ACTA ARITHMETICA
XXIII (1973)

A method in diophantine approximation (VI)

by

CHARLES F. OSGOOD (Washington, D.C.)

To the memory of J. Popken

1. Introduction. In the following paper we shall first extend Theorem I of [6] so as to remove in that theorem the unnatural condition that a certain effectively computable polynomial does not vanish. To be precise we obtain the following result:

Let z denote a complex variable; let D denote $\frac{d}{dz}$; let l denote a fixed integer larger than or equal to one; and let each $g_j(z)$, for $1 \leqslant j \leqslant l$, denote a polynomial of degree at most $j-1$ that has coefficients in the Gaussian field; i.e. $Q(i)$. Suppose that $y_1, \ldots, y_l$ denote any l linearly independent solutions of

$$y = \sum_{j=1}^{l} g_j(z) D^j y. \tag{1}$$

Suppose further, that for some $0 \leqslant t \leqslant l-1$, $y_1, \ldots, y_t$ belong to the vector space over C generated by all differences of two branches of a solution of (1). Let $z_1, \ldots, z_m$ denote any $m \geqslant 1$ distinct points of $Q(i)$ none of which are zeros of $g_l(z)$.

THEOREM I. *The field F generated over $Q(i)$ by the numbers $D^{\varphi} y_j(z_k)$, for $1 \leqslant j \leqslant l$, $1 \leqslant k \leqslant m$, and $0 \leqslant \varphi < \infty$, has dimension over $Q(i)$ at least $ml(l-t)^{-1}$.*

Also we shall prove:

THEOREM II. *If $y \not\equiv 0$ satisfies an equation of type* (1) *and z_1 and z_2 are two distinct points in $Q(i)$ such that $g_l(z_1) g_l(z_2) \neq 0$, then there exists a power series coefficient of $y(z)$ at either $z = z_1$ or $z = z_2$ which is not in $Q(i)$.*

In [6] the conjecture was made that if $g_l(z)$ has $l-1$ distinct roots then we would "usually" be able to choose functions $y_1, \ldots, y_l$ above so that $t = l-1$. (Some qualification is necessary since one may pick the various coefficients in our equation so that each solution of (1) is entire.) Suppose now that $X_1, \ldots, X_{l-1}$ are $l-1$ fixed distinct points in C such that $\prod_{j=1}^{l}(z-X_j)$ belongs to $Q[i,z]$. Consider the equation

$$\gamma y = \sum_{k=1}^{l-1}\Big(\sum_{0\leqslant j<k}\gamma_{j,k}z^j\Big)D^k y + \prod_{l=1}^{l-1}(z-X_l)D^l y \tag{2}$$

where the $l(l+1)/2+1$ parameters γ and the $\gamma_{j,k}$ take values in C. Let G denote C minus cuts from each X_j to $z=\infty$.

Theorem III. *Given any z_0 in $Q(i)\cap G$ there exist $l-1$ functions $Y_1, \ldots, Y_{l-1}$ of γ, the $\gamma_{j,k}$, and z which satisfy (1), which satisfy $Y_j^{(s)}(z_0) = \delta_{j-1}^{s}$, for all $0\leqslant s\leqslant l-2$ (and all points in $C^{\frac{l(l+1)}{2}+1}$) and which are such that there exist a simple closed curve c, that does not pass through any of the points $X_1, \ldots, X_{l-1}$, and solutions W_j $(1\leqslant j\leqslant l-1)$ of (1), which are analytic on $C^{\frac{l(l+1)}{2}}\times G$ such that the difference between any W_j and its continuation around c is h_jY_j, where each h_j is a non-zero entire function of γ and the $\gamma_{j,k}$. Therefore, if the parameters in (2) have a value $\Gamma\in(Q(i))^{\frac{l(l+1)}{2}+1}$, the $y_1, \ldots, y_l$ are chosen as linearly independent solutions of (2) at the point Γ which are in the vector space over C spanned by the Y_j $(1\leqslant j\leqslant l-1)$, and the $y_{l+1}, \ldots, y_l$ are chosen so that, at Γ, $y_1, \ldots, y_l$ are a fundamental system of solutions of (2), then we are in the case described by Theorem I above, unless the entire function $\gamma\big(\prod_{j=1}^{l-1}h_j\big)$ vanishes at Γ.*

