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On the number of integers which are sums of two squares

by

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1. Let $b(n)$ be defined to be 1 or 0 according as n be or not be expressed as a sum of two integral squares. As is well-known Landau has proved the asymptotic formula

$$\sum_{n\leqslant N} b(n) = \big(1+o(1)\big)C\frac{N}{\sqrt{\log N}} \quad (\text{as } N\to\infty). \tag{1}$$

On account of this formula we may introduce the problem to prove the asymptotic formula

$$\sum_{N\leqslant n\leqslant N+M} b(n) = \big(1+o(1)\big)C\frac{M}{\sqrt{\log N}}, \tag{2}$$

where M is in the range $N^{a}\leqslant M<N$ with a constant $a<1$. Although this problem has the aspect similar to the theorem of Hoheisel in the theory of prime numbers, it seems extreemly difficult to adapt any methods there to our problem. Thus it is very desirable to prove a good lower estimation of the left side of (2), and this has been recently done by Hooley in the following form(¹): we have

$$\sum_{N\leqslant n\leqslant N+N^{\theta}} b(n) \gg \frac{N^{\theta}}{\sqrt{\log N}} \tag{3}$$

for any $\theta>\frac{12}{37}$. This lower bound of θ comes from the fact that this is the hitherto best exponent of the remainder term of the circle problem [1], and so it seems very difficult to improve (3).

The purpose of the present paper is to prove a non-trivial, but slightly weaker than (3), estimation for "almost all" intervals of very small length. More precisely we shall prove

(¹) In the first draft of the present paper we have proved an estimation weaker than this (by a factor of $(\log\log N)^{-C}$), and we are indebted to Prof. Schinzel who informed us of Prof. Hooley's strong result.

THEOREM. *Let ε be an arbitrarily small positive constant. Then there are two constants C_ε and D_ε such that they depend on ε at most and the number of integers in the interval $[n, n+n^\varepsilon]$ that can be expressed as a sum of two squares exceeds the quantity*

$$C_\varepsilon \frac{n^\varepsilon}{\sqrt{\log n}\,(\log\log n)^{D_\varepsilon}}$$

for all but $o(N)$ integers $n \leqslant N$ as $N \to \infty$.

Here we should remark that in the recent paper [3] Hooley has developed a very ingeneous idea to attack the problem of the estimation of the moment of differences between consecutive integers that can be expressed as a sum of two squares, and between his work and ours there are many similar aspects. Especially his formula (23) of [3] might be used to deduce a result similar to our theorem, but to do this we have to prove the inequality (12) of [3] for *every* short interval of length x^ε. This seems difficult, although it might be possible to modify the definition of Hooley's neutralizer $t(n)$ and to prove such a result.

We hope we shall return elsewhere to the difficult problem of the elimination of the factor $(\log\log n)^{D_\varepsilon}$ of our theorem.

Notation. Throughout this paper N is assumed to be sufficiently large. x is a positive variable. p denote generally a prime number. We denote by $\omega(n)$ the number of different prime factors of n. The function $d_k(n)$ denotes the number of representations of n as a product of k factors, especially $d(n) = d_2(n)$ is the number of divisors of n. We define $r(n)$ as usual to be the number of representations of n as a sum of two squares, and then we have

$$r(n) = 4\sum_{d|n} \varrho(d),$$

where ϱ is the non-principal character mod 4. The positive constants ε and A are assumed to be sufficiently small and large, respectively, and all constants involved in the symboles "$\ll$" and "O" depend on them at most.

2. We define $\bar{N}$ the fundamental quantity in this paper by

$$\bar{N} = N^{(\log\log N)^{-2}} \tag{4}$$

and we introduce the symbols Δ_0, Δ_1 and Γ which represent three sets of positive integers that are composed entirely of prime factors not exceeding $\bar{N}$, of prime factors congruent to $-1 \bmod 4$ and not exceeding $\bar{N}$, of prime factors exceeding $\bar{N}$, respectively. Here we assume that Δ_0, Δ_1 and Γ contain the number 1.

We decompose any integer into two factors;

$$n = n^{(1)} n^{(2)}$$

where $n^{(1)} \in \Delta_0$ and $n^{(2)} \in \Gamma$. Further we define $\beta(n)$ to be 1 or 0 according as n be or not be composed entirely of the prime number 2 and prime factors congruent to 1 mod 4.

