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Density estimates for the zeros of L-functions*

by

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0. Introduction. We shall follow the notations introduced in [2]; in particular any product $\omega(n) = \chi(n)n^{it}$ is called a *generalized Dirichlet character*. We set $\|\omega\| = q(|t|+1)$, where q is the modulus of the Dirichlet character χ. If Ω is a finite set of generalized characters, we denote by $|\Omega|$ the cardinality of Ω and by $D = D(\Omega)$ the $\sup\limits_{\substack{\omega\neq\omega'\\ \omega,\omega'\in\Omega}} \|\omega'\overline{\omega}\|$. Ω is said to be *δ-well spaced* if for each $\omega, \omega' \in \Omega$, $\omega \neq \omega'$, $\omega = \chi(n)n^{it}$, $\omega' = \chi'(n)n^{it'}$ we have either $|t-t'| \geqslant \delta$ or $\chi'\overline{\chi}$ non-principal. $\mathscr{N}$ denotes the set of integers between $N/2$ and N. Moreover we define the Dirichlet series operator $\mathscr{D} = \mathscr{D}(\mathscr{N}, \Omega)$ as in [2]; thus one has the inequality

$$\Big(\sum_{\Omega}\Big|\sum_{\mathscr{N}} a_n\omega(n)\Big|^q\Big)^{1/q} \leqslant \|\mathscr{D}\|_{p,q}\Big(\sum_{\mathscr{N}}|a_n|^p\Big)^{1/p}.$$

We denote by $N(\alpha, T; \chi)$ the number of zeros $\varrho = \beta+i\gamma$ of $L(s,\chi)$ in the rectangle $\alpha \leqslant \beta \leqslant 1$, $|\gamma| \leqslant T$; $N(\alpha, T)$ denotes the number of zeros of $\zeta(s)$ in the same rectangle. For any finite set X of Dirichlet characters χ we set

$$N(\alpha, T;\ X) = \sum_{\chi\in X} N(\alpha, T;\ \chi).$$

We associate to each zero $\varrho = \beta+i\gamma$ of $L(s,\chi)$ the generalized character $\omega(n) = \chi(n)n^{-i\gamma}$; following the method introduced in [1] we shall determine certain coefficients $a_n \ll n^{\varepsilon}$ such that the Dirichlet polynomial $\sum\limits_{\mathscr{N}} a_n\omega(n)$, for a suitable N between $D^{\varepsilon'}$ and $D^{\frac{1}{2}+\varepsilon''}$, assumes large values on a well spaced set $\Omega = \Omega(\alpha, T;\ X)$ of generalized characters of cardinality approximately $N(\alpha, T;\ X)$. At this point one may readily apply the results of [2] for the Dirichlet series operator to obtain the following

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THEOREM 1.1. *For any $p, q \geqslant 2$ and for a suitable constant $C \geqslant 0$ one has*

$$N(\alpha, T; X) \ll \|\mathscr{D}\|_{p,q}^{q} N^{\frac{q}{p}-q\alpha+\varepsilon} \log^{C} D.$$

Note that if the conjecture

$$\|\mathscr{D}\|_{p,p} \ll (N^{\frac{1}{2}} + D^{\frac{1}{p}+\varepsilon}) N^{\frac{1}{2}-\frac{1}{p}+\varepsilon}$$

in [2] Section 4 held, one would obtain from Theorem 1.1, with $N = D^{\xi}$, $p = 2/\xi$, the density hypothesis $N(\alpha, T; X) \ll D^{2(1-\alpha)+\varepsilon}$.

Using the estimates proved in [2], we deduce first of all for the L-functions (cf. [6]) the classical result of Ingham for the zeta-function:

$$\mu_1(\alpha) \leqslant \frac{3(1-\alpha)}{2-\alpha},$$

where $\mu_1(\alpha)$ is the least exponent such that

$$N(\alpha, T; X) \ll D^{\mu_1(\alpha)+\varepsilon}$$

uniformly in $\frac{1}{2} \leqslant \alpha \leqslant 1$.

Continuing in this direction we obtain the following density theorems, which improve some of the recent results of Montgomery and Huxley (cf. [5] and [6]):

THEOREM 2.1. *We have*

$$N(\alpha, T; X) \ll D^{\mu_1(\alpha)+\varepsilon},$$

where

$$\mu_1(\alpha) \leqslant \begin{cases} \dfrac{45}{20-\sqrt{3}}(1-\alpha) & \text{for} \quad \frac{1}{2} \leqslant \alpha \leqslant 1, \\ 2(1-\alpha) & \text{for} \quad \alpha \geqslant \dfrac{5+\sqrt{3}}{8} = .8415\ldots \end{cases}$$

Remark. Arguing a little more carefully in the proof of the theorem, one may replace the constant $\frac{45}{20-\sqrt{3}} = 2.4633\ldots$ by a slightly smaller one. The corresponding constant found by Montgomery [6] was $\frac{5}{2}$.