However, more is true:

Theorem IV. *Even if $\big(\prod_{j=1}^{l-1}h_j\big)$ vanishes at Γ, the field F generated over $Q(i)$ by the power series coefficients of the y_j, $1\leqslant j\leqslant l$, at the points $z_1, \ldots, z_m$ (where each z_k belongs to $Q(i)$ and no z_k is an X_t, $1\leqslant t\leqslant l-1$) has dimension over $Q(i)$ at least $ml(l-t)^{-1}$.*

Comments. In [6] we already gave several examples of Theorem I. Also Theorem IV allows us to see that Theorem I applies in very many cases with $t=l-1$. There is, however, loss of information in our present more general case as to exactly which elements of F can not be simultaneously approximated very well. Thus it is difficult to formulate a theorem analogous to Theorem IV of [6].

The results in this series of papers extend work done by Popken, in his thesis [7], on the simultaneous diophantine approximation of power series coefficients of an entire solution of an equation of type (1) at a rational point.

In the Addendum at the end of this paper we use Theorems III and IV, along with their proofs, to see that under certain circumstances collections of $l-1$ elements of F can be approximated better than almost all elements of R^{l-1}, despite the theorems in [2]–[6] and the present paper which might suggest otherwise.

In the second section of this paper we apply one of the new methods used in the proof of Theorem I to an analysis of the algebraic structure of some of the solutions of equations of type (1). (A much more complicated approach was used in [4].)

Definitions. By a *purely formal power series* (henceforth a p.f.p.s.) we shall mean a formal sum of the form $\sum_{j=1}^{k}\sum_{n=0}^{\infty}a_{n,j}z^{n+c_j}\big(\Gamma(n+c_j+1)\big)^{-1}$ where the c_j's and the $a_{n,j}$'s are any complex numbers and we view each $z^{\theta}\big(\Gamma(\theta+1)\big)^{-1}$ as a formal object satisfying merely

$$D\big(z^{\theta}(\Gamma(\theta+1))^{-1}\big) = z^{\theta-1}\big(\Gamma(\theta)\big)^{-1} \tag{i}$$

and

$$zD\big(z^{\theta}(\Gamma(\theta+1))^{-1}\big) = \theta\big(z^{\theta}(\Gamma(\theta+1))^{-1}\big). \tag{ii}$$

Using multiplication by powers of z and D we may change any linear homogeneous differential equation with coefficients in $Q[i,z]$ into one of the type

$$\sum_{j=1}^{l}h_j(zD)D^jy = 0 \tag{3}$$

where each $h_j(zD)$ belongs to $Q[i,zD]$ and l is a non-negative integer. Let R denote the set of all p. f. p. s. which are solutions of equations of type (3). Notice that any equation of type (1) may be rewritten as identially equal to an equation of type (3). Let $M\subseteq R$ denote the set of all p. f. p. s. which are solutions of equations of type (1). Let $R'\subseteq R$ denote the subset of R consisting of all p. f. p. s. which satisfy an equation of type (3) with, at worst, a regular singular point at $z=\infty$.