Then we consider the expression

$$R(N, h) = \sum_{n \leqslant N}\Big\{\sum_{0 \leqslant j < h} \beta\big((n+j)^{(1)}\big) r\big((n+j)^{(2)}\big) - \pi\delta(N) h\Big\}^2, \tag{5}$$

where

$$\delta(N) = \prod_{\substack{p \equiv 1 \,(\mathrm{mod}\, 4)\\ p \leqslant \bar{N}}}\Big(1-\frac{1}{p}\Big) \prod_{\substack{p \equiv -1 \,(\mathrm{mod}\, 4)\\ p \leqslant \bar{N}}}\Big(1-\frac{1}{p^2}\Big), \tag{6}$$

and the size of h is to be determined later.

In the right side of (5) $\beta(n^{(1)})$ simulates the behaviour of $b(n)$ and the factor $r(n^{(2)})$ has the effect to eliminate the strong difficulty which would be caused by the factor $b(n^{(2)})$ if we treat the problem in its crude form, and moreover $\beta(n^{(1)}) r(n^{(2)})$ has the favourable feature that it vanishes if $b(n) = 0$.

Since $\beta(n^{(1)}) r(n^{(2)}) = O(n^\varepsilon)$, we have easily

$$R(N, h) = 2\sum_{0 \leqslant j_1 < j_2 < h} I(N, j_2 - j_1) + hS(N) - 2\pi\delta(N) h^2 T(N) + \pi^2 \delta^2(N) h^2 N + O(h^3 N^\varepsilon), \tag{7}$$

where

$$I(N, a) = \sum_{n \leqslant N} \beta(n^{(1)}) r(n^{(2)}) \beta\big((n+a)^{(1)}\big) r\big((n+a)^{(2)}\big),$$
$$S(N) = \sum_{n \leqslant N} \beta(n^{(1)}) r^2(n^{(2)}), \quad T(N) = \sum_{n \leqslant N} \beta(n^{(1)}) r(n^{(2)}). \tag{8}$$

3. First we shall estimate $T(N)$, and to do this we remark that

$$\beta(n^{(1)}) = \sum_{\substack{l|n\\ l \in \Delta_1}} \mu(l). \tag{9}$$

By this we devide $T(N)$ into two parts as follows:

$$T(N) = \sum_{n \leqslant N} r(n^{(2)}) \Big\{\sum_{\substack{l|n\\ l \in \Delta_1\\ \omega(l) \leqslant A \log\log N}} \mu(l) + \sum_{\substack{l|n\\ l \in \Delta_1\\ \omega(l) > A \log\log N}} \mu(l)\Big\} = \Sigma_1 + \Sigma_2, \text{ say.} \tag{10}$$

We have

$$|\Sigma_2| \leqslant \sum_{\substack{n\leqslant N \\ \omega(n^{(1)})>A\log\log N}} r(n^{(2)})\, d(n^{(1)}) \leqslant 4 \sum_{\substack{n\leqslant N \\ \omega(n)\geqslant A\log\log N}} d(n),$$

and so we have

$$(11)\qquad |\Sigma_2| \ll 2^{-A\log\log N} \sum_{n\leqslant N} 2^{\omega(n)} d(n)$$

$$\ll (\log N)^{-A\log 2} \sum_{n\leqslant N} d^2(n) \ll N(\log N)^{3-A\log 2}.$$

To estimate the sum Σ_1, we remark the simple fact that we have

$$(12)\qquad r(n^{(2)}) = \sum_{\substack{mu=n \\ u\in\Delta_0}} r(m)\,\varrho(u)\,\mu(u),$$

which can be easily seen from the equality

$$\sum_{n=1}^{\infty} \frac{r(n^{(2)})}{n^s} = 4\prod_{p\leqslant\tilde N}\left(1-\frac{\varrho(p)}{p^s}\right)\zeta(s)L(s,\varrho) = \left\{\sum_{n\in\Delta_0}\frac{\varrho(n)\mu(n)}{n^s}\right\}\left\{\sum_{n=1}^{\infty}\frac{r(n)}{n^s}\right\},$$

where $\zeta(s)$ and $L(s,\varrho)$ stand respectivly for Riemann's zeta-function and Dirichlet's L-function attached to the character ϱ.