For the zeta-function we obtain (compare with [5]):

THEOREM 3.1. *Let $\alpha^* = .8079\ldots$ be the greatest root of the polynomial $64\alpha^3 - 28\alpha^2 - 60\alpha + 33$. Then*

$$N(\alpha, T) \ll T^{\mu_1(\alpha)+\varepsilon},$$

where

$$\mu_1(\alpha) \leqslant \begin{cases} \frac{12}{5}(1-\alpha) & \text{for} \quad \frac{1}{2} \leqslant \alpha \leqslant 1, \\ 2(1-\alpha) & \text{for} \quad \alpha \geqslant \alpha^*. \end{cases}$$

Remark. Huxley [5] had obtained the density hypothesis for the zeta-function for $\alpha \geqslant \frac{5}{6}$.

As is well known, density estimates for the zeros of the L-functions and the zeta-function have important arithmetical applications. For example, using in the arguments of Gallagher [3] the above value $c < \frac{45}{20-\sqrt{3}}$, one obtains quantitative improvements of Theorems 2 and 3 in [3] concerning the distribution of primes in progressions having a prime power as modulus. In particular one sees that the smallest prime in an arithmetic progression of modulus p^r is $\ll_p p^{\frac{45r}{20-\sqrt{3}}}$.

The above results are based only on trivial estimates for the order of the functions L and ζ in the critical strip. Using deeper estimates (see e.g. [4], [7]) we can extend slightly the interval of validity of the density hypothesis, as well as improve the bound $\mu_1(\alpha) \leqslant 2(1-\alpha)$ in restricted ranges. In this way, we have both proved the density hypothesis for the zeta-function for $\alpha \geqslant .8059$ and obtained some bounds slightly better than Huxley's for $\alpha > \frac{3}{4}$.

Added 20. 9. 1972. After having finished this paper the authors learned of further improvements made independently by M. Jutila and M. N. Huxley. In particular, Jutila's results allow one to replace the constant $\frac{45}{20-\sqrt{3}}$ by $\frac{3}{16}(9+\sqrt{17}) = 2.4605\ldots$ in the statement of Theorem 2.1 above, as well as $\frac{5+\sqrt{3}}{8}$ by $\frac{5}{6}$. Results of the same strength have been obtained by Huxley.

The process which we have used, based on the general estimate (1.15) below, differs from the methods of Jutila and Huxley and may be considered a direct application of the ideas developed in [2].

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1. Preliminary results.

LEMMA 1.1. *For any α, T, X there exists a set of zeros of $L(s, \chi)$, $\chi \in X$, in the rectangle $\alpha \leqslant \sigma \leqslant 1$, $|t| \leqslant T$, such that the set Ω of associate characters satisfies the following conditions:*

(i) *Ω is $\log^2 D$-well spaced;*
(ii) $|\Omega| \gg N(\alpha, T; X)\log^{-A} D$ *for a suitable constant* $A \geq 0$;
(iii) *there exist coefficients* $a_n \ll n^{\varepsilon}$ *such that for each* $\omega \in \Omega$

$$\sum_{N/2}^{N} a_n \omega(n) \gg N^{\alpha} \log^{-1} D \tag{1.1}$$

for a suitable N *between* $D^{\nu(1-\lambda)}$ *and* $D^{(\frac{1}{2}+\nu+\varepsilon_1)(1+\lambda)}$, *with* λ, ν *arbitrary,* $0<\lambda<1$, $\nu>0$. *In particular we may take* N *between* $D^{\varepsilon'}$ *and* $D^{\frac{1}{2}+\varepsilon''}$ $(\varepsilon' < \varepsilon'')$.

Proof. Subdivide the rectangle $\alpha \leq \sigma \leq 1$, $|t| \leq T$ in horizontal strips of height $\log^2 D$. Since $N(\alpha, t+1; \chi) - N(\alpha, t; \chi) \ll \log D$, each strip contains $O(\log^3 D)$ zeros of $L(s, \chi)$ for every $\chi \in X$. Let us choose a zero from every third strip. The set Ω_χ of characters associated to these zeros obviously has cardinality $|\Omega_\chi| \gg N(\alpha, T; \chi)\log^{-3} D$. Setting $\Omega_1 = \bigcup_{\chi \in X} \Omega_\chi$, we see that Ω_1 is $\log^2 D$-well spaced, and that

$$|\Omega_1| \gg \sum_{\chi \in X} N(\alpha, T; \chi)\log^{-3} D = N(\alpha, T; X)\log^{-3} D.$$

