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A sharpening of the bounds for linear forms in logarithms II

by

A. BAKER (Cambridge)

Dedicated to Professor C. L. Siege
on his 75 th birthday

1. Introduction. Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers with degrees at most d and let the heights of $\alpha_1, \ldots, \alpha_{n-1}$ and α_n be at most A' and A ($\geqslant 2$) respectively. It was proved in [1] that, for some effectively computable number $C > 0$ depending only on n, d and A', the inequalities

$$0 < |b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n| < C^{-\log A \log B}$$

have no solution in rational integers $b_1, \ldots, b_n$ with absolute values at most B ($\geqslant 2$). In the present paper we shall establish the following generalization:

THEOREM 1. *There is an effectively computable number C, depending only on n, d and A', such that, for any δ with $0 < \delta < \frac{1}{2}$, the inequalities*

$$(1) \qquad 0 < |b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n| < (\delta/B')^{C \log A} e^{-\delta B}$$

have no solution in rational integers $b_1, \ldots, b_{n-1}$ and b_n ($\neq 0$) with absolute values at most B and B' respectively.

It is clear that, on taking $\delta = 1/B$ and assuming that $B' \leqslant B$ one obtains the result of [1]. Furthermore one sees at once that the case $b_n = -1$ of Theorem 1 furnishes the following corollary:

THEOREM 2. *If, for some $\varepsilon > 0$, there exist rational integers $b_1, \ldots, b_{n-1}$ with absolute values at most B such that*

$$0 < |b_1 \log \alpha_1 + \ldots + b_{n-1} \log \alpha_{n-1} - \log \alpha_n| < e^{-\varepsilon B},$$

then $B < C \log A$ for some effectively computable number C depending only on n, d, A' and ε.

Theorem 2 improves upon the analogous result implied by [1] to the extent of the elimination of a factor $\log\log A$ from the bound for B, and the strengthened conclusion is plainly best possible. Moreover, it leads to similar improvements in applications; in particular, in the light of the work of [2], it shows that the inequality

$$|\alpha - p/q| > cq^{-\varkappa}$$

is valid for any algebraic number α with degree $n \geqslant 3$ and all rationals p/q $(q > 0)$, where c, $\varkappa$ are positive effectively computable numbers depending only on α, and $\varkappa < n$. A special case of Theorem 2 involving certain restrictions on α_n was proved recently by Feldman [3] using rather different adaptations in the basic theory of linear forms in logarithms; and the result sufficed to establish the theorem on rational approximations to algebraic numbers just quoted. The removal of the subsidiary conditions simplifies the subsequent application and, it is hoped, will be helpful in connexion with further researches in this field (1).

2. Main theory. We suppose that $\alpha_1, \ldots, \alpha_n$ are defined as in § 1 and we signify by $c, c_1, c_2, \ldots$ numbers, greater than 1, which can be specified explicitly in terms of n, d and A' only. We suppose further that $0 < \delta < \frac{1}{2}$ and that there exist rational integers $b_1, \ldots, b_{n-1}$ and b_n $(\neq 0)$ with absolute values at most B and B' respectively such that (1) holds for some number $C > 0$ depending only on n, d and A'. We proceed to prove that, if C is sufficiently large, then there exist further rational integers $b_1', \ldots$ $\ldots, b_{n-1}'$ and b_n' $(\neq 0)$ with absolute values at most $c_1 B$ and $c_1 B'$ respectively, and an algebraic number α_n' in the field generated by the α's over the rationals with height at most $c_2 A^{1/2}$ such that (1) remains valid with $b_1, \ldots, b_n$ and α_n replaced by $b_1', \ldots, b_n'$ and α_n' respectively. An inductive argument will then complete the proof of Theorem 1.