Now we have

$$\Sigma_1 = \sum_{\substack{l\in\Delta_1 \\ \omega(l)\leqslant A\log\log N}} \mu(l) \sum_{n\leqslant N/l} r(n^{(2)}),$$

since we have for the above l (square-free) $l \leqslant N^{A/\log\log N}$ and also we have $l^{(2)} = 1$. Thus inserting the expression (12) into the inner-sum we get

$$(13)\qquad \Sigma_1 = \sum_{\substack{l\in\Delta_1 \\ \omega(l)\leqslant A\log\log N}} \mu(l) \sum_{\substack{mu=n \\ u\in\Delta_0 \\ n\leqslant N/l}} r(m)\,\mu(u)\,\varrho(u)$$

$$= \sum \mu(l)\Big\{\sum_{\omega(u)\leqslant A\log\log N} + \sum_{\omega(u)>A\log\log N}\Big\} = \Sigma_3+\Sigma_4, \text{ say.}$$

We have, as is easily seen,

$$(14)\qquad |\Sigma_4| \ll \sum_{l\leqslant N^\varepsilon} \sum_{\substack{n\leqslant N/l \\ \omega(n)>A\log\log N}} d^2(n)$$

$$\ll (\log N)^{-A\log 2} \sum_{l\leqslant N^\varepsilon} \sum_{n\leqslant N/l} d^3(n) \ll N(\log N)^{8-A\log 2}.$$

Since

$$\sum_{n\leqslant x} r(n) = \pi x + O(x^{1/3}),$$

we have

$$\Sigma_3 = \sum_{\substack{l\in\Delta_1 \\ \omega(l)\leqslant A\log\log N}} \mu(l) \sum_{\substack{u\in\Delta_0 \\ \omega(u)\leqslant A\log\log N}} \mu(u)\,\varrho(u) \sum_{m\leqslant N/lu} r(m)$$

$$= \pi N \sum_{\substack{l\in\Delta_1 \\ \omega(l)\leqslant A\log\log N}} \frac{\mu(l)}{l} \sum_{\substack{u\in\Delta_0 \\ \omega(u)\leqslant A\log\log N}} \frac{\varrho(u)\mu(u)}{u} + O(N^{1/3+\varepsilon}).$$

Here we see easily

$$\sum_{\substack{u\in\Delta_0 \\ \omega(u)\leqslant A\log\log N}} \frac{\varrho(u)\mu(u)}{u} = \prod_{p\leqslant\tilde N}\left(1-\frac{\varrho(p)}{p}\right) + O\big((\log N)^{2-A\log 2}\big),$$

and

$$\sum_{\substack{l\in\Delta_1 \\ \omega(l)\leqslant A\log\log N}} \frac{\mu(l)}{l} = \prod_{\substack{p\leqslant\tilde N \\ p\equiv-1\,(\mathrm{mod}\,4)}} \left(1-\frac{1}{p}\right) + O\big((\log N)^{2-A\log 2}\big).$$

And so we get

$$(15)\qquad \Sigma_3 = \pi N\delta(N) + O\big(N(\log N)^{2-A\log 2}\big).$$

Thus from (13), (14) and (15) we have

$$\Sigma_1 = \pi N\delta(N) + O\big(N(\log N)^{8-A\log 2}\big),$$

which, with (10) and (11), gives rize to

LEMMA 1. *We have the asymptotic equality*

$$T(N) = \pi N\delta(N) + O\big(N(\log N)^{-E_1}\big),$$

where E_1 can be taken arbitrarily large.

4. The next problem is the estimation of $S(N)$, but we prove here an upper estimation of the more difficult sum

$$S(N,H) = \sum_{N-H\leqslant n\leqslant N} \beta(n^{(1)})\, r^2(n^{(2)}),$$

where H is in the range $N^\varepsilon \leqslant H < N$. We shall encounter this sum at the last step of this paper.

Our proof depends on a recent result [5] of Wolke, which is embodied in

LEMMA 2. *For any $0<\xi<\frac{1}{2}$ there is an absolute constant $c\geqslant 1$ such that*

$$d(n) \ll \Big\{\sum_{\substack{v\mid n \\ v\leqslant n^\xi}} 1\Big\}^{c(\xi\log\xi^{-1})^{-1}}$$

where the constant in the symbol "$\ll$" depends on ξ at most.