Let Ω_b be the set of those characters $\omega(n) = \chi(n) n^{-i\gamma} \in \Omega_1$ such that

$$L(s, \chi) \neq 0 \quad \text{for} \quad \sigma \geq \alpha + \varepsilon_0, \quad |t-\gamma| \leq \log^2 D, \tag{1.2}$$

and set $\Omega_c = \Omega_1 - \Omega_b$. It follows that $|\Omega_c| \leq N(\alpha+\varepsilon_0, T; X)$. Therefore, either

$$N(\alpha, T; X) \ll |\Omega_b| \log^3 D \tag{1.3}$$

or

$$N(\alpha, T; X) \ll N(\alpha+\varepsilon_0, T; X);$$

in the second case one can argue as above for the zeros in $\alpha+\varepsilon_0 \leq \sigma \leq 1$, $|t| \leq T$. Thus we may assume that (1.3) holds. From (1.2) it follows (see e.g. [1]), for $\eta > \varepsilon_0$,

$$L(s, \chi) \ll D^{\varepsilon}, \quad \frac{1}{L(s, \chi)} \ll D^{\varepsilon} \quad \text{in } \sigma \geq \alpha+\eta, \ |t-\gamma| \leq \tfrac{1}{2}\log^2 D. \tag{1.4}$$

Now let

$$g(s, \chi) = \frac{1}{L(s, \chi)} = \sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s}$$

and

$$g_{Y,k}(s, \chi) = \sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s} e^{-\left(\frac{n}{Y}\right)^k}.$$

If $Y \ll D^B$, we obtain from [2], Lemma 2.1, choosing $\delta = \alpha+\eta-\sigma$, and from (1.2), (1.4) above, that:

$$g_{Y,k}(s, \chi) \ll D^{\varepsilon} Y^{\alpha+\eta-\sigma} \quad \text{in } \sigma < \alpha+\eta, \ |t-\gamma| \leq \tfrac{1}{8}\log^2 D.$$

We observe that

$$L(s, \chi) g_{Y,k}(s, \chi) = \sum_{n=1}^{\infty} b_{n,k}(Y)\chi(n) n^{-s},$$

with

$$b_{n,k}(Y) = \sum_{d|n} \mu(d) e^{-\left(\frac{d}{Y}\right)^k} \ll n^{\varepsilon}.$$

If $Y \leq Z \ll D^B$, we then have (see [2], Lemma 2.1)

$$\begin{aligned}(L(s, \chi) g_{Y,k}(s, \chi))_{Z,k} &= \sum_{n=1}^{\infty} \frac{b_{n,k}(Y)\chi(n)}{n^s} e^{-\left(\frac{n}{Z}\right)^k} \\ &= L(s, \chi) g_{Y,k}(s, \chi) + O\big(D^{\varepsilon} Z^{\delta} Y^{\alpha+\eta-\sigma-\delta} \max_{|t-\gamma| \leq \frac{1}{8}\log^2 D} |L(\sigma+\delta+it, \chi)|\big),\end{aligned} \tag{1.5}$$

for $-k/2 \leq \delta < 0$, $\sigma < \alpha+\eta-\delta$.

If $n \leq Y^{1-\lambda}$ $(0<\lambda<1)$, we have $e^{-\left(\frac{d}{Y}\right)^k} = 1 + O(Y^{-\lambda k})$, from which it follows that

$$b_{n,k}(Y) = \begin{Bmatrix} 1 & \text{for } n = 1 \\ 0 & \text{for } n > 1 \end{Bmatrix} + O(Y^{-\lambda k+\varepsilon}).$$

Hence

$$\sum_{n=1}^{\infty} \frac{b_{n,k}(Y)\chi(n)}{n^s} e^{-\left(\frac{n}{Z}\right)^k} = 1 + \sum_{Y^{1-\lambda}}^{Z^{1+\lambda}} \frac{b_{n,k}(Y)\chi(n)}{n^s} e^{-\left(\frac{n}{Z}\right)^k} + O(Y^{1-\alpha-\lambda k+\varepsilon}). \tag{1.6}$$