The work involves only minor modifications in the discussion of [1]. In particular, the definitions given there remain unaltered except that, having, as before, signified by k an integer exceeding a sufficiently large number c as above, we put

$$h = L_{-1} + 1 = [\log(B' C \delta^{-1})],$$

$$L = L_0 = \ldots = L_{n-1} = [k^{1-1/(4n)} \log M], \qquad L_n = [k^{1/2}],$$

where

$$M = \max\{A, \exp(\delta B/(Ch))\}.$$

In other words, the notation of [1] is unchanged except for the replacement of A and B in the old definitions of L and h by M and $B' C \delta^{-1}$ respectively. It is then readily verified that all the lemmas enunciated in [1]

(1) Added in proof. R. Tijdeman has recently used Theorem 1 to solve an old problem of Wintner; see his paper *On integers with many small prime factors* to appear in Compositio Math.

remain valid provided that A (which, in fact, occurs in the statements only of Lemmas 5, 6 and 7) is replaced by M throughout. For the proof of Lemma 5 we require now the observation that since

$$|b_n \lambda_r - b_r \lambda_n| \leqslant k(B + B' \log M) \leqslant hk \log M (C\delta^{-1} + B' h^{-1})$$

and, furthermore, the last expression in parenthesis is at most $2B' C\delta^{-1} \leqslant e^{4h}$, we have

$$|\Delta(b_n \lambda_r - b_r \lambda_n; m_r)| \leqslant e^{4hm_r} \Delta(hk \log M; m_r);$$

the required estimate $M^{c_7 hk}$ for U then follows since

$$\Delta(x; m_r) \leqslant e^{x+m_r} \qquad (x > 0).$$

In the proof of the subsequent lemmas one has naturally to keep in mind that there is a new expression on the right of (1); since, however, by hypothesis, $\delta < \frac{1}{2}$, we have

$$(B'/\delta)^{\sqrt{C}} \geqslant B' C/\delta \geqslant e^h,$$

whence $(\delta/B')^{C \log A} \leqslant M^{-\sqrt{C}h}$ if $M = A$, and clearly $e^{-\delta B} = M^{-Ch}$ if $M > A$. Thus, from (1), we obtain

$$|\log \alpha_n - \log \alpha_n'| < M^{-\sqrt{C} + 1h},$$

and the arguments leading to Lemmas 6, 7 and 8 plainly hold if the number on the right of the last inequality is substituted for $C^{-\log A \log B}$, that is, if M and $\sqrt{C}h$ are substituted for A and $\log B \log C$ respectively (2).

As in [1], we take $q = p$, where p is a prime between L_n and $2L_n$ exclusive, and we deduce from Lemma 8 that

$$\alpha_n = \alpha_1^{j_1} \ldots \alpha_{n-1}^{j_{n-1}} \alpha_n'^{p}$$

where α_n' is an element of the field K generated by $\alpha_1, \ldots, \alpha_n$ over the rationals and $j_1, \ldots, j_{n-1}$ are integers with $0 \leqslant j_r < p$. On assuming, as we may without loss of generality, that $\alpha_1 = -1$, taking logarithms and substituting for $\log \alpha_n$ in (1) we obtain

$$0 < |b_1' \log \alpha_1 + \ldots + b_n' \log \alpha_n| < M^{-\sqrt{C}h},$$

where $b_1', \ldots, b_n'$ are rational integers as defined in [1]; in particular $b_n' = pb_n$. Thus $b_1', \ldots, b_{n-1}'$ and b_n' have absolute values at most $6nk^{1/2}B$

(2) Note that a factor 4 has been omitted from the exponent of q in the denominator for Q specified on page 126 of [1] and that h should be replaced by l in the exponents of the α's.

and $2k^{1/2}B'$ respectively. Further, as in [1], we see that the height of α_n' is at most $(2dA')^{4nD}A^{2D/p}$, whence, since $2D/p < \frac{1}{2}$ when $p > k^{1/2}$, it follows that $b_1', \ldots, b_n'$ and α_n' have the properties asserted at the beginning.