Following the assertion of [4] we put in the above inequality

(16) $$\xi^{-1} = \exp\left(\frac{16c}{\varepsilon}\right).$$

Then we have

(17) $$2c(\xi\log\xi^{-1})^{-1} = \frac{\varepsilon}{8\xi} \leqslant \left[\frac{\varepsilon}{8\xi}\right]+1 = \eta, \text{ say}.$$

And thus we have

(18) $$S(N,H) \ll \sum_{N\leqslant n\leqslant N+H} \beta(n^{(1)})\, d^2(n^{(2)}) \ll \sum_{N\leqslant n\leqslant N+H}\beta(n^{(1)})\Big\{\sum_{\substack{v|n\\ v\in\Gamma\\ v\leqslant N^{\xi}}} 1\Big\}^{\eta},$$

where Γ is the set of integers defined in the second paragraph. We have

$$\Big\{\sum_{\substack{v|n\\ v\in\Gamma\\ v\leqslant N^{\xi}}} 1\Big\}^{\eta} \leqslant \sum_{\substack{t|n\\ t\in\Gamma\\ t\leqslant N^{\xi\eta}}}\ \sum_{t=[v_1,v_2,\ldots,v_\eta]} 1 \leqslant \sum_{\substack{t|n\\ t\in\Gamma\\ t\leqslant N^{\varepsilon}}} d^{\eta}(t),$$

since we have, from (16) and (17),

$$\xi\eta \leqslant \frac{\varepsilon}{8}+\xi\leqslant\frac{\varepsilon}{4}.$$

Thus we have, from (18),

$$S(N,H)\ll \sum_{\substack{t\in\Gamma\\ t\leqslant N^{\varepsilon/4}}} d^{\eta}(t) \sum_{\substack{N\leqslant n\leqslant N+H\\ n\equiv 0\,(\mathrm{mod}\, t)}} \beta(n^{(1)}) = \sum_{\substack{t\in\Gamma\\ t\leqslant N^{\varepsilon/4}}} d^{\eta}(t)\sum_{N/t\leqslant n\leqslant (N+H)/t}\beta(n^{(1)}),$$

since $t\in\Gamma$. The inner-sum is estimated analogous as in the case of $T(N)$ and we find

$$\sum_{N/t\leqslant n\leqslant(N+H)/t}\beta(n^{(1)}) = (1+o(1))\frac{N}{t}\prod_{\substack{p\leqslant\tilde N\\ p\equiv -1\,(\mathrm{mod}\,4)}}\Big(1-\frac1p\Big),$$

which gives

$$S(N,H)\ll H\prod_{\substack{p\leqslant\tilde N\\ p\equiv -1\,(\mathrm{mod}\,4)}}\Big(1-\frac1p\Big)\sum_{\substack{t\in\Gamma\\ t\leqslant N}}\frac{d^{\eta}(t)}{t}.$$

Here we have

$$\sum_{\substack{t\in\Gamma\\ t\leqslant N}}\frac{d^{\eta}(t)}{t}\leqslant\prod_{\tilde N<p\leqslant N}\Big\{\sum_{m=0}^{\infty}\frac{d^{\eta}(p^m)}{p^m}\Big\}\leqslant\prod_{\tilde N<p\leqslant N}\Big(1-\frac1p\Big)^{-2^{\eta}}\ll(\log\log N)^{2^{\eta+1}},$$

which gives rise to

LEMMA 3. *We have the inequality*

$$S(N,H)\ll H(\log\log N)^{B}\prod_{\substack{p\leqslant\tilde N\\ p\equiv-1\,(\mathrm{mod}\,4)}}\Big(1-\frac1p\Big),$$

where the constant B depends only on ε.

5. We now enter into the estimation of the most difficult sum $I(N,a)$. From (9) we have

(19) $$I(N,a) = \sum_{n\leqslant N} r(n^{(2)})\, r((n+a)^{(2)}) \sum_{\substack{l|n\\ l_1|n+a\\ l,l_1\in\Delta_1}}\mu(l)\mu(l_1) = \sum_{n\leqslant N} r(n^{(2)})\, r((n+a)^{(2)})\times$$

$$\times\Big\{\sum_{\max(\omega(l),\omega(l_1))\leqslant A\log\log N} + \sum_{\max(\omega(l),\omega(l_1))>A\log\log N}\Big\} = \Sigma_5+\Sigma_6, \text{ say}.$$