Given $\omega \in \Omega_b$, let $\varrho = \beta+i\gamma$ be the associated zero. Setting

$$a_n = b_{n,k}(Y) e^{-\left(\frac{n}{Z}\right)^k} \ll n^{\varepsilon},$$

we have from (1.5) and (1.6), for $s = \varrho$,

$$\begin{aligned}&\sum_{Y^{1-\lambda}}^{Z^{1+\lambda}} n^{-\beta} a_n \omega(n) \\ &= -1 + O(Y^{1-\alpha-\lambda k+\varepsilon}) + O\big(D^{\varepsilon} Z^{\delta} Y^{\alpha+\eta-\beta-\delta} \max_{|t-\gamma| \leq \frac{1}{8}\log^2 D} |L(\beta+\delta+it, \chi)|\big).\end{aligned} \tag{1.7}$$

Now let $\sigma^* \geq \frac{1}{2}$; following [1], we define $\mu(\sigma, \sigma^*)$ as the least exponent such that

$$L(\sigma+it, \chi) \ll (qT)^{\mu(\sigma, \sigma^*)+\varepsilon}$$

uniformly in χ, for t in the intervals $|t-\gamma| \leq \frac{1}{8}\log^2 D$, under the condition that $L(s, \chi) \neq 0$ for $\sigma > \sigma^*$, $|t-\gamma| \leq \log^2 D$. Obviously $\mu(\sigma, \sigma^*)$ is a convex function of σ; furthermore $\mu(\sigma, \sigma^*) = \frac{1}{2} - \sigma + \mu(1-\sigma, \sigma^*)$ and, by (1.4),

$$\mu(\sigma, \sigma^*) = \begin{cases} 0 & \text{for} \quad \sigma \geqslant \sigma^*, \\ \frac{1}{2} - \sigma & \text{for} \quad \sigma \leqslant 1 - \sigma^*. \end{cases}$$

It then follows from (1.7) that

$$\sum_{Y^{1-\lambda}}^{Z^{1+\lambda}} n^{-\beta} a_n \omega(n) = -1 + O(Y^{1-\alpha-\lambda k+\varepsilon}) + O(Z^\delta Y^{-\delta+\eta} D^{\mu(\alpha+\delta, \alpha+\eta)+\varepsilon}).$$

By the convexity of $\mu(\sigma, \sigma^*)$ we obtain that $\mu(\alpha+\delta, \alpha+\eta) \leqslant -\frac{\delta}{2} + \frac{\eta}{2}$.

Hence, for $Y = D^\nu, Z = D^{\frac{1}{2}+\nu+\varepsilon_1}$,

$$\sum_{Y^{1-\lambda}}^{Z^{1+\lambda}} n^{-\beta} a_n \omega(n) = -1 + O(D^{\nu(1-\alpha-\lambda k+\varepsilon)}) + O(D^{\delta\varepsilon_1 + (\nu+\frac{1}{2})\eta+\varepsilon}).$$

Choosing $k > \frac{1}{\lambda}$ and $\delta < -\frac{(\nu+\frac{1}{2})\eta+\varepsilon}{\varepsilon_1}$, we see that

$$\sum_{D^{\nu(1-\lambda)}}^{D^{(\frac{1}{2}+\nu+\varepsilon_1)(1+\lambda)}} n^{-\beta} a_n \omega(n) = -1 + o(1). \tag{1.8}$$

For the sake of brevity we set $\nu(1-\lambda) = a$, $(\frac{1}{2}+\nu+\varepsilon_1)(1+\lambda) = b$; we subdivide the interval $D^a \leqslant n \leqslant D^b$ into subintervals

$$2^{k-1} D^a \leqslant n \leqslant 2^k D^a \qquad (1 \leqslant k \leqslant m;\ m \ll \log D).$$

For every $\omega \in \Omega_b$, by (1.8), there exists a k such that

$$\sum_{2^{k-1}D^a}^{2^k D^a} n^{-\beta} a_n \omega(n) \gg \log^{-1} D. \tag{1.9}$$

Thus, for a suitable k_0, the set Ω of those $\omega \in \Omega_b$ for which (1.9) holds is such that

$$|\Omega| \gg |\Omega_b| \log^{-1} D. \tag{1.10}$$

For $N = 2^{k_0} D^a$, it follows from (1.3), (1.9) and (1.10) that Ω and N satisfy the required conditions. Q.E.D.

Remark. Let $\gamma(\alpha) = \lim_{h \to 0-} \frac{\mu(\alpha+h, \alpha)}{-h}$. We have

$$\frac{\mu(\alpha+\delta, \alpha+\eta)}{\eta - \delta} \leqslant \gamma(\alpha+\eta) + \varepsilon \leqslant \gamma(\alpha) + \varepsilon;$$

we may therefore replace the exponent $\frac{1}{2} + \nu + \varepsilon_1$ in (iii) of Lemma 1.1 by $\gamma(\alpha) + \nu + \varepsilon_1$.

