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On a theorem of Segre*

by

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As a generalization of a theorem of Hurwitz [2], B. Segre [5] proved the following

THEOREM. *Let α be an irrational number, τ a real number satisfying $0 < \tau \leqslant 1$. Then at least one of the inequalities*

$$(1)\qquad -\frac{1}{(1+4\tau)^{1/2}} \leqslant x(\alpha x - y) < 0, \qquad 0 < x(\alpha x - y) \leqslant \frac{\tau}{(1+4\tau)^{1/2}}$$

has an infinity of solutions with natural x and integer y.

Further, he showed that for $\tau = k^{-1}$ $(k = 1, 2, \ldots)$ there is no improvement possible and conjectured that for

$$\tau = (k+c)^{-1} \qquad (k = 1, 2, \ldots, 0 < c < 1)$$

one can replace the numbers

$$(1+4\tau)^{-1}, \quad \text{respectively} \quad \tau(1+4\tau)^{-1}$$

occurring in (1) by smaller ones. Segre's proof was based on Geometry of Numbers. Using continued fractions, I give a simple proof of the above result. Further, I give improvements in two directions, one of them conjectured by Segre.

1. Notations. The notations of the present paper coincide generally with those of O. Perron [4]. α denotes an irrational number, which can be assumed to satisfy $0 < \alpha < 1$. Further we put

$$(1.1)\qquad \alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}} = [0;\ a_1, a_2, \ldots],$$

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* After the present paper has been submitted, Dr. A. Schinzel kindly called my attention to the paper [3] by W. J. LeVeque, which considers the same problem. In an entirely different way LeVeque showed a statement analogous to my Theorems 2.3 and 2.4. Instead of (2.6), resp. (2.8) he needed five inequalities; on the other hand he did not need such restrictions as my restrictions (2.5) or (2.7).

(1.2) $$\frac{A_n}{B_n} = [0;\ a_1, \ldots, a_n], \quad (A_n, B_n) = 1,$$

(1.3) $$\zeta_n = [a_n;\ a_{n+1}, a_{n+2}, \ldots],$$

(1.4) $$t_n = \frac{B_{n-1}}{B_n} = [0;\ a_n, a_{n-1}, \ldots, a_1].$$

2. Formulation of the results and some remarks. The following theorems are proved.

THEOREM 2.1. *Suppose that* $0 < \tau \leqslant 1$ *and that for some* n *we have*

(2.1) $$a_{n+2} \geqslant 1/\tau.$$

Then at least one of the inequalities

(2.2) $$\begin{aligned} \zeta_{n+1}+t_n &\geqslant (1+4\tau)^{1/2}, \\ \zeta_{n+2}+t_{n+1} &\geqslant \tau^{-1}(1+4\tau)^{1/2}, \\ \zeta_{n+3}+t_{n+2} &\geqslant (1+4\tau)^{1/2} \end{aligned}$$

is satisfied.

THEOREM 2.2. *Suppose that* (2.1) *holds and that* τ *has the form*

(2.3) $$\tau = (k+c)^{-1} \quad (k = 1, 2, \ldots, 0 < c < 1).$$

Then there is a τ' $(\tau' < \tau)$ *depending only on* k *and* c *in* (2.3) *such that at least one of the inequalities*

(2.4) $$\begin{aligned} \zeta_{n+1}+t_n &\geqslant (1+4\tau)^{1/2}, \\ \zeta_{n+2}+t_{n+1} &\geqslant \frac{(1+4\tau)^{1/2}}{\tau'}, \\ \zeta_{n+3}+t_{n+2} &\geqslant (1+4\tau)^{1/2} \end{aligned}$$

holds.

THEOREM 2.3. *Suppose that* τ *has the form*

(2.5) $$\tau = k^{-1} \quad (k = 1, 2, \ldots).$$

Then at least one of the inequalities

(2.6) $$\begin{aligned} \zeta_n+t_{n-1} &\geqslant \tau^{-1}(1+4\tau)^{1/2}, \\ \zeta_{n+1}+t_n &\geqslant (1+4\tau)^{1/2}, \\ \zeta_{n+2}+t_{n+1} &\geqslant \tau^{-1}(1+4\tau)^{1/2}, \\ \zeta_{n+3}+t_{n+2} &\geqslant (1+4\tau)^{1/2} \end{aligned}$$

holds.

THEOREM 2.4. *Suppose that*

(2.7) $$1-\frac{(1+4\tau)^{1/2}-1}{2\tau} < \left\{\frac{1}{\tau}\right\},$$

$\{z\}$ *denoting the fractional part of* z. *Then with some* τ' $(\tau' < \tau)$ *depending only on* τ *we have at least one of the inequalities*

(2.8) $$\begin{aligned} \zeta_n+t_{n-1} &\geqslant \tau'^{-1}(1+4\tau)^{1/2}, \\ \zeta_{n+1}+t_n &\geqslant (1+4\tau)^{1/2}, \\ \zeta_{n+2}+t_{n+1} &\geqslant \tau'^{-1}(1+4\tau)^{1/2}, \\ \zeta_{n+3}+t_{n+2} &\geqslant (1+4\tau)^{1/2}. \end{aligned}$$

In the proof of Theorems 2.2, respectively 2.4 the expression of τ' in terms of τ will be given (see (3.5)).

Before proving the above theorems, I would like to deduce from them the mentioned theorem of Segre and its improvements. If A_n and B_n have the meaning (1.2), then, according to a well-known relation (see for instance Perron [4], p. 43)

(2.9) $$B_n(B_n a - A_n) = (-1)^n(\zeta_{n+1}+t_n)^{-1}.$$

Therefore, in order to show that there is an infinity of pairs (x, y) satisfying at least one of the inequalities (1), one has to show that there is an infinity of odd n's such that at least one of the inequalities (2.2) hold. By Theorem 2.1 this can be achieved if one can show that for an infinity of odd n's (2.1) holds. Now suppose that for all odd n's (or for all sufficiently large odd n's)

(2.10) $$a_{n+2} < 1/\tau$$

holds. Then

(2.11) $$\zeta_{n+1}+t_n > [1;\ 1/\tau, 1, 1/\tau, 1, \ldots]+[0;\ 1/\tau, 1, \ldots] = (1+4\tau)^{1/2},$$

which means that at least one of the inequalities (2.2) holds. Because of (2.9) this means that at least one of the inequalities (1) holds.

Therefore, we may suppose that for infinitely many odd n's (2.1) holds. Applying Theorem 2.1 and (2.9), Segre's theorem follows at once.

Theorem 2.2 is, with respect to (2.9), a conjecture of Segre, namely, that (1) can be improved for $\tau \neq k^{-1}$ $(k = 1, 2, \ldots)$.

Theorems 2.3 and 2.4 are "localization theorems". For $\tau = 1$ a classical theorem of Borel [1] asserts that for any n at least one of the inequalities

$$(2.12)\qquad \begin{aligned} \zeta_{n+1}+t_n &> 5^{1/2},\\ \zeta_{n+2}+t_{n+1} &> 5^{1/2},\\ \zeta_{n+3}+t_{n+2} &> 5^{1/2} \end{aligned}$$

has to hold. Theorems 2.3 and 2.4 make analogous statements for τ's of special kinds, but instead of three consecutive n, one has to admit four.

The last section of the present paper contains the proofs of our theorems.

3. Proofs.

Proof of Theorem 2.1. Suppose that, in contrary to our assertion, Theorem 2.1 is not true, that is,

$$(3.1)\qquad \begin{aligned} \zeta_{n+1}+t_n &< (1+4\tau)^{1/2},\\ \zeta_{n+2}+t_{n+1} &< \tau^{-1}(1+4\tau)^{1/2},\\ \zeta_{n+3}+t_{n+2} &< (1+4\tau)^{1/2}. \end{aligned}$$

Using the relations

$$\zeta_{n+1} = a_{n+1}+\frac{1}{\zeta_{n+2}} \quad\text{and}\quad t_{n+1} = \frac{1}{a_{n+1}+t_n}$$

we obtain by the first inequality of (3.1)

$$\frac{1}{\zeta_{n+2}} < (1+4\tau)^{1/2}-\frac{1}{t_{n+1}};$$

multiplying by

$$\zeta_{n+2} < \tau^{-1}(1+4\tau)^{1/2}-t_{n+1}$$

(second inequality of (3.1)) we obtain

$$1 < \left((1+4\tau)^{1/2}-\frac{1}{t_{n+1}}\right)\left(\tau^{-1}(1+4\tau)^{1/2}-t_{n+1}\right),$$

or, after an easy computation

$$(3.2)\qquad \tau t_{n+1}^2-(1+4\tau)^{1/2}t_{n+1}+1 < 0.$$

Similarly, using the second and third inequalities of (3.1) instead of the first and second ones, we obtain

$$(3.3)\qquad t_{n+2}^2-(1+4\tau)^{1/2}t_{n+2}+\tau < 0.$$

Solving the inequalities (3.2) and (3.3) we obtain

$$(3.4)\qquad t_{n+1} > \frac{(1+4\tau)^{1/2}-1}{2\tau},\qquad t_{n+2} > \frac{(1+4\tau)^{1/2}-1}{2}.$$

On the other hand, we get by (2.1)

$$t_{n+2} = \frac{1}{a_{n+2}+t_{n+1}} < \frac{1}{\tau^{-1}+\dfrac{(1+4\tau)^{1/2}-1}{2\tau}} = \frac{(1+4\tau)^{1/2}-1}{2}$$

which contradicts to (3.4).

Proof of Theorem 2.2. Let τ be of the form (2.3). Determine τ' ($\tau' < \tau$) such that

$$(3.5)\qquad \frac{1}{k+1} = \frac{\left(1+4(\tau-\tau')\right)^{1/2}}{\tau'}.$$

I am going to prove Theorem 2.2 with this τ'.

Suppose that our Theorem is not true, that is, (2.1) holds and still we have

$$(3.6)\qquad \begin{aligned} \zeta_{n+1}+t_n &< (1+4\tau)^{1/2},\\ \zeta_{n+2}+t_{n+1} &< \tau'^{-1}(1+4\tau)^{1/2},\\ \zeta_{n+3}+t_{n+2} &< (1+4\tau)^{1/2}. \end{aligned}$$

We obtain, by the relations

$$\zeta_{n+1} = a_{n+1}+\frac{1}{\zeta_{n+2}} \quad\text{and}\quad t_{n+1} = \frac{1}{a_{n+1}+t_n},$$

as in the proof of Theorem 2.1, using the first two inequalities of (3.6)

$$(3.7)\qquad t_{n+1} > \frac{(1+4\tau)^{1/2}-\sqrt{1+4(\tau-\tau')}}{2\tau'}$$

and using the second and third inequalities of (3.6)

$$(3.8)\qquad t_{n+2} > \frac{(1+4\tau)^{1/2}-\sqrt{1+4(\tau-\tau')}}{2}.$$

Now since (2.1) holds and a_{n+2} is an integer, we have, by (3.5) sharper

$$(3.9)\qquad a_{n+2} \geqslant \frac{\left(1+4(\tau-\tau')\right)^{1/2}}{\tau'}.$$

Therefore, by (3.7) and (3.9) we obtain

$$t_{n+2}=\frac{1}{a_{n+2}+t_{n+1}}<\frac{1}{\frac{\left(1+4(\tau-\tau')\right)^{1/2}}{\tau'}+\frac{(1+4\tau)^{1/2}-\left(1+4(\tau-\tau')\right)^{1/2}}{2\tau'}}$$

$$=\frac{(1+4\tau)^{1/2}-\left(1+4(\tau-\tau')\right)^{1/2}}{2},$$

contrary to (3.8).

Proof of Theorem 2.3. If

$$\zeta_n+t_{n-1}>\tau^{-1}(1+4\tau)^{1/2}$$

or

$$\zeta_{n+1}+t_n\geqslant(1+4\tau)^{1/2},$$

then there is nothing to prove. Therefore, we may suppose that the inequalities

$$\zeta_n+t_{n-1}<\tau^{-1}(1+4\tau)^{1/2},\qquad \zeta_{n+1}+t_n<(1+4\tau)^{1/2} \tag{3.10}$$

hold. We prove our Theorem 2.3 by showing that (3.10) and (2.5) imply (2.1) and Theorem 2.3 follows from Theorem 2.1.

Indeed, putting again

$$\zeta_n=a_n+\frac{1}{\zeta_{n+1}};\qquad t_n=\frac{1}{a_n+t_{n-1}},$$

we obtain, as in the proof of Theorem 2.1 (formula (3.3)) that

$$t_n>\frac{(1+4\tau)^{1/2}-1}{2}. \tag{3.11}$$

Suppose now that

$$a_{n+1}\geqslant 2. \tag{3.12}$$

Then by (3.10) and (3.11)

$$\zeta_{n+1}\leqslant(1+4\tau)^{1/2}-t_n\leqslant\frac{(1+4\tau)^{1/2}+1}{2},$$

or by (3.12)

$$\frac{1}{\zeta_{n+2}}<\frac{(1+4\tau)^{1/2}+1}{2}-2=\frac{(1+4\tau)^{1/2}-3}{2},$$

which is impossible, since the right-hand side is negative for $\tau\leqslant 1$. Therefore, $a_{n+1}=1$. Applying again the second inequality of (3.10) we obtain

$$\zeta_{n+1}=1+\frac{1}{\zeta_{n+2}}<(1+4\tau)^{1/2}-t_n<\frac{(1+4\tau)^{1/2}+1}{2},$$

or

$$\zeta_{n+2}>\frac{(1+4\tau)^{1/2}+1}{2\tau}. \tag{3.13}$$

Since the right-hand side of (3.13) is greater than $1/\tau$ and $1/\tau$ is an integer, we get

$$a_{n+2}=[\zeta_{n+2}]\geqslant 1/\tau$$

and our Theorem 2.3 is proved.

The proof of Theorem 2.4 can be carried out similarly by showing that, if (2.7) holds, then with τ' defined by (3.5) the inequalities

$$\zeta_n+t_{n-1}<\tau'^{-1}(1+4\tau)^{1/2},$$

$$\zeta_{n+1}+t_n<(1+4\tau)^{1/2}$$

imply (2.1) and the proof is completed, by application of Theorem 2.2. I omit the details.

References

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