By a *formal power series* (f. p. s.) we shall mean a formal sum of the type $\sum_{j=1}^{k}\sum_{n=0}^{\infty}a_{n,j}z^{n+c_j}$ for complex numbers c_j and $a_{n,j}$. (If w is any f. p. s. which is a solution of a linear homogeneous differential equation with coefficients in $Q[i,z]$ then w satisfies an equation of type (3). If no c_j is a negative integer then we may write w as $\overline{w}$, a p. f. p. s., in an obvious way. As a p. f. p. s. $\overline{w}$ satisfies the same equation of type (3) as w except that, in general, there is a non-homogeneous term which is a p. f. p. s. with a finite number of terms each of the form $b_{-\theta}z^{-\theta}\big(\Gamma(1-\theta)\big)^{-1}$ where θ is a positive integer. Thus, multiplying through by $(zD-1)\ldots(zD-N)$ for some positive integer N, we see that $\overline{w}\in R$. Note also that if $\bar{\bar{w}}$ is a p. f. p. s. formed from w by first deleting all terms with $n+c_j$ equal to a negative integer and then rewriting what is left as a p.f.p.s., if follows that $\bar{\bar{w}} \in R$.)

Let us define $y_1 * y_2$ for y_1 and y_2 in R by

$$z^{\theta_1}\big(\Gamma(\theta_1+1)\big)^{-1} * z^{\theta_2}\big(\Gamma(\theta_2+1)\big)^{-1} = z^{\theta_1+\theta_2+1}\big(\Gamma(\theta_1+\theta_2+1)\big)^{-1}$$

for each pair of complex numbers θ_1 and θ_2 and extending by linearity.

THEOREM V. *The set R is a ring under $*$ and $+$.*

THEOREM VI. (i) *If $y_1(z) \in R$ and $y_2(z) \in M$ then $y_1(z)*y_2(tz) \in M$ for all algebraic values of t with but at most a finite number of exceptions.*

(ii) *If $y_1(z) \in R'$ and $y_2(z) \in M$ then we always have that $y_1(z)*y_2(z) \in M$.*

Clearly in the above Theorem we may apply Theorem II if $y_1(z)*y_2(tz)$ does belong to M and the p. f. p. s. for $y_1(z)*y_2(tz)$ does converge to a function on some deleted neighborhood of $z=0$. However, more is true.

THEOREM VII. *Suppose that $y \in M$ and that y represents a not identically zero function on some deleted neighborhood of $z=0$. Suppose that y can be continued analytically to some non-zero point $z_1 \in K$ where $[K:Q(i)] < \infty$. If there exists an equation of type (1) which is satisfied by y and of which z_1 is not a singular point then*

$$\max_{0\leqslant k\leqslant l-1}\{|y^{(k)}(z_1)-p_k q^{-1}|\} > |q|^{-d} \tag{4}$$

for all Gaussian integers p_k and q with $|q|$ sufficiently large, where $d>0$, $l>0$, and the lower bound on $|q|$ each depend on y and z_1.

THEOREM VIII. *Suppose that $y_1 \not\equiv 0$ belongs to R, $y_2 \not\equiv 0$ belongs to M and that $y = y_1 * y_2$ belongs to M. Suppose that y_1 and y_2 each represent functions in a deleted neighborhood of $z=0$. Let X denote the largest region which is starshaped about zero and which contains no non-zero singularities of the respective equations for y_1 and y_2. Let K denote any finite extension field of $Q(i)$. Then (4) holds for $d>0$, $l>0$, and the lower bound on $|q|$ each depending on y and z_1.*

Comments (continued). The theorems in Section II allow us to see that one may construct many functions to which we can apply Theorem I (or Theorem VII) and which are not entire functions at all. One would like to prove a theorem showing that either M or the set of all solutions of equations of type (1) is closed under ordinary multiplication, since then a transcendentality proof might be possible. This is trivially impossible since $1 = e^z \cdot e^{-z} = e^{w_j(z)} e^{-w_j(z)}$ where $w_j(z)$ is a solution to $p(w) = z$ for any $p(w) \in Q[i, w]$. Using asymptotic expansions about $z=\infty$ one can find many other such examples. To see that, in general, M is not closed under $*$ note that

$$\frac{1}{z^2} = \left(\frac{1}{z}e^{-1/z}\right)\left(\frac{1}{z}e^{1/z}\right)$$

and take inverse Laplace transforms of both sides of the above equation. This yields

$$z = \sum_{n=0}^{\infty}\frac{(-z)^n}{(n!)^2} * \sum_{n=0}^{\infty}\frac{z^n}{(n!)^2}.$$

The latter two functions satisfy $y = \pm DzDy$ while the function z can not satisfy an equation of type (1) by, say, Theorem II.