From (1.1) we obtain, for $q \geqslant 0$,

$$\sum_{\Omega} \Big| \sum_{N/2}^{N} a_n \omega(n) \Big|^q \gg |\Omega| N^{q\alpha} \log^{-q} D,$$

from which follows immediately ($\mathscr{D} = \mathscr{D}(N, \Omega)$):

THEOREM 1.1. *For any $p, q \geqslant 2$ and for a suitable constant $C \geqslant 0$ one has*

$$N(\alpha, T;\ X) \ll \|\mathscr{D}\|_{p,q}^q N^{\frac{q}{p} - q\alpha + \varepsilon} \log^C D. \tag{1.11}$$

For the applications of Theorem 1.1, we shall use the results of Sections 3 and 4 of [2], in particular the estimates (Theorems 4.1 and 4.2 of [2]):

$$\|\mathscr{D}(\mathscr{N})\|_{p,p}^p \ll N^{\frac{p}{2}-1+\varepsilon} \|\mathscr{D}(\mathscr{N}^{k+1})\|_{2,2}^{2\vartheta} \|\mathscr{D}(\mathscr{N}^k)\|_{2,2}^{2(1-\vartheta)}, \tag{1.12}$$

$$\text{with } k = \left[\frac{p}{2}\right], \vartheta = \left\{\frac{p}{2}\right\};$$

$$\|\mathscr{D}(\mathscr{N})\|_{2k,2k}^{2k} \ll N^{k-1+\varepsilon} \|\mathscr{D}(\mathscr{N}^k)\|_{2,2}^2, \tag{1.13}$$

together with the estimates of $\|\mathscr{D}\|_{2,2}$ contained in Section 3 of [2], in particular

$$\|\mathscr{D}\|_{2,2}^2 \ll N + D\log^2 D, \tag{1.14}$$

$$\|\mathscr{D}\|_{2,2}^2 \ll N + N^\delta D^{\frac{1}{2}-\delta+\varepsilon}|\Omega| + N^\delta D^{\frac{1}{2}-\frac{\delta}{2}+\left(1-\frac{\delta}{2}\right)\mu_1(1-\delta)+\varepsilon} \qquad \text{for } 0 \leqslant \delta \leqslant \tfrac{1}{2}, \tag{1.15}$$

(Theorems 3.2 and 3.4 of [2]), where $\mu_1(\alpha)$ denotes, as in the sequel, the least exponent for which $N(\alpha, T;\ X) \ll D^{\mu_1(\alpha)+\varepsilon}$.

2. Density of the zeros of L-functions. From now on we let $N = D^\xi$. Hence we may choose ξ so that

$$\varepsilon' \leqslant \xi \leqslant \tfrac{1}{2} + \varepsilon'' \qquad (\varepsilon' < \varepsilon''). \tag{2.1}$$

From (1.11), (1.13) and (1.14) we obtain

$$N(\alpha, T;\ X) \ll D^{2k\xi(1-\alpha)+\varepsilon} + D^{1+k\xi(1-2\alpha)+\varepsilon},$$

whence

$$\mu_1(\alpha) \leqslant \max\{2k\xi(1-\alpha), 1 + k\xi(1-2\alpha)\},$$

for any choice of the positive integer k. Since $2k\xi(1-a) \geqslant 1+k\xi(1-2a)$ if and only if $k\xi \geqslant 1$, we have, with

$$k = \left[\frac{1}{\xi}\right], \tag{2.2}$$

$$\mu_1(a) \leqslant \min\{2(k+1)\xi(1-a),\, 1+k\xi(1-2a)\}. \tag{2.3}$$

Since $2(k+1)\xi(1-a) = 1+k\xi(1-2a)$ for $\xi = \dfrac{1}{2-2a+k}$, it follows from (2.3), for a suitable k, that

$$\mu_1(a) \leqslant \frac{2(k+1)(1-a)}{2-2a+k}. \tag{2.4}$$

Note that one may assume $k \geqslant 2$, because were $k = 1$ it would follow from (2.2) that $\xi > \frac{1}{2}$, contradicting (2.1). Moreover the right-hand side of (2.4) decreases as k increases; thus, with $k = 2$, we obtain (cf. [6]):

$$\mu_1(a) \leqslant \frac{3(1-a)}{2-a}. \tag{2.5}$$

From (1.11), (1.13) and (1.15) it follows that

$$N(a, T;\ X) \ll D^{2k\xi(1-a)+\varepsilon} + D^{\frac{1}{2}-\delta+k\xi(1+\delta-2a)+\varepsilon} N(a, T;\ X) + \\ + D^{\frac{1}{2}-\frac{\delta}{2}+\left(1-\frac{\delta}{2}\right)\mu_1(1-\delta)+k\xi(1+\delta-2a)+\varepsilon},$$

whence, with the assumption

$$\tfrac{1}{2}-\delta+k\xi(1+\delta-2a) < 0, \tag{2.6}$$

we conclude that

$$\mu_1(a) \leqslant \max\left\{2k\xi(1-a),\, \frac{1}{2}-\frac{\delta}{2}+\left(1-\frac{\delta}{2}\right)\mu_1(1-\delta)+k\xi(1+\delta-2a)\right\} \tag{2.7}$$

for every positive integer k and every δ such that $0 \leqslant \delta \leqslant \frac{1}{2}$.

