ACTA ARITHMETICA
XXIII (1973)

Local behaviour of a class of multiplicative functions

by

W. NARKIEWICZ (Wrocław)

1. We shall consider the number $N_k(x)$ of solutions $n \leqslant x$ of

(1) $$f(n) = k$$

where k is a given number and $f(n)$ a positive, integer-valued multiplicative function, about which we assume the following facts:

(i) There are numbers t such that for $j = 1, 2, \ldots, t$ one has for prime p the equality $f(p^j) = a_j$, where a_j does not depend on p. Let T_1 be the maximal such t and, if every t is such, put $T_1 = \infty$.

(ii) If there are numbers $u \geqslant 1$ with $f(p^u) = 1$ for all primes p then $u \leqslant T_1$. If there are no such u's, then $T_1 = \infty$. In the first case we denote the minimal value of u by T.

The functions

$$d_r(n) = \sum_{x_1 \ldots x_r = n} 1 \quad (r = 2, 3, \ldots)$$

obviously satisfy (i) and (ii), and so does every multiplicative function equal to unity at primes. In the last case a recent result of A. S. Fainleib [2] gives $N_k(x) = c_k x + O(x^{1/2})$, where c_k is non-negative and vanishes only if (1) has no solutions.

2. To state our result define for $n = \prod_{i=1}^{r} p_i^{a_i}$ $(a_i \geqslant 1)$:

$$v(n) = \min(a_i\colon\ i = 1, 2, \ldots, r),$$
$$s_j(n) = \mathcal{N}\ (1 \leqslant i \leqslant r\colon\ a_i = j)$$

and for given k let $m = m(k)$ be defined by

$$m = \min\big(v(n)\colon f(n) = k\big).$$

(Note that this implies $m \leqslant T$.)

We prove the following

THEOREM. *Let f be a multiplicative function satisfying* (i) *and* (ii) *and such that* (1) *is solvable. Then we have the following two possibilities:*

(a) *If $m < T$ then $s = \max(s_m(n): f(n) = k)$ is finite and*

$$N_k(x) = \big(C_k + o(1)\big)x^{1/m}(\log\log x)^{s-1}(\log x)^{-1}$$

with C_k positive;

(b) *If $m = T$, then s is infinite and*

$$N_k(x) = \big(C_k + o(1)\big)x^{1/T}$$

with C_k positive.

Let us note that in the case where T_1 is infinite it is possible to make m and s more explicit. Indeed, if $f(p^j) = a_j$ and S is the set of all solutions of $a_{x_1}\dots a_{x_l} = k$, then m is the minimal value of x_i appearing in S and s is the maximal number of $x_i = m$ which can appear in a solution. It follows that a_m^s divides k; hence if $m < T$ and k is squarefree, one gets $s = 1$.

One easily sees that for the divisor function $d(n)$ one has $a_j = 1+j$; thus for $k \geqslant 2$ the number $1+m$ equals the minimal prime divisor q of k and s is defined by $q^s \mid k$, $q^{s+1} \nmid k$. So we get a result obtained by L. Mirsky [3], which we state as

Corollary. *If $k \geqslant 2$, q is the minimal prime dividing k, and $q^s \,\|\, k$, then*

$$\big(n \leqslant x:\ d(n) = k\big) = \big(C_k + o(1)\big)x^{1/(q-1)}(\log\log x)^{s-1}(\log x)^{-1}$$

with some positive C_k.

3. The proof is based on two lemmas, the first of which is a slight extension of one proved by Faînleib ([2], formula (12)) and of which we give a proof for the convenience of the reader.

Lemma 1. *Let A_r be the set of all numbers of the form $n = a^r b$ with squarefree a, $(r+1)$-full b and $(a, b) = 1$. Let $F(n)$ be a function defined on A_r which is bounded and depends only on b, i.e. $F(a^r b) = F(a_1^r b)$ as long as $(a, b) = (a_1, b) = 1$ and $\mu^2(a)\mu^2(a_1) = 1$. Then for x tending to infinity one has*

$$S(x) = \sum_{\substack{n\leqslant x\\ n\in A_r}} F(n) = C_F x^{1/r} + O(M_F x^{1/(1+r)})$$

with $M_F = \max_n |F(n)|$ and the implied constant does not depend on F. If $F \geqslant 0$ and is not identically zero, then $C_F > 0$. (For $r = 1$ this is the result of Faînleib.)

Proof. We start with the following elementary result:

Proposition 1. *If the integer M is given, then*

$$\sum_{\substack{n\leqslant x\\ (n,M)=1}} \mu^2(n) = \sum_{\substack{m\leqslant x\\ a(m)\mid a(M)}} (-1)^{\Omega(m)} \sum_{k\leqslant x/m} \mu^2(k)$$

where $\Omega(m)$ is the number of prime divisors of m, each counted according to its multiplicity, and $a(m)$ is the product of all distinct primes, dividing m.

Proof. In fact, we shall prove the following identity, from which the assertion follows at once:

$$(2)\qquad \sum_{\substack{k,m\\ km=n\\ a(m)\mid a(M)}} \mu^2(k)(-1)^{\Omega(m)} = \begin{cases} 1, & \mu^2(n) = 1,\ (n, M) = 1,\\ 0, & \text{otherwise.}\end{cases}$$

Write $M = P_1^{c_1}\dots P_u^{c_u}$ $(c_i \geqslant 1)$ and let $n = p_1^{b_1}\dots p_t^{b_t}P_1^{d_1}\dots P_u^{d_u}$ $(b_i \geqslant 1$, $d_i \geqslant 0$, $p_i \nmid M)$.

If at least one of the b_i's exceeds 1, then in every factorization $km = n$ with $a(m)\mid a(M)$ one has $p_1^{b_1}\dots p_t^{b_t} \mid k$ and our sum vanishes; hence (2) holds in this case.

If all exponents d_i are zero, then the only possible factorization $km = n$ with $a(m) \mid a(M)$ is given by $k = p_1\dots p_t$, $m = 1$ and our sum is equal to 1. On the other hand, in this case n is squarefree and relatively prime to M; thus (2) is true.

There remains the case $b_1 = \dots = b_t = 1$ and at least one exponent d_i is non-zero. Assume that $d_1, \dots, d_z$ are non-zero whereas the remaining d_i's vanish. Then every factorization $km = n$ with $a(m) \mid a(M)$ has the form

$$k = p_1\dots p_t \prod_{i\in S} P_i, \qquad m = \prod_{i\notin S} P_i^{d_i} \prod_{i\in S} P_i^{d_i-1}$$

where $S \subset \{1, 2, \dots, z\}$, and so the corresponding term in the left-hand side of (2) equals

$$\mu^2(k)(-1)^{\Omega(m)} = (-1)^{\Sigma d_i - |S|} = \begin{cases} 1, & |S| \equiv \sum d_i \pmod 2,\\ -1, & |S| \not\equiv \sum d_i \pmod 2.\end{cases}$$

But the number of subsets of a finite set of z elements with the cardinality of a given parity equals 2^{z-1}; hence in our sum the same number of $+1$ and -1 appears, and so it vanishes and (2) is satisfied also in this case.

Now we return to the proof of the lemma. For a given number r and all n's we shall write

$$q_n^{(r)} = \prod_{\substack{p^c\| n\\ c>r}} p^c.$$

With this notation the sum $S(x)$ which we want to evaluate equals

$$S(x) = \sum_{\substack{a^r b\leqslant x\\ b=q_b^{(r)}\\ (a,b)=1}} \mu^2(a)F(b) = \sum_{b=q_b^{(r)}\leqslant x} F(b) \sum_{\substack{a\leqslant (x/b)^{1/r}\\ (a,b)=1}} \mu^2(a)$$

$$= \sum_{b=q_b^{(r)}\leqslant x} F(b) \sum_{\substack{m\leqslant (x/b)^{1/r}\\ a(m)\mid a(b)}} (-1)^{\Omega(m)} \sum_{k\leqslant \left(\frac{x}{b}\right)^{1/r}\cdot\frac{1}{m}} \mu^2(k)$$

by Proposition 1. We shall put aside the case $r = 1$, which is fully proved in [2] and which needs some extra efforts connected with the evaluation of the remainder term. We now utilize the classical evaluation

$$\sum_{k\leqslant x}\mu^2(k) = \frac{6}{\pi^2}x + O(\sqrt{x}),$$

which is sufficient for our purpose in the case of $r > 1$. Using it, we can write our sum in the form

$$(3)\qquad S(x) = \frac{6}{\pi^2}x^{1/r}\sum_{\substack{m^r b\leqslant x\\ b=q_b^{(r)}\\ a(m)\mid a(b)}} F(b)(-1)^{\Omega(m)}(mb^{1/r})^{-1} + \\ + O\Big(M_F x^{1/2r}\sum_{\substack{m^r b\leqslant x\\ b=q_b^{(r)}\\ a(m)\mid a(b)}} m^{-1/2}b^{-1/2r}\Big).$$