We estimate Σ_6 first, and we see that

(20) $$|\Sigma_6| \ll \sum_{\substack{\max(\omega(n),\omega(n+a))>A\log\log N\\ n\leqslant N}} d(n)\,d(n+a)$$

$$\ll (\log N)^{-A\log 2}\sum_{n\leqslant N}2^{\omega(n)}d(n)\,2^{\omega(n+a)}d(n+a)$$

$$\ll (\log N)^{-A\log 2}\Big\{\sum_{n\leqslant N}d^4(n)\Big\}^{1/2}\Big\{\sum_{n\leqslant N}d^4(n+a)\Big\}^{1/2}$$

$$\ll N(\log N)^{15-A\log 2}.$$

Now turning to the sum Σ_5, we remark that l and l_1 can be assumed to be square-free and so $l, l_1 \leqslant N^{A/\log\log N}$. Thus, noticing that $l^{(2)} = l_1^{(2)} = 1$, we have

(21) $$\Sigma_5 = \sum_{\substack{\max(\omega(l),\omega(l_1))\leqslant A\log\log N\\ (l,l_1)|a\\ l,l_1\in\Delta_1}}\mu(l)\mu(l_1)\sum_{\substack{l_1m_1=lm+a\\ lm\leqslant N}} r(m^{(2)})\,r(m_1^{(2)}).$$

Inserting the expression (12), we have

(22) $$\sum_{\substack{l_1m_1=lm+a\\ lm\leqslant N}} r(m^{(2)})\,r(m_1^{(2)}) = \sum_{\substack{l_1u_1v_1=luv+a\\ luv\leqslant N\\ u_1,u\in\Delta_0}}\mu(u)\varrho(u)\mu(u_1)\varrho(u_1)\,r(v)\,r(v_1)$$

$$= \sum_{\max(\omega(u),\omega(u_1))\leqslant A\log\log N}+\sum_{\max(\omega(u),\omega(u_1))>A\log\log N}$$

$$= \Sigma_7+\Sigma_8, \text{ say}.$$

We have

$$(23)\quad \Sigma_7 = \sum_{\substack{(ul,u_1l_1)|a\\ u,u_1\in\Delta_0\\ \max(\omega(u),\omega(u_1))\le A\log\log N}} \mu(u)\varrho(u)\mu(u_1)\varrho(u_1)\sum_{\substack{l_1u_1v_1=luv+a\\ luv\le N}} r(v)r(v_1).$$

Here we quote the following generalization of Estermann's result [2]: if $k, k_1 \le x^\varepsilon$ we have uniformly, denoting by $[n, m]$ the least common multiple of n and m,

$$\sum_{\substack{k_1v_1=kv+a\\ kv\le x}} r(v)r(v_1) = 16x\sum_{\substack{(kt,k_1t_1)|a\\ t,t_1\le\sqrt{x}}}\frac{\varrho(t)\varrho(t_1)}{[kt,k_1t_1]}+O(x^{\frac{11}{12}+\varepsilon}).$$

This can be established by following Estermann's argument closely, and so the proof may be omitted.

Inserting the above result into the inner sum of (23), we have

$$(24)\quad \Sigma_7 = 16N\sum_{\substack{u,u_1\in\Delta_0\\ \max(\omega(u),\omega(u_1))\le A\log\log N}}\mu(u)\varrho(u)\mu(u_1)\varrho(u_1)\times$$

$$\times\sum_{\substack{(lut,l_1u_1t_1)|a\\ t,t_1\le\sqrt{N}}}\frac{\varrho(t)\varrho(t_1)}{[lut,l_1u_1t_1]}+O(N^{\frac{11}{12}+2\varepsilon}).$$

Now we have from (19), (20), (21) and (22)

$$(25)\quad I(N,a) = \sum_{\substack{l,l_1\in\Delta_1\\ (l,l_1)|a\\ \max(\omega(l),\omega(l_1))\le A\log\log N}}\mu(l)\mu(l_1)\{\Sigma_7+\Sigma_8\}+O\big(N(\log N)^{15-A\log 2}\big),$$

where as in the case of Σ_6 we have easily

$$(26)\quad \sum_{\substack{l,l_1\in\Delta_1\\ (l,l_1)|a\\ \max(\omega(l),\omega(l_1))\le A\log\log N}}\mu(l)\mu(l_1)\Sigma_8$$

$$\ll\sum_{\substack{n\le N\\ \max(\omega(n),\omega(n+a))\le A\log\log N}} d^2(n)d^2(n+a)\ll N(\log N)^{63-A\log 2}.$$