From now on we suppose that (2.2) holds. If

$$\mu_1(1-\delta) \leqslant 2\delta, \tag{2.8}$$

we obtain the density hypothesis

$$\mu_1(a) \leqslant 2(1-a), \tag{2.9}$$

provided that a satisfies

$$\begin{cases} 2a > 1+\delta+\dfrac{\frac{1}{2}-\delta}{k\xi}, \\ 2a \leqslant 1+\delta+\dfrac{\frac{1}{2}-\frac{5}{2}\delta+\delta^2}{1-k\xi}. \end{cases} \tag{2.10}$$

It being obvious that $k\xi > \frac{2}{3}$, the inequalities (2.10) imply that $1-8\delta+ +4\delta^2 \geqslant 0$. The largest value of δ is, therefore, $\delta = 1-\dfrac{\sqrt{3}}{2}$, and thus we have the following

LEMMA 2.1. *For* $a \geqslant \dfrac{5+\sqrt{3}}{8}$ *the density hypothesis* $\mu_1(a) \leqslant 2(1-a)$ *holds.*

We will now obtain an estimate of the type

$$\mu_1(a) \leqslant c(1-a) \tag{2.11}$$

(c constant), uniformly in $\frac{1}{2} \leqslant a \leqslant 1$. By (2.3), (2.11) is obviously valid if $\xi \leqslant c/(2k+2)$; otherwise $k\xi > c/3$, whence, by (2.6) and (2.7), (2.11) holds whenever

$$\frac{1}{2}-\delta+\frac{c}{3}(1+\delta-2a) \leqslant 0,$$

and

$$\frac{1}{2}-\frac{\delta}{2}+\left(1-\frac{\delta}{2}\right)\mu_1(1-\delta)+\frac{c}{3}(1+\delta-2a) \leqslant c(1-a).$$

Let us suppose now

$$\mu_1(1-\delta) \leqslant b\delta; \tag{2.12}$$

in order that (2.11) hold it suffices that

$$\begin{cases} 2a \geqslant 1+\delta+\dfrac{3}{c}\left(\dfrac{1}{2}-\delta\right), \\ 2a \leqslant 4-2\delta-\dfrac{3}{c}\{1+(2b-1)\delta-b\delta^2\}. \end{cases} \tag{2.13}$$

The inequalities (2.13) are compatible if

$$c \geqslant \frac{\frac{3}{2}+2(b-1)\delta-b\delta^2}{1-\delta}. \tag{2.14}$$

By Lemma 2.1, (2.12) holds for $b_0 = 2$, $\delta_0 = \dfrac{3-\sqrt{3}}{8}$. From (2.14) we then obtain that $c \geqslant c_0 = \dfrac{30-\sqrt{3}}{2(5+\sqrt{3})}$, and from the first inequality of (2.13) that $a \geqslant a_0 = \dfrac{381-5\sqrt{3}}{16(30-\sqrt{3})}$. We may therefore put $b_1 = c_0$, $\delta_1 = \frac{7}{40} < 1-a_0$ in (2.12), (2.13) and (2.14) and verify directly that (2.12) holds for

$$b_2 = \frac{47}{20}, \qquad \delta_2 = \frac{5(2-\sqrt{3})}{2(5-\sqrt{3})}.$$

On the other hand it follows from (2.5) that (2.11) holds provided that $\alpha \leqslant 2-3/c$; from the first inequality in (2.13), (2.11) will hold in the whole interval $\frac{1}{2} \leqslant \alpha \leqslant 1$ if $4-\dfrac{6}{c} \geqslant 1+\delta+\dfrac{3}{c}(\frac{1}{2}-\delta)$, or

$$c \geqslant \frac{3(5-2\delta)}{2(3-\delta)}. \tag{2.15}$$

For $\delta = \delta_2$, (2.15) gives $c \geqslant \dfrac{45}{20-\sqrt{3}}$, which is greater than the value given by (2.14) for $b = b_2$, $\delta = \delta_2$.