First we show that the main term in (3) equals $Cx^{1/r}+O(M_F x^{1/(r+1)})$ where C is a constant which in the case of a non-negative function F can vanish only if F itself vanishes. To obtain this, observe that a trivial estimation gives

$$\Big|\sum_{\substack{m^r b> x\\ b=q_b^{(r)}\\ a(m)\mid a(b)}} F(b)(-1)^{\Omega(m)}(mb^{1/r})^{-1}\Big| \leqslant M_F\sum_{k>x}\frac{g(k)}{k^{1/r}},$$

where

$$g(k) = \sum_{\substack{bm^r=k\\ b=q_b^{(r)}\\ a(m)\mid a(b)}} 1.$$

Since obviously

$$g(k) \leqslant \sum_{\substack{b\mid k\\ b=q_b^{(r)}\\ a(b)=a(k)}} 1 = \sum_{\substack{b\mid k\\ a^{1+r}(k)\mid b}} 1,$$

one gets, writing $b = Ba^{1+r}(k)$, the inequality

$$g(k) \leqslant \sum_{B\mid k/a^{1+r}(k)} 1 = d\big(k/a^{1+r}(k)\big).$$

(Note that if $k\neq q_k^{(r)}$ then $g(k) = 0$.)

Now consider the multiplicative function

$$h(k) = \begin{cases} d\big(k/a^{1+r}(k)\big) & \text{if}\quad k = q_k^{(r)},\\ 0 & \text{otherwise}.\end{cases}$$

By Euler's factorization we get for $\operatorname{Re}s > 1/(1+r)$

$$\sum_{n=1}^{\infty} h(n)n^{-s} = \prod_p\Big(1+\frac{1}{p^{(1+r)s}}+\ldots\Big) = G(s)\Big(s-\frac{1}{1+r}\Big)^{-1}$$

with $G(s)$ regular for $\operatorname{Re}s > 1/(2+r)$ and non-vanishing at $s = 1/(1+r)$.

By [1] this gives

$$\sum_{n\leqslant x} h(n) = \big(C+o(1)\big)x^{1/(1+r)}$$

with some positive constant C. Finally we obtain

$$\sum_{k>x}\frac{g(k)}{k^{1/r}} \leqslant \sum_{k>x}\frac{h(k)}{k^{1/r}} \ll \sum_{k>x} h(k)\int_k^{\infty}\frac{dt}{t^{1+1/r}} = \int_x^{\infty}\sum_{x<k\leqslant t} h(k)\frac{dt}{t^{1+1/r}} \ll x^{-1/r(r+1)}$$

which shows that the main term in (3) is equal to

$$\frac{6}{\pi^2}x^{1/r}\sum_{\substack{m,b\\ b=q_b^{(r)}\\ a(m)\mid a(b)}} F(b)(-1)^{\Omega(m)}(mb^{1/r})^{-1} + O(x^{1/(1+r)}).$$

One immediately sees that the series occurring here equals

$$\sum_{b=q_b^{(r)}}\frac{F(b)}{b^{1/r}}\prod_{p\mid b}\frac{1}{1+\dfrac{1}{p}}$$

and so in the case of non-negative F vanishes only if F does.

Now we turn to the error term in (3). Utilizing the same function $g(k)$ as before, one sees that it is

$$\ll M_F x^{1/2r}\sum_{k\leqslant x}\frac{g(k)}{k^{1/2r}} \ll M_f x^{1/2r}\int_2^x t^{-1-\frac{1}{2r}+\frac{1}{1+r}}dt \ll M_f x^{1/(1+r)}.$$

This concludes the proof of Lemma 1.

LEMMA 2. *Let B be any non-void set of $(1+m)$-full natural numbers. (We assume throughout that 1 is $(1+m)$-full.) Then for every non-negative integer j one has in the halfplane $\operatorname{Re}s > 1/m$ the identity:*

$$S_j(s) = \sum_{\omega(n)=j}\mu^2(n)n^{-ms}\sum_{\substack{n_1\in B\\ (n,n_1)=1}} n_1^{-s} = V_j\big(\log\big(1/(s-1/m)\big)\big)$$

where $V_j(x)$ is a polynomial of degree j over Ω_m, the ring of functions regular for $\operatorname{Re}s \geqslant 1/m$, with leading coefficient positive at $s = 1/m$.

Proof. Write

$$S_j(s) = \sum_{N\in B} N^{-s} \sum_{\substack{n \\ \omega(n)=j \\ (n,N)=1}} \mu^2(n) n^{-ms}$$

and evaluate the inner sum as follows: For $|z|<1$, $\mathrm{Re}\, s > 1/m$ and given N one has

$$\sum_{\substack{n \\ (n,N)=1}} \mu^2(n) z^{\omega(n)} n^{-ms} = \prod_p \left(1+\frac{z}{p^{ms}}\right) \prod_{p|N} \left(1+\frac{z}{p^{ms}}\right)^{-1}$$

$$= \exp\left\{z \log\frac{1}{s-1/m}\right\} g(s,z) \prod_{k=1}^{t} \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{p_k^{msj}},$$

where $p_1, \ldots, p_t$ are all primes dividing N and $g(s,z) = \sum_j g_j(s) z^j$, $g_j(s) \in \Omega_m$, $g_0(1/m) \neq 0$.