Section I

DEFINITIONS. By a *formal series* expansion about $z=\infty$ (f. s.) we mean an expression of the form

$$\sum_{h=1}^{r} g_h(\log(z)) \sum_{j=1}^{k} \exp\big(p_{j,h}(z^{1/n})\big) f_{j,h}(z)$$

where $k \geqslant 1$, $n \geqslant 1$, and $r \geqslant 1$ are each integers, each $p_{j,h}(x)$ and each $g_h(x) \in C[x]$, and each $f_{j,h}(z)$ is a f. p. s. in z^{-1}. Two f. s. *are equal* iff the g_h's, the $p_{j,h}$'s, and the $f_{j,h}$'s are identical (see [1]).

Proof of Theorem I. In this proof we shall consider a more general class of algebraic functions than we considered in [6]. Let $m>1$ be a positive integer. For $Y_1, \ldots, Y_m$ and z each sufficiently near zero

$$\prod_{k=1}^{m}(w - Y_k z - z_k) = z \tag{5}$$

defines m distinct algebraic functions (note if $z=0$ we have that $\prod_{k=1}^{m}(w-z_k) = 0$, and the z_k are distinct by assumption). Regarding the $Y_1, \ldots, Y_m$ as indeterminants and letting L denote the field of algebraic numbers with $Y_1, \ldots, Y_m$ adjointed we know that there exist m f. p. s. expansions in z^{-1} which satisfy (5). In each expansion we see that beginning with a dominant term other than one of the $Y_k z$ is impossible, since otherwise (5) could not possibly hold. (Recall $m>1$.) Also interchanging the Y_k's can only send one expansion into another. Thus the m different expansions start $Y_k z + \ldots$ for $1 \leqslant k \leqslant m$. Using similar reasoning we see that there are m f. p. s. expansions in w^{-1} of the form $Y_k^{-1} w + \ldots$ which are also solutions to (5). Let us denote by $w_k(z)$ and $z_k(w)$, respectively, the f. p. s. above, as well as the functions to which they converge, with the enumeration being the obvious one.

In [1] it is shown that given any linear homogeneous differential equation of order l with coefficients meromorphic at $z = \infty$ it possesses l linearly independent solutions which are each f. s. expansions involving only one exponential. Also in [1] it is shown that no more than l such linearly independent f. s. solutions exist. In an equation of type (1) the formal solutions must each have a non-constant exponential factor since, as one may verify immediately upon replacing y in (1) by $z^a(\ln(z))^a$, any expansion in terms of powers of z and a finite number of powers of $\ln(z)$ can not be a solution.