And thus we have, from (24), (25) and (26),

LEMMA 4. *We have*

$$I(N,a) = 16N\sum_{\substack{l,l_1\in\Delta_1\\ \max(\omega(l),\omega(l_1))\le A\log\log N}}\mu(l)\mu(l_1)\times$$

$$\times\sum_{\substack{u,u_1\in\Delta_0\\ \max(\omega(u),\omega(u_1))\le A\log\log N}}\mu(u)\varrho(u)\mu(u_1)\varrho(u_1)\sum_{\substack{(lut,l_1u_1t_1)|a\\ t,t_1\le\sqrt{N}}}\frac{\varrho(t)\varrho(t_1)}{[lut,l_1u_1t_1]}+$$

$$+O\big(N(\log N)^{63-A\log 2}\big).$$

6. Now from Lemma 4 we have

$$\sum_{0\le j_1<j_2<h} I(N,j_2-j_1)$$

$$=16N\sum_{\substack{l,l_1\in\Delta_1\\ \max(\omega(l),\omega(l_1))\le A\log\log N}}\mu(l)\mu(l_1)\sum_{\substack{u,u_1\in\Delta_0\\ \max(\omega(u),\omega(u_1))\le A\log\log N}}\mu(u)\varrho(u)\mu(u_1)\varrho(u_1)\times$$

$$\times\sum_{t,t_1\le\sqrt{N}}\frac{\varrho(t)\varrho(t_1)}{[lut,l_1u_1t_1]}\sum_{\substack{j_2\equiv j_1(\mathrm{mod}(lut,l_1u_1t_1))\\ 0\le j_1<j_2<h}}1+O\big(h^2N(\log N)^{63-A\log 2}\big).$$

Here the inner sum is equal to

$$\frac{h^2}{2(lut,l_1u_1t_1)}+O(h),$$

where (n, m) denotes the greatest common divisor of n and m.

Thus we have, since $[n,m](n,m) = nm$,

$$(27)\quad \sum_{0\le j_1<j_2<h} I(N,j_2-j_1)$$

$$=8h^2N\Big\{\sum_{\substack{l,l_1\in\Delta_1\\ \max(\omega(l),\omega(l_1))\le A\log\log N}}\frac{\mu(l)\mu(l_1)}{ll_1}\Big\}\times$$

$$\times\Big\{\sum_{\substack{u,u_1\in\Delta_0\\ \max(\omega(u),\omega(u_1))\le A\log\log N}}\frac{\mu(u)\varrho(u)\mu(u_1)\varrho(u_1)}{uu_1}\Big\}\Big\{\sum_{t\le\sqrt{N}}\frac{\varrho(t)}{t}\Big\}^2+$$

$$+O\Big\{hN\sum_{l,l_1,u,u_1\le N^\varepsilon}\ \sum_{t,t_1\le\sqrt{N}}\frac{1}{[lut,l_1u_1t_1]}\Big\}+O\big(h^2N(\log N)^{63-A\log 2}\big)$$

$$=8h^2N\{\Sigma_9\}\{\Sigma_{10}\}\{\Sigma_{11}\}^2+O(hN\Sigma_{12})+O\big(h^2N(\log N)^{63-A\log 2}\big),\ \text{say}.$$

Now we have

$$(28)\qquad \Sigma_9 = \sum_{l,l_1\in\Delta_1}\frac{\mu(l)\mu(l_1)}{ll_1}+O\Big\{\sum_{\substack{l,l_1\in\Delta_1\\ \max(\omega(l),\omega(l_1))>A\log\log N}}\frac{1}{ll_1}\Big\}$$

$$= \prod_{\substack{p\equiv -1\,(\mathrm{mod}\,4)\\ p\leqslant N}}\Big(1-\frac{1}{p}\Big)^2+O\big((\log N)^{4-A\log 2}\big),$$

since we have

$$\sum_{\substack{l,l_1\in\Delta_1\\ \max(\omega(l),\omega(l_1))>A\log\log N}}\frac{1}{ll_1}\leqslant 2^{-A\log\log N}\sum_{l,l_1\in\Delta_1}\frac{d(l)d(l_1)}{ll_1}$$

$$\leqslant (\log N)^{-A\log 2}\prod_{p\leqslant N}\Big(1-\frac{1}{p}\Big)^{-4}\ll (\log N)^{4-A\log 2}.$$

