(l)| \geqslant S_1^{-nLh_2}(2SL)^{-2k} - \beta S_1^{7nLh_2}(2SL)^{7k}.$$

Set

$$w = |(\log \alpha_1)^{m_1} \ldots (\log \alpha_{n-1})^{m_{n-1}}|.$$

Now we shall approximate $f(l)$ from above. By Hermite's interpolation formula

$$(7) \quad \frac{1}{2\pi i} \int_{\Gamma: |z| = \Delta} \frac{f(z) F(l)}{(z-l) F(z)} dz = f(l) + \sum_{r=1}^{h_1} \sum_{m=0}^{k_1 - k_2} \frac{f^{(m)}(r)}{m! 2\pi i} \int_{\substack{\Gamma_r: \\ |z-r| = 1/2}} \frac{(z-r)^m F(l)}{(z-l) F(z)} dz$$

where

$$F(z) = \prod_{u=1}^{h_1} (z-u)^{k_1 - k_2 + 1} \quad \text{and} \quad \Delta = 5h_2 \exp\left(\frac{1}{A} \log S_1\right).$$

Notice that

$$\max_{|z|=\Delta} |f(z)| \leqslant w(L+1)^n S_1^{4nLh} (2SL)^{5k} (2SL)^{2k} \max_{\substack{(\lambda_1, \ldots, \lambda_n) \\ |z| = \Delta}} |\alpha_1^{\gamma_1 z} \ldots \alpha_{n-1}^{\gamma_{n-1} z}|.$$

Observe that

$$(8) \quad |\gamma_1 \log \alpha_1 + \ldots + \gamma_{n-1} \log \alpha_{n-1}| = |\lambda_n(\beta_1 \log \alpha_1 + \ldots + \beta_{n-1} \log \alpha_{n-1} - \log \alpha_n) + (\lambda_1 \log \alpha_1 + \ldots + \lambda_n \log \alpha_n)| \leqslant L\beta + nL \exp\left(-\frac{1}{A} \log S_1\right) \leqslant 2nL \exp\left(-\frac{1}{A} \log S_1\right),$$

since β can be assumed to be less than $\exp\left(-\frac{1}{A} \log S_1\right)$. (Otherwise we are through.)

Hence

$$\max_{|z|=\Delta} |f(z)| \leqslant w S_1^{6nLh_2} (2SL)^{7k}.$$

Now when $|z| = \Delta$,

$$\left|\frac{F(l)}{F(z)}\right| \leqslant \exp\left(-\frac{h_1(k_1 - k_2 + 1)}{A} \log S_1\right).$$

Therefore the integral on the left-hand side of (7), in absolute value, does not exceed

$$(9) \quad w S_1^{6nLh_2} (2SL)^{7k} \exp\left(-\frac{h_1(k_1 - k_2 + 1)}{A} \log S_1\right).$$

Notice that

$$f^{(m)}(r) = \sum_{\substack{t_1, \ldots, t_{n-1} \\ t_1 + \ldots + t_{n-1} = m}} \frac{m!}{t_1! \ldots t_{n-1}!} \Phi_{m_1 + t_1, \ldots, m_{n-1} + t_{n-1}}(r, \ldots, r).$$

For $0 \leqslant m \leqslant k_1 - k_2$, $1 \leqslant r \leqslant h_1$ and $t_1 + \ldots + t_{n-1} = m$, we know

$$q(r, m_1 + t_1, \ldots, m_{n-1} + t_{n-1}) = 0,$$

since $m_1 + t_1 + \ldots + m_{n-1} + t_{n-1} \leqslant k_2 + m \leqslant k_1$. Hence in the same way as we got (4), we get

$$|\Phi_{m_1 + t_1, \ldots, m_{n-1} + t_{n-1}}(r, \ldots, r)| \leqslant \beta S_1^{7nLh_1} (2SL)^{7k} w.$$

The double sum on the right-hand side of (7), in absolute value, does not exceed

$$(10) \quad n^k h_1 (k_1 - k_2 + 1) \beta w S_1^{7nLh_1} (2SL)^{7k} (2h_2)^{h_1(k_1 - k_2 + 1)} \leqslant \beta w n^k S_1^{8nLh_1} (2SL)^{8k} (2h_2)^{h_1(k_1 - k_2 + 1)}.$$

Assume that

$$(11) \quad \frac{k_1}{n^2} > 2.$$

Set $k_2 = \left[\left(1-\frac{1}{n^2}\right)k_1\right]$ and obtain from (7), (9) and (10)

(12) $|f(l)|$

$$\leqslant w\left(S_1^{6nLh_2}(2SL)^{7k}\exp\left(-\frac{h_1k_1}{An^2}\log S_1\right)+\beta n^k S_1^{8nLh_1}(2SL)^{8k}(2h_2)^{2h_1k_1/n^2}\right).$$

We shall choose h_1, h_2, k_1, L suitably and fix β quite small and conclude that (6) and (12) are inconsistent. This contradiction would establish

$$q(l, m_1, \ldots, m_{n-1}) = 0$$

for all l, $1 \leqslant l \leqslant h_2$ and for all non-negative integers $m_1, \ldots, m_{n-1}$ with $m_1+\ldots+m_{n-1} \leqslant k_2$. For this, it is sufficient to secure that

$$\exp\left(\frac{h_1k_1}{An^2}\log S_1\right) > S_1^{9nLh_2}(2SL)^{10k}\{1+2\beta n^k(2h_2)^{2h_1k_1/n^2}S_1^{2h_1k_1/An^2}\}.$$

Let E, δ, b be positive constants (i.e. independent of S_1, A and n) to be suitably chosen. Define

$$\tilde{r}_1 = \left[\frac{(n-2)(1+nE)+\delta}{b}\right]+2.$$

Let $B = \left(1-\frac{1}{n^2}\right)^{-1}$. Let c_1 be a large positive constant (depending on E, δ and b) to be suitably chosen. Put

$$h = [2c_1B^{2\tilde{r}_1}n^8A^2], \quad k = [\tfrac{1}{9}h^{1+nE}], \quad L = [h^{1+(n-1)E}],$$
$$h_1 = h, \quad h_r = [2h_{r-1}h^b], \quad r = 2, \ldots, \tilde{r}_1,$$
$$k_1 = k, \quad k_r = [B^{-1}k_{r-1}], \quad r = 2, \ldots, \tilde{r}_1.$$

Notice that (3) and (11) are satisfied. Observe that $h_r \geqslant 10$, $k_r \geqslant 10n^2$ $(1 \leqslant r \leqslant \tilde{r}_1)$. Assume that

(13) $$\beta \leqslant \tfrac{1}{2}n^{-k}(2h_{r+1})^{-2h_rk_r/n^2}S_1^{-2h_rk_r/An^2}, \quad r = 1, \ldots, \tilde{r}_1-1.$$

We proceed by induction and claim that

$$q(l, m_1, \ldots, m_{n-1}) = 0$$

for $1 \leqslant l \leqslant h_{\tilde{r}_1}$ and for non-negative integers $m_1, \ldots, m_{n-1}$ such that $m_1+\ldots+m_{n-1} \leqslant k_{\tilde{r}_1}$, if the following inequalities are satisfied:

$$\exp\left(\frac{h_rk_r}{An^2}\log S_1\right) > 2S_1^{9nLh_{r+1}}(2SL)^{10k}, \quad r = 1, \ldots, \tilde{r}_1-1.$$

Set $S = S_1$ in the above inequalities. Denote by $u_1, u_2, \ldots$ positive constants > 1, independent of n, A and S_1. Let r be an integer such that $1 \leqslant r < \tilde{r}_1$. It is sufficient to secure the following inequalities:

$$h_rk > u_1AB^{\tilde{r}_1}n^3Lh_{r+1}, \quad h_1 > u_2AB^{\tilde{r}_1}n^2, \quad h_1\log S_1 > u_3AB^{\tilde{r}_1}n^2\log L.$$

Notice that the second inequality is trivially satisfied for large c_1. For the first, it is enough to secure that

$$h^{E-b} > u_4AB^{\tilde{r}_1}n^3.$$

Choose $E > 1+b$ and the above inequality is satisfied for large c_1. The right-hand side of the last inequality $\leqslant AB^{\tilde{r}_1}n^5\log B\log A\log n$ and so the last one is also satisfied.

Hence $q(l, m_1, \ldots, m_{n-1}) = 0$ for $1 \leqslant l \leqslant h_{\tilde{r}_1}$ and for non-negative integers $m_1, \ldots, m_{n-1}$ such that $m_1+\ldots+m_{n-1} \leqslant k_{\tilde{r}_1}$. Define

$$\Phi_1(z) = \Phi(z, \ldots, z).$$

Hence for $1 \leqslant l \leqslant h_{\tilde{r}_1}$, $0 \leqslant m \leqslant k_{\tilde{r}_1}$, we get

$$|\Phi_1^{(m)}(l)| \leqslant \beta n^k S_1^{7nLh_{\tilde{r}_1}}(2S_1L)^{7k} \quad \text{with} \quad S = S_1.$$

In the notation of Lemma 2, set $E(z) = \Phi_1(z)$, $m = (L+1)^n$, $s = h_{\tilde{r}_1}$, $t = k_{\tilde{r}_1}+1$, $r = h_{\tilde{r}_1}(k_{\tilde{r}_1}+1)$. Arrange $1, \ldots, h_{\tilde{r}_1}$ as $\beta_0, \ldots, \beta_{s-1}$ (not to be confused with our previous β_i). Further arrange $\gamma_1\log\alpha_1+\ldots+\gamma_{n-1}\log\alpha_{n-1}$ as $\alpha_0, \ldots, \alpha_{m-1}$ (not to be confused with our previous α_i). Notice that when $|\lambda_i| \leqslant L$, $1 \leqslant i \leqslant n$ and $(\lambda_1, \ldots, \lambda_n) \neq (0, \ldots, 0)$,

$$|\gamma_1\log\alpha_1+\ldots+\gamma_{n-1}\log\alpha_{n-1}|$$
$$= |\lambda_n(\beta_1\log\alpha_1+\ldots+\beta_{n-1}\log\alpha_{n-1}-\log\alpha_n)+(\lambda_1\log\alpha_1+\ldots+\lambda_n\log\alpha_n)|$$
$$\geqslant \tfrac{1}{2}S_1^{-4nL}-L\beta > \tfrac{1}{4}S_1^{-4nL},$$

if

(14) $$\beta < \frac{1}{4L}S^{-4nL}.$$

In the notation of the same lemma:

$$a = \max\left(2nL\exp\left(-\frac{1}{A}\log S_1\right), 1\right), \quad b = h_{\tilde{r}_1}, \quad a_0 \geqslant \tfrac{1}{4}S_1^{-4nL},$$

(for a, see (8))

$$b_0 = 1, \quad A \geqslant 1 \quad \text{and} \quad E \leqslant \beta n^k S_1^{7nLh_{\tilde{r}_1}}(2S_1L)^{7k}.$$

Before we may apply Lemma 2, we must check that the inequality (1) is satisfied. It is sufficient to secure the following:

$$h_{\tilde{r}_1}k_{\tilde{r}_1} \geqslant 6(L+1)^n, \quad h_{\tilde{r}_1}k_{\tilde{r}_1} \geqslant 78nL\exp\left(-\frac{1}{A}\log S_1\right)h_{\tilde{r}_1}, \quad h_{\tilde{r}_1}k_{\tilde{r}_1} \geqslant 39h_{\tilde{r}_1}.$$

The last is trivially satisfied. For the first one,

$$6(L+1)^n \ll L^n\left(1+\frac{1}{L}\right)^n \ll L^n$$

and

$$h_{\tilde{r}_1} k_{\tilde{r}_1} \gg B^{-\tilde{r}_1} h^{1+(n-2)(1+nE)+\delta+1+nE} \gg h^\delta L^n B^{-\tilde{r}_1}.$$

Hence it is sufficient to secure that $h^\delta \gg B^{\tilde{r}_1}$. Notice that $B^{\tilde{r}_1} \ll 1$. So the first inequality is satisfied, if c_1 is large enough. For the second, it is sufficient to secure that $h^E \gg n$, which is satisfied if $E > 1$ and c_1 is large enough. Hence by Lemma 2,

$$1 \leqslant A \leqslant h_{\tilde{r}_1} e^{14nLh_{\tilde{r}_1}} \exp(20L^{4n}\log S_1)\,(72h_{\tilde{r}_1})^{2h_{\tilde{r}_1}k_{\tilde{r}_1}} \beta n^k S_1^{7nLh_{\tilde{r}_1}} (2S_1L)^{7k}$$
$$\leqslant \beta \exp\left((nA)^{u_5 n^2}\log S_1\right).$$

Assume that

$$\beta \leqslant \exp\left(-(nA)^{u_6 n^2}\log S_1\right),$$

where $u_6 > u_5$ and is large enough so that (13) and (14) are satisfied. We get $1 \leqslant A < 1$, which is not possible. Hence

$$\beta > \exp\left(-(nA)^{cn^2}\log S_1\right)$$

where c is an effectively computable positive constant.

Remarks. (i) The bound obtained in theorem is the best possible as a function of S_1 in the sense that it is false if A and n are fixed and $\log S_1$ is replaced by $(\log S_1)^f$ with $f < 1$.

(ii) Theorem can be proved independently of Lemma 2.

(iii) Theorem can easily be generalized to linear forms in the logarithms of algebraic numbers with algebraic coefficients.

The proof of the main theorem, stated in the introduction is now complete.

Added at the time of proofs: Lemma 2 has been improved by M. Jutila by using Vinogradoff's method (this will appear in Journ. Indian Math. Soc.). Results more suitable for our purpose, than the theorem of § 3 have since been proved by T. N. Shorey (to appear). Using this two results it follows by our method that

$$f(k) = O\left(\frac{k\log\log\log k}{\log k \log\log k}\right).$$

As far as we are aware the best known lower bound for $f(k)$ is

$$f(k) \geqslant \frac{\log k \log\log k \log\log\log\log k}{(\log\log\log k)^2}$$

deduced by Erdös from a result of Rankin.

References

[1] A. Baker, *Linear forms in the logarithms of algebraic numbers II*, Mathematika 14 (1967), pp. 102–107.

[2] P. Erdös, *A theorem of Sylvester and Schur*, J. London Math. Soc. 9 (1934), pp. 282–288.

[3] — *On consecutive integers*, Nieuw Arch. Voor Wisk. 3 (1955), pp. 124–128.

[4] M. N. Huxley, *On the difference between consecutive primes*, Invent. Math. 15 (1972), pp. 164–170.

[5] K. Ramachandra, *A note on numbers with a large prime factor II*, J. Indian Math. Soc. 34 (1970), pp. 39–48.

[6] — *A note on numbers with a large prime factor III*, Acta Arith. 19 (1971), pp. 49–62.

[7] — *Application of Baker's theory to two problems considered by Erdös and Selfridge*, J. Indian Math. Soc. (to appear).

[8] — *Largest prime factor of the product of k-consecutive integers*, Reports of an International Conference on Number Theory (Moscow, 1971), (to appear).

[9] — *Lectures on Transcendental Numbers*, Ramanujan Institute, Madras 1969.

[10] T. N. Shorey, *Linear forms in the logarithms of algebraic numbers with small coefficients I*, J. Indian Math. Soc. (to appear).

[11] R. Tijdeman, *On the maximal distance of numbers with a large prime factor*, J. London Math. Soc. 5 (1972), pp. 313–320.

[12] — *An auxiliary result in the theory of transcendental numbers*, Journal of Number Theory (to appear).

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ERRATA

Page, line	For	Read
35^{16}	$M^{-\sqrt{C}+1h}$	$M^{-\sqrt{C}}h$
39_8	$s \geqslant 4nk$	$s > 4nk$
43_{10}	$\alpha \bmod p^{0}$	$\alpha \bmod p^{l_0}$
50_6	$\prod_{m=r_1+r}^{r_1+r_2}$	$\prod_{m=r_1+1}^{r_1+r_2}$