With the y_j, $1 \leqslant j \leqslant l$, as in the hypotheses and the functions $w_k(z)$, $1 \leqslant k \leqslant m$, as above we wish to show that the composite functions $y_j(w_k(z))$ have a not identically zero Wronskian, as a function of $z, Y_1, \ldots, Y_m$. Suppose not. We may write their Wronskian matrix as $A(z, Y_1, \ldots, Y_m)$ times $\left(y_j^{(\theta)}(w_k(z))w_k^{\varphi}(z)\right)$, for $1 \leqslant j \leqslant l$, $0 \leqslant \theta \leqslant l-1$, $1 \leqslant k \leqslant m$, and $0 \leqslant \varphi \leqslant m-1$. It was shown in [6] that the determinant of this latter matrix is a power product of the Vandermonde determinant of $w_1(z), \ldots$ $\ldots, w_m(z)$ and the Wronskians of the $y_1(w_k(z)), \ldots, y_l(w_k(z))$, for $1 \leqslant k \leqslant m$. Thus $A(z, Y_1, \ldots, Y_m)$ must be singular. It follows, since $A(z, Y_1, \ldots, Y_m)$ has entries in $Q(i, z, Y_1, \ldots, Y_m)$ that the rows of $A(z, Y_1, \ldots, Y_m)$, and hence of the Wronskian of the $y_j(w_k(z))$, are linearly dependent over $Q[i, z, Y_1, \ldots, Y_m]$. Thus there exists a linear homogeneous differential equation with coefficients in $Q[i, z, Y_1, \ldots, Y_m]$ which has order less than ml and which is satisfied by each $y_j(w_k(z))$. We shall show that this is impossible. One may choose $\bar{y}_1, \ldots, \bar{y}_l$ to be l linearly independent f. s. solutions of (1). Then we notice that for each choice of $Y_1, \ldots, Y_m$ the composites $\bar{y}_j(w_k(z))$ may be written as f. s. also, using the f. p. s. expansion about $z = \infty$ for $w_k(z)$. Thus for each choice of $Y_1, \ldots, Y_m$ the $\bar{y}_j(w_k(z))$ are f. s. solutions to our differential equation of order less than ml. We shall obtain a contradiction by showing that for some choice of $Y_1, \ldots, Y_m$ for which our differential equation is not identically zero the $\bar{y}_j(w_k(z))$ are linearly independent. This will show that the order of our equation is actually larger than or equal to ml. Recall the definition of equality of two f. s. is that the series be exactly identical termwise. We notice that setting each

$$\bar{y}_j = \exp(a_{h(j)}z^{m^{-1}h(j)} + \ldots) \text{ times a series free of exponentials},$$

where $a_{h(j)}h(j) \neq 0$, then every

$$\bar{y}_j(w_k(z)) = \exp\left(a_{h(j)}(Y_k z)^{m^{-1}h(j)} + \ldots\right) \text{ times a series free of exponentials}.$$

If we choose $Y_1, \ldots, Y_m$ to be algebraically independent over $Q(a_{h(1)}, \ldots$ $\ldots, a_{h(l)})$ then the differential equation does not vanish and the $\bar{y}_j(w_k(z))$, if they are linearly dependent, must satisfy a minimal dependence relation

(having a minimal number of non-zero terms) in which only one value of k, say k_1, appears. Now substitute $z_{k_1}(w)$ for z in the dependence relation and we have a dependence relation among the $\bar{y}_j$. This contradiction proves that our differential equation has order at least ml which contradicts the assumption that the Wronskian of the $y_j(w_k(z))$ vanishes identically. Thus we have seen that the $y_j(w_k(z))$ have a Wronskian which is a not identically zero function of $z, Y_1, \ldots, Y_n$.

At the point $0 = Y_1 = \ldots = Y_m = z$ the Wronskian must be analytic, since the z_k are distinct and none of them are singularities of the $y_j(z)$. Also there must exist a region D_1 in C^m which contains $(0, \ldots, 0)$ and a region D_2 in C which contains 0 such that the Wronskian is analytic on $D_1 \times D_2$. Further there must exist a ray in D_1, given by $(Y_1, \ldots, Y_m)$ $= (K_1 s, \ldots, K_m s)$ for $0 \leqslant s \leqslant 1$ where $(K_1, \ldots, K_m)$ is a non-zero vector in $(Q(i))^m$, such that on $[0, 1] \times D_2$ the Wronskian, $W(s, z)$, is not identically zero. Thus there must exist a non-negative integer M such that $s^{-M}W(s, z)$ is analytic and not identically zero when $s = 0$. Also we may choose our ray so that our linear differential equation for the $y_j(w_k(s, z))$ with coefficients in $Q[i, s, z]$ is not identically zero. Then dividing through by an appropriate power of s and taking the limit as $s \to 0$ we have a non-zero linear differential equation with coefficients in $Q[i, z]$ which is satisfied by the $y(w_k(0, z))$.

Suppose that above $M > 0$. Then at $s = 0$ the rank of the Wronskian matrix is less than ml. This says that at $s = 0$ there is a linear relation among the columns, with constant coefficients, since the $y_j(w_k(0, z))$ satisfy a common linear differential equation which is not identically zero. The coefficients of the dependence relation may be chosen to be in the field F since the power series coefficients of the $y_j(w_k(0, z))$ at $z = 0$ are in F, by definition. Suppose that we have actually carried out all of the above procedure on the Wronskian of the functions

$$\{f_1, \ldots, f_{ml}\} \stackrel{\text{def}}{=} \{y_j(w_k(s, z)) - y_j(w_1(s, z)), \text{ for } 1 \leqslant j \leqslant l \text{ and } 2 \leqslant k \leqslant m,$$
$$\text{and the functions } y_1(w_1(s, z)), \ldots, y_l(w_1(s, z)), \text{ enumerated in this order}\}.$$

Then at $s = 0$ there is a dependence relation which involves only columns in an initial segment of $1, \ldots, ml$ of minimal length. In the Wronskian we may replace the function in the last column involved in the dependence relation by s^{-1} times the linear combination of functions which is, at $s = 0$, identically zero. Then these functions have a Wronskian which vanishes to the order $M-1$ at $s = 0$. Continuing, we arrive at a set of ml functions $\Phi_1(s, z), \ldots, \Phi_{ml}(s, z)$ which satisfy our original linear differential equation, which are analytic on $[0, 1] \times D_2$, which have a Wronskian that does not vanish identically at $s = 0$, which have each $\left(\dfrac{\partial}{\partial z}\right)^n \Phi_j(0, 0)$

in F, and which are such that each

$$\Phi_j(0, z) \stackrel{\text{def}}{=} \varphi_j(z) = \varphi_j = \sum_{0\leqslant r\leqslant j}\Big(\sum_{\theta\geqslant 0} A_{r,\theta}\Big(\frac{\partial}{\partial s}\Big)^{\theta} f_r(0, z)\Big)$$

for a collection of $A_{r,\theta}$ in F. From now on let $w_k(z) = w_k$ denote $w_k(0, z)$. We recall from differential equations that

$$W(\lambda\Phi_1, \ldots, \lambda\Phi_{ml}) = \lambda^{ml} W(\Phi_1, \ldots, \Phi_{ml}),$$

where W denotes the Wronskian, if λ is analytic. Set λ equal to a power of

$$\prod_{k=1}^{m}\Big(\frac{\partial}{\partial w_k}\prod_{j=1}^{m}(w_k - z_j)\Big).$$

If the power is sufficiently high then each $\lambda\varphi_j$ may be written as a linear combination over C of different $w_k^{\alpha}(z)y_j^{(\beta)}(w_k(z))$, for $1\leqslant j\leqslant l$, $1\leqslant k\leqslant m$, $\alpha\geqslant 0$, and $\beta\geqslant 0$, and each of $\lambda\varphi_1, \ldots, \lambda\varphi_{(m-1)l+t}$ may be written as a linear combination over C of differences of branches of the $w_k^{\alpha}(z)y_j^{(\beta)}(w_k(z))$ (see the proof of Theorem II in [6]). Clearly $W(\lambda\varphi_1, \ldots, \lambda\varphi_{ml}) \not\equiv 0$ and the $\lambda\varphi_j$ have derivatives in F at $z = 0$. From the statements about the $\lambda\varphi_j$ in the last two sentences we shall ultimately prove Theorem I.

First we must see that the $w_k^{\alpha}(z)y_j^{(\beta)}(w_k(z))$ satisfy a common equation of type (1). Now as we shall next see, the statement that y satisfies an equation of type (1) is equivalent to the statement that there exist a sequence of integrals $E^1y, \ldots, E^Ny, E^{N+1}y$, such that $D(E^{N+1}y) = E^Ny$ for each positive integer N and the functions $y, Ey, \ldots$ generate a finitely generated module over the Noetherian ring $Q[i, zD]$. If y satisfies an equation of type (1) then integrating the equation once and applying integration by parts we may determine Ey such that Ey is a linear combination over $Q[i, zD]$ of $y, Dy, \ldots, D^{l-1}y$ and, extending by induction, we may choose E^Ny such that it is a linear combination over $Q[i, zD]$ of $E^{N-1}y = DE^Ny, \ldots, E^{N-l+1}y = D^{l-1}E^Ny$. The other way is easier. By the ascending chain condition in $Q[i, zD]$ there must exist an equation of type (1) which is satisfied by E^cy for some positive integer c. Thus y satisfies an equation of type (1). It suffices now to show that if y satisfies an equation of type (1) then zy satisfies an equation of type (1) also. This is true since then we would have that each $z^{\alpha}y^{(\beta)}(z)$ satisfies an equation of type (1), so by Theorem V of [6] each $(w_k(z))^{\alpha}y^{(\beta)}(w_k(z))$ satisfies an equation of type (1) also, and finally by the a. c. c. of $Q[i, zD]$ a linear combination of all of these above functions, for bounded α and β, with coefficients which are arbitrary constants satisfies an equation of type (1). Now we may define each $E^N(zy)$ to be $-NE^{N+1}y + (zD)E^{N+1}y$ and we have shown that zy satisfies an equation of type (1).

We may next show that our equation of type (1) may be assumed to have a regular point at $z = 0$. All that we need to do is to show that there exists some linear differential equation with coefficients in $Q[i, z]$ which is satisfied by all of the above functions and which has a regular point at $z = 0$, since we may multiply this latter equation by a high power of D and add it to our equation of type (1) above. Now the

$$\Big(\prod_{k=1}^{m}\big(g_l(w_k(z))\big)\Big(\frac{\partial}{\partial w_k}\Big(\prod_{j=1}^{m}(w_k - z_j)\Big)\Big)^2\Big)^N D^N\big(w_k^{\alpha}(z)y_j^{(\beta)}(w_k(z))\big)$$

generate a finitely generated module over $Q[i, z]$. By the a. c. c. of $Q[i, z]$ we see that our $w_k^{\alpha}(z)y_j^{(\beta)}(w_k(z))$ satisfy a linear differential equation with coefficients in $Q[i, z]$ with a regular point at $z = 0$.

Little remains to be done. We may now apply Theorem I of [3] (as strengthened in the proof of Theorem II of [6]) to our equation of type (1) satisfied by $\lambda\varphi_1, \ldots, \lambda\varphi_{(m-1)l+t}$ and such that the $\lambda\varphi_j$, $1\leqslant j\leqslant (m-1)l+t$, are each linear combinations of differences of branches of solutions of the equation. Then we conclude that for all choices of c_j (except each c_j equal to zero) some derivative of $\sum_{j=1}^{(m-1)l+t} c_j\lambda\varphi_j$ is not in $Q(i)$. We have a non-zero linear differential equation with coefficients in $Q[i, z]$ and of order at most ml which is satisfied by the φ_j's as was seen by setting $s = 0$ in an equation satisfied by the different Φ_j's.

Obviously this former equation must have order exactly ml. If $z = 0$ is not a singular point of the above equation then we know that every derivative of $\sum_{j=1}^{(m-1)l+t} c_j\lambda\varphi_j$ may be written as a linear combination over $Q(i)$ of at most ml distinct derivatives. (We shall be able to show this last, later, even if there is a singularity at $z = 0$.) Since the $\lambda\varphi_j$ each satisfy a differential equation with a regular point at $z = 0$ it follows that there are $0\leqslant\theta_1 < \theta_2 < \ldots < \theta_{(m-1)l+t}$ non-negative integers, such that the matrix $\big((\lambda\varphi_j)^{(\theta_k)}\big)$ is non-singular at $z = 0$. Without loss of generality one may assume that the θ_k, $1\leqslant k\leqