

If $l > 4$ our $l-1$ numbers $Y_j^{(l-1)}(0)$ may be approximated better than almost all $(l-1)$ -tuples in K^{l-1} . It is not difficult to show, by the methods of these papers, that one can never approximate much better in the above case, i.e. with a somewhat larger exponent on the $\log|q_N|$ the last inequality could only be satisfied finitely often for any choice of q_N and $P_{N,j}$.

References

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Received on 22. 4. 1972

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On the difference of consecutive terms of sequences defined by divisibility properties, II

by

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In a paper of the same title P. Erdős proved the following theorem:
 Let $b_1 < b_2 < \dots$ be an infinite sequence of integers satisfying

$$\sum \frac{1}{b_i} < \infty \quad (b_i, b_j) = 1.$$

Denote by a_1, a_2, \dots the sequence of integers not divisible by any b_i . Then there is an absolute constant c , independent of our sequence $b_1 < b_2 < \dots$ so that for all sufficiently large x the interval $(x, x + x^{1-c})$ contains a 's.

P. Erdős conjectured that perhaps $a_{i+1} - a_i = o(a_i)^\varepsilon$ holds for every $\varepsilon > 0$. We are unable to prove this at present, but we are going to prove the following sharpening of the result of P. Erdős.

THEOREM. Let $B = \{b_1 < b_2 < \dots\}$ be an increasing sequence of positive integers such that

$$(i) \quad \sum_{i=1}^{\infty} \frac{1}{b_i} < \infty$$

and

$$(ii) \quad (b_i, b_j) = 1 \quad \text{if } i \neq j.$$

Then for every $\varepsilon > 0$, if x is large enough, the interval $(x, x + x^{1/2+\varepsilon})$ contains a number a which is divisible by no b_j .

Proof. We can assume $b_1 > 1$. Let us define ε_1 and α so that

$$(1) \quad \varepsilon_1 = \min \left\{ \prod_{j=1}^{\infty} \left(1 - \frac{1}{b_j}\right), \varepsilon^2 \right\}$$

and

$$(2) \quad \sum_{j=\alpha}^{\infty} \frac{1}{b_j} < \varepsilon_1^2 < \varepsilon/8.$$

We shall assume that x is greater than a suitable function of ε , ε_1 and α .

Denote by P the set of all primes $p \notin B$. Put

$$H(x) = \{pt \in (x, x + x^{1/2+\varepsilon}) : p \in P \cap (2x^{1/2}, x^{1/2+\varepsilon})\},$$

$$\tilde{H}(x) = \{y \in H(x), b_j \nmid y \text{ for } j \leq \alpha\}$$

and then for $p \in P \cap (2x^{1/2}, x^{1/2+\varepsilon})$

$$H(x, p) = \{y \in H(x) : p \mid y\},$$

$$\tilde{H}(x, p) = H(x, p) \cap \tilde{H}(x).$$

Let

$$B(x) = \{b_i \in B, b_i = pt \geq x^{1/2+\varepsilon}, 2x^{1/2} \leq p < b_i\},$$

$$\tilde{B}(x) = \{y \in (x, x + x^{1/2+\varepsilon}); \exists_{b_j \in B(x)} b_j \mid y\}$$

and

$$L(x) = \{y \in (x, x + x^{1/2+\varepsilon}); \exists_{b_i} b_i \in (b_\alpha, x^{1/2+\varepsilon}), b_i \mid y\},$$

$$T(x) = \tilde{H}(x) - L(x) - \tilde{B}(x).$$

If $y \in T(x)$ and $b_j \in B$, then $b_j \nmid y$. Indeed

$$T(x) = \{y \in H(x); \forall_{j \leq \alpha} b_j \nmid y, \forall_{b_i < b_j < x^{1/2+\varepsilon}} b_i \nmid y, \forall_{\substack{b_j = qt \geq x^{1/2+\varepsilon} \\ 2x^{1/2} \leq q < b_j}} b_j \nmid y\}$$

$$= \{y \in H(x); b_i \mid y \Rightarrow b_i \geq x^{1/2+\varepsilon}, b_i \neq qt, 2x^{1/2} \leq q < b_i\}$$

$$\subseteq \{y = pt \in (x, x + x^{1/2+\varepsilon}); 2x^{1/2} < p < x^{1/2+\varepsilon}, p \notin B, b_i \mid y \Rightarrow b_i \geq x^{1/2+\varepsilon}, b_i \nmid y\}$$

$$\subseteq \{y \in H(x) \mid \forall_{b_i \in B} b_i, b_i \nmid y\}.$$

Therefore, it suffices to show $T(x) \neq \emptyset$. Since

$$(3) \quad \sum_{p \in (2x^{1/2}, x^{1/2+\varepsilon})} \frac{1}{p} \geq \frac{\varepsilon}{4}$$

it follows by (2) that

$$(4) \quad \sum_{p \in P \cap (2x^{1/2}, x^{1/2+\varepsilon})} \frac{1}{p} \geq \frac{\varepsilon}{8}.$$

Using the sieve of Erathostenes for all $p \in P \cap (2x^{1/2}, x^{1/2+\varepsilon})$ we obtain

$$(5) \quad |\tilde{H}(x, p)| \geq H(x, p) \prod_{i=1}^{\alpha} \left(1 - \frac{1}{b_i}\right) - 2^\alpha \geq \left(\frac{x^{1/2+\varepsilon}}{p} - 1\right) \varepsilon_1 - 2^\alpha.$$

For different p 's in question the sets $H(x, p)$ are disjoint. Since

$$\tilde{H}(x) = \bigcup_{p \in P \cap (2x^{1/2}, x^{1/2+\varepsilon})} H(x, p) \cap \tilde{H}(x) = \bigcup_{p \in P \cap (2x^{1/2}, x^{1/2+\varepsilon})} \tilde{H}(x, p)$$

we get from (4) and (5)

$$(6) \quad |\tilde{H}(x)| \geq \sum_{p \in P \cap (2x^{1/2}, x^{1/2+\varepsilon})} |\tilde{H}(x, p)|$$

$$\geq x^{1/2+\varepsilon} \varepsilon_1 \frac{\varepsilon}{8} - (2^\alpha + \varepsilon_1) \pi(x^{1/2+\varepsilon}) > \frac{\varepsilon_1 \cdot \varepsilon}{16} x^{1/2+\varepsilon}.$$

On the other hand, we obtain from (2)

$$(7) \quad |L(x)| \leq 2x^{1/2+\varepsilon} \sum_{b_i \in (b_\alpha, x^{1/2+\varepsilon}) \cap B} \frac{1}{b_i} \leq 2\varepsilon_1^2 x^{1/2+\varepsilon}.$$

Since $(b_i, b_j) = 1$ for $i \neq j$ we have by the definition of $B(x)$

$$(8) \quad |B(x) \cap (0, 2x)| = \sum_{\substack{b_i \in [x^{1/2}, x^{1/2+\varepsilon}] \cap B \\ b_i = p_j t_j, 2x^{1/2} < p_j < b_i}} 1 \leq \sum_{1 < b_i < x^{1/2}} \sum_{\substack{b_i \in B \\ b_i \mid b}} 1 \leq \sum_{1 < l < x^{1/2}} 1 < x^{1/2}.$$

Hence

$$(9) \quad |\tilde{B}(x)| \leq \sum_{b \in B(x)} \left(\sum_{y \in (x, x + x^{1/2+\varepsilon})} 1 \right) \leq \sum_{\substack{b \in B(x) \\ b < 2x}} 1 < x^{1/2}.$$

Finally, the estimates (6), (7) and (9) give

$$|T(x)| \geq |\tilde{H}(x)| - |L(x)| - |\tilde{B}(x)| > 0$$

which completes the proof of the theorem.

I express my thanks to the referee for the helpful criticism.

Reference

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Received on 22. 4. 1972

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