3. Proof of Theorem 1. The proof is completed by induction. There is plainly no loss of generality in assuming that $B' \geqslant c_1^8$; to begin with we assume that also $A \geqslant c_2^8$, whence

$$\log A \log(\delta^{-1}B') > \log(c_2 A^{1/2})\log(c_1^2\delta^{-1}B').$$

Then all the hypotheses recorded in § 2 hold with $b_1', \ldots, b_n'$ and α_n' in place of $b_1, \ldots, b_n$ and α_n respectively; the new values of A, B, B' are $c_2A^{1/2}$, c_1B, c_1B', and δ is replaced by δ/c_1. We suppose further, as we may, that d is taken, at the outset, as the degree of K so that the values of c_1, c_2 remain unaltered in the inductive discussion.

We now repeat the previous argument and obtain for each $s = 1, 2, \ldots$ a set of integers $b_1^{(s)}, \ldots, b_{n-1}^{(s)}$ and $b_n^{(s)}$ with absolute values at most c_1^sB and c_1^sB' respectively, and an element $\alpha_n^{(s)}$ in K with height at most $c_2^{1+\frac{1}{2}+\ldots+(\frac{1}{2})^{s-1}}A^{(\frac{1}{2})^s}$, such that (1) holds with $b_1^{(s)}, \ldots, b_n^{(s)}$ and $\alpha_n^{(s)}$ in place of $b_1, \ldots, b_n$ and α_n respectively. The algorithm terminates for some $s \leqslant 2\log\log A$ when the height of $\alpha_n^{(s)}$ is at most c_2^8; and $b_1^{(s)}, \ldots, b_n^{(s)}$ then have absolute values at most

$$H = (\log A)^{c_3}\max(B, B').$$

But the number on the right of (1) is at most $H^{-\sqrt{C}}$; for if $B \geqslant B'$ then either $H \leqslant B^2$ or $H \leqslant (\log A)^{2c_3}$, and the assertion is obvious if $B < B'$. Thus one concludes from the result of [1] that the inequality is untenable if C is sufficiently large, and the theorem follows.

References

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On Waring's problem in algebraic number fields

by

TIKAO TATUZAWA (Tokyo)

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Since Siegel succeeded in dealing with Waring's problem in algebraic number fields, there have appeared many works under almost the same title (see [8], [9], [12], [5] and [1]). I should like to use this opportunity to add some corrections and justification for my former results. The result presented at the United States–Japan seminar at Tokyo in 1971 is not the best. Nevertheless, we seem to be able to place our hope in its further developments, because the result is based on some remarkable new research due to Mitsui (see [7]). My concern is mainly the treatment of Fourier analysis.

1. Preliminaries and basic domains. In the first place, we shall summarize the main results obtained so far. Let K be an algebraic number field of degree n, let $K^{(l)}$ $(1 \leqslant l \leqslant r_1)$ be r_1 real conjugate fields, and let $K^{(m)}$, $K^{(m+r_2)}$ $(r_1+1 \leqslant m \leqslant r_1+r_2)$ be r_2 pairs of complex conjugate fields, so that $n = r_1+2r_2$. Let $\mathfrak{d}$ be the different of K, and d the discriminant of K. We can choose $\omega_1, \omega_2, \ldots, \omega_n$ as an integral basis of K and $\varrho_1, \varrho_2, \ldots, \varrho_n$ as a basis of $\mathfrak{d}^{-1}$, satisfying

$$\operatorname{trace}(\varrho_r\omega_s) = \begin{cases} 1 & (r = s), \\ 0 & (r \neq s). \end{cases}$$

We denote by $\mathfrak{o}$ the integral domain of all algebraic integers in K. We denote by $P(T)$ the set of $(z_1, \ldots, z_n)$ satisfying

$$0 \leqslant \lambda^{(l)} \leqslant T, \quad |\lambda^{(m)}| \leqslant T$$

where

$$\lambda = \omega_1z_1 + \ldots + \omega_nz_n,$$

the indices l and m being over the set of numbers cited above. On the other hand,

$$\sum_{\lambda \in P(T)}$$