In the same way we have

$$(29)\qquad \Sigma_{10} = \prod_{p\leqslant N}\Big(1-\frac{\varrho(p)}{p}\Big)^2+O\big((\log N)^{4-A\log 2}\big).$$

Also we have

$$(30)\qquad \Sigma_{11} = \frac{\pi}{4}+O(N^{-1/2}).$$

Thus, from (27), (28), (29) and (30) we get

$$(31)\qquad \sum_{0\leqslant j_1<j_2<h} I(N, j_2-j_1)$$

$$= \frac{\pi^2}{2}h^2N\delta^2(N)+O(hN\Sigma_{12})+O\big(h^2N(\log N)^{63-A\log 2}\big),$$

where $\delta(N)$ is defined by (6).

Now we have

$$\Sigma_{12}\leqslant \sum_{q,q_1\leqslant N}\frac{d_3(q)d_3(q_1)}{[q,q_1]}\leqslant \sum_{k\leqslant N^2}\frac{1}{k}\sum_{[q,q_1]=k}d_3(q)d_3(q_1)$$

$$\leqslant \sum_{k\leqslant N^2}\frac{1}{k}\sum_{\substack{q|k\\ q_1|k}}d_3(q)d_3(q_1) = \sum_{k\leqslant N^2}\frac{d_4^2(k)}{k}.$$

And hence we have

$$\Sigma_{12}\ll (\log N)^{16},$$

which, with (31), gives

LEMMA 5. *We have the asymptotic equality*

$$\sum_{0\leqslant j_1<j_2<h} I(N, j_2-j_1) = \frac{\pi^2}{2}h^2\delta^2(N)N+O\big(hN(\log N)^{16}\big)+O\big(h^2N(\log N)^{-E_2}\big),$$

where E_2 can be taken arbitrarily large.

7. Finally inserting the results of Lemmas 1, 3 and 5 into the right side of (7) we find

$$(32)\qquad R(N,h) = O\big(hN(\log N)^{16}\big)+O\big(h^2N(\log N)^{-100}\big)+O(h^3N^{\varepsilon}).$$

Now let $N^{1-2\varepsilon}\geqslant h\geqslant (\log N)^{17}$ and $Q_N(h)$ denote the number of integers $n\leqslant N$ such that

$$\tfrac{1}{2}\pi\delta(N)h\leqslant\Big|\sum_{0\leqslant j<h}\beta\big((n+j)^{(1)}\big)\gamma\big((n+j)^{(2)}\big)-\pi\delta(N)h\Big|.$$

Then we have from (32)

$$Q_N(h)\ll Nh^{-1}\delta^{-2}(N)(\log N)^{16}+N\delta^{-2}(N)(\log N)^{-100}+hN^{\varepsilon}\delta^{-2}(N)$$

$$\ll \frac{N}{(\log\log N)^2},$$

since

$$\delta(N) = \big(1+o(1)\big)C'\frac{\log\log N}{\sqrt{\log N}}.$$

Thus we have proved the result that, if $N^{1-2\varepsilon}\geqslant h\geqslant(\log N)^{17}$, then for almost all integers $n\leqslant N$ we have the inequality

$$\sum_{0\leqslant j<h}\beta\big((n+j)^{(1)}\big)r\big((n+j)^{(2)}\big)>\frac{\pi}{2}\delta(N)h.$$

And for such integers n we have, by the Cauchy–Schwarz inequality,

$$(33)\qquad \frac{\pi^2}{4}\delta^2(N)h^2\leqslant\sum_{0\leqslant j<h}b(n+j)\sum_{0\leqslant j<h}\beta\big((n+j)^{(1)}\big)r^2\big((n+j)^{(2)}\big).$$

Now, if h is in the range $N^{1-2\varepsilon}\geqslant h\geqslant N^{\varepsilon}$, we have from Lemma 3

$$\sum_{0\leq j<h} \beta\big((n+j)^{(1)}\big) r^2\big((n+j)^{(2)}\big) \ll h \frac{(\log\log N)^{B'}}{\sqrt{\log N}}.$$

Hence, from (33) and this, we get the inequality

$$\sum_{0\leq j<h} b(n+j) > h \frac{1}{\sqrt{\log N}\,(\log\log N)^{D_6}}$$

for almost all $n \leq N$. This ends the proof of our theorem.

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