This leads to

$$\sum_{\substack{n \\ (n,N)=1}} \mu^2(n) z^{\omega(n)} n^{-ms}$$

$$= \sum_{r=0}^{\infty} z^r \Big(\sum_{a+b+i_1+\ldots+i_t=r} g_a(s) \frac{1}{b!} \log^b \frac{1}{s-1/m} (-1)^{r-a-b} (p_1^{i_1} \ldots p_t^{i_t})^{-ms} \Big);$$

thus

$$\sum_{\substack{n \\ \omega(n)=j \\ (n,N)=1}} \mu^2(n) n^{-ms} = \sum_{a+b+i_1+\ldots+i_t=j} g_a(s) \frac{(-1)^{j-a-b} \log^b \dfrac{1}{s-1/m}}{b!(p_1^{i_1} \ldots p_t^{i_t})^{ms}}.$$

It follows that

$$S_j(s) = \sum_{a+b+c=j} g_a(s) \frac{1}{b!} \log^b \frac{1}{s-1/m} (-1)^c \sum_{N\in B} N^{-s} \sum_{i_1+\ldots+i_t=c} (p_1^{i_1} \ldots p_t^{i_t})^{-ms}$$

and, as the sum

$$\sum_{N\in B} N^{-s} \sum_{i_1+\ldots+i_t=c} (p_1^{i_1} \ldots p_t^{i_t})^{-ms}$$

obviously lies in Ω_m, our lemma follows.

4. Proof of the theorem. Case (a). $m < T$. Let

$$A = \{n: f(n) = k,\ v(n) = m\}.$$

For any n in A write $n = n_1^m n_2$ with n_1 2-free, n_2 $(1+m)$-full and $(n_1, n_2) = 1$. Then $\omega(n_1) \leqslant s$ and for some n one must have equality here. For $j = 0, 1, \ldots, s$ let $A_j = \{n: n\in A, \omega(n_1) = j\}$ and observe that, for n in A_j, $f(n_1^m)$ is constant, being in fact equal to a_m^j. It follows that $f(n_2) = k/a_m^j$.

Now if $B = \{n: n(1+m)\text{-full}, f(n) = k/a_m^j\}$, then by our assumption B is non-void and

$$\sum_{n\in A_j} n^{-s} = \sum_{\omega(n_1)=j} \mu^2(n_1) n_1^{-ms} \sum_{\substack{n_2\in B \\ (n_1,n_2)=1}} n_2^{-s} = S_j(s).$$

Applying Lemma 2 and the Tauberian Theorem of H. Delange [1] one obtains

$$\mathcal{N}(n \leqslant x: n\in A_j) = \big(C_j + o(1)\big) x^{1/m} (\log\log x)^{j-1} (\log x)^{-1};$$

thus

$$\mathcal{N}(n \leqslant x: n\in A) = \big(C_s + o(1)\big) x^{1/m} (\log\log x)^{s-1} (\log x)^{-1}.$$

It now suffices to observe that for $\mathcal{N}(n \leqslant x: f(n) = k, v(n) > m)$ one has the evaluation $O(x^{1/(1+m)})$.

Case (b). $T = m$. If A is the set of solutions of $f(n) = k$ with $(m+1)$-full n, then

$$\{n: f(n) = k\} = \{n: n = a^m b, (a,b) = 1, \mu^2(a) = 1, b \text{ in } A\}.$$

Applying Lemma 1 to the function $F(n)$ defined by

$$F(n) = \begin{cases} 1, & n = q_1^{a_1} \ldots q_s^{a_s} Q,\ a_i \leqslant m,\ Q \in A,\ (Q, q_1 \ldots q_s) = 1, \\ 0 & \text{otherwise}, \end{cases}$$

we get our assertion.

References

[1] H. Delange, *Généralisation du théorème de Ikehara*, Ann. Sci. Ec. Norm. Sup. 71 (1954), pp. 213–242.

[2] A. S. Fainleib, *Local theorems with remainder term for a class of arithmetical functions* (in Russian), Acta Math. Acad. Sci. Hung. 21 (1970), pp. 271–281.

[3] L. Mirsky, *On the distribution of integers having a prescribed number of divisors*, Simon Stevin 26 (1949), pp. 168–175.

WROCŁAW UNIVERSITY, INSTITUTE OF MATHEMATICS
UNIVERSITÉ BORDEAUX I, U. E. R. MATHÉMATIQUES ET INFORMATIQUE

Received on 14. 5. 1972 (286)