This proves the following

Lemma 2.2. *For $\frac{1}{2} \leqslant \alpha \leqslant 1$ we have*

$$\mu_1(\alpha) \leqslant \frac{45}{20-\sqrt{3}}(1-\alpha).$$

We summarize the preceding results in the following

Theorem 2.1. *We have*

$$N(\alpha, T;\ X) \ll D^{\mu_1(\alpha)+\varepsilon},$$

where

$$\mu_1(\alpha) \leqslant \begin{cases} \dfrac{45}{20-\sqrt{3}}(1-\alpha) & \text{for} \quad \frac{1}{2} \leqslant \alpha \leqslant 1, \\ 2(1-\alpha) & \text{for} \quad \alpha \geqslant \dfrac{5+\sqrt{3}}{8} = .8415\ldots \end{cases}$$

3. Density of the zeros of the zeta-function. In the special case of the zeta-function Theorem 2.1 can be improved, keeping in mind Remark 3.5 of [2] (cf. [5]): from (3.14) of [2], with $V = T^{\eta}$ $(0 \leqslant \eta \leqslant 1)$, we obtain,

for $0 \leqslant \delta \leqslant \frac{1}{2}$,

$$\|\mathscr{D}\|_{2,2}^2 \ll NT^{1-\eta} + N^{\delta} T^{\eta\left(\frac{1}{2}-\delta\right)+\varepsilon} |\Omega| + N^{\delta} T^{1-\eta\left\{\frac{1}{2}+\frac{\delta}{2}-\left(1-\frac{\delta}{2}\right)\mu_1(1-\delta)\right\}+\varepsilon}.$$

Such an estimate, combined with (1.11) and (1.13), yields

$$N(\alpha, T) \ll T^{2k\xi(1-\alpha)+1-\eta+\varepsilon} + T^{\eta\left(\frac{1}{2}-\delta\right)+k\xi(1+\delta-2\alpha)+\varepsilon} N(\alpha, T) + T^{1-\eta\left\{\frac{1}{2}+\frac{\delta}{2}-\left(1-\frac{\delta}{2}\right)\mu_1(1-\delta)\right\}+k\xi(1+\delta-2\alpha)+\varepsilon}. \tag{3.1}$$

Hence, for the zeta-function, we have $\mu_1(\alpha) \leqslant 2(1-\alpha)$ provided that

$$\begin{aligned} &\eta(\tfrac{1}{2}-\delta)+k\xi(1+\delta-2\alpha) < 0, \\ &2k\xi(1-\alpha)+1-\eta \leqslant 2(1-\alpha), \\ &1-\eta(\tfrac{1}{2}-\tfrac{3}{2}\delta+\delta^2)+k\xi(1+\delta-2\alpha) \leqslant 2(1-\alpha), \end{aligned} \tag{3.2}$$

where $k = \left[\dfrac{1}{\xi}\right]$ and $\mu_1(1-\delta) \leqslant 2\delta$.

Inasmuch as it is convenient to choose η as large as possible in (3.2), we will set

$$\eta = \min\left\{1,\ k\xi\,\frac{2\alpha-1-\delta}{\frac{1}{2}-\delta}\right\}. \tag{3.3}$$

If $\eta = 1$ we reobtain the conditions (2.10); otherwise we have

$$\begin{aligned} &2\alpha \leqslant 1+\delta+\frac{\frac{1}{2}-\delta}{k\xi}, \\ &2\alpha \geqslant 1+\frac{k\xi}{k\xi(3-2\delta)-1+2\delta}, \\ &2\alpha \geqslant 1+\delta+\frac{\delta}{k\xi(2-\delta)-1}. \end{aligned} \tag{3.4}$$

For $k\xi = \frac{2}{3}$ it suffices that

$$\begin{aligned} &2\alpha \geqslant \frac{5+2\delta}{3+2\delta}, \\ &2\alpha \geqslant \frac{1+2\delta-2\delta^2}{1-2\delta}. \end{aligned} \tag{3.5}$$

From the first inequality of (3.5) we obtain that $\delta \geqslant \dfrac{5-6\alpha}{4\alpha-2}$. Thus, for

$\alpha \leqslant \frac{5}{6}$, we may put

(3.6) $$\delta = \frac{5-6\alpha}{4\alpha-2},$$

obtaining

(3.7) $$64\alpha^3 - 28\alpha^2 - 60\alpha + 33 \geqslant 0.$$

Let $\alpha^* = .8079\ldots$ be the greatest root of the polynomial (3.7); since, with the value of δ in (3.6), the inequalities (2.10) are compatible for $\alpha \geqslant \alpha^*$, and keeping in mind that the first inequality of (3.4) is complementary to the first of (2.10), we have proved the following

LEMMA 3.1. *For* $\alpha \geqslant \alpha^* = .8079\ldots$ *the density hypothesis* $\mu_1(\alpha) \leqslant 2(1-\alpha)$ *holds for the zeta-function.*

Note that, with the choice (3.3) for η, the exponent $2k\xi(1-\alpha)+1-\eta$ in (3.1) decreases as δ increases, while $1-\eta\left\{\frac{1}{2}+\frac{\delta}{2}-\left(1-\frac{\delta}{2}\right)\mu_1(1-\delta)\right\}+$ $+k\xi(1+\delta-2\alpha)$ increases as δ increases; if we impose the condition that the second exponent does not exceed the first, we obtain

$$2\alpha \leqslant 1+\delta+\frac{1-3\delta+2\delta^2}{1+(2b-1)\delta-b\delta^2},$$

provided that (2.12) holds. It follows that

$$\mu_1(\alpha) \leqslant 2k\xi(1-\alpha)+1-\eta = \max\left\{2k\xi(1-\alpha),\ 1+\frac{k\xi}{\frac{1}{2}-\delta}\,[2-\delta-(3-2\delta)\alpha]\right\}.$$

Since we have $\mu_1(\alpha) \leqslant 2(k+1)\xi(1-\alpha)$, we may choose ξ so that

$$2(k+1)\xi(1-\alpha) = 1+\frac{k\xi}{\frac{1}{2}-\delta}\,[2-\delta-(3-2\delta)\alpha],$$

whence, for $k=2$, $\xi = \dfrac{\frac{1}{2}-\delta}{(3+2\delta)\alpha-1-4\delta}$. Thus we have proved the following:

LEMMA 3.2. *If* δ, α *are such that*

$$\mu_1(1-\delta) \leqslant b\delta, \qquad 2\alpha \leqslant 1+\delta+\frac{1-3\delta+2\delta^2}{1+(2b-1)\delta-b\delta^2},$$

then

$$\mu_1(\alpha) \leqslant \max\left\{2(1-\alpha),\ \frac{3-6\delta}{(3+2\delta)\alpha-1-4\delta}(1-\alpha)\right\}.$$

Putting $\delta = 0$ in the above lemma, we reobtain the following result of Huxley [5]:

COROLLARY 3.1. *For* $\alpha \leqslant \frac{5}{6}$ *we have*

$$\mu_1(\alpha) \leqslant \frac{3}{3\alpha-1}(1-\alpha).$$

Combining this corollary with Theorem 2.1 we have (cf. [5]):

COROLLARY 3.2. *For* $\frac{1}{2} \leqslant \alpha \leqslant 1$ *we have*

$$\mu_1(\alpha) \leqslant \tfrac{12}{5}(1-\alpha).$$

From Lemma 3.1 and Corollary 3.2 we deduce

THEOREM 3.1. *Let* $\alpha^* = .8079\ldots$ *be the greatest root of the polynomial* $64\alpha^3-28\alpha^2-60\alpha+33$. *Then*

$$N(\alpha, T) \ll T^{\mu_1(\alpha)+\varepsilon},$$

where

$$\mu_1(\alpha) \leqslant \begin{cases} \frac{12}{5}(1-\alpha) & \text{for } \frac{1}{2} \leqslant \alpha \leqslant 1, \\ 2(1-\alpha) & \text{for } \alpha \geqslant \alpha^*. \end{cases}$$

References

[1] E. Bombieri, *Density theorems for the zeta-function*, Proc. of Symposia in Pure Math., vol. XX, Amer. Math. Soc. (1971), pp. 352–358.

[2] M. Forti and C. Viola, *On large sieve type estimates for the Dirichlet series operator*, Proc. of Symposia in Pure Math., vol. XXIV, Amer. Math. Soc. (to appear).

[3] P. X. Gallagher, *Primes in progressions to prime-power modulus*, Invent. Math. 16 (1972), pp. 191–201.

[4] W. Haneke, *Verschärfung der Abschätzung von* $\zeta(\frac{1}{2}+it)$, Acta Arith. 8 (1963), pp. 357–430.

[5] M. N. Huxley, *On the difference between consecutive primes*, Invent. Math. 15 (1972), pp. 164–170.

[6] H. L. Montgomery, *Topics in multiplicative number theory*, Lecture Notes in Mathematics 227, Berlin-Heidelberg-New York 1971.

[7] H.-E. Richert, *Zur Abschätzung der Riemannschen Zetafunktion in der Nähe der Vertikalen* $\sigma = 1$, Math. Ann. 169 (1967), pp. 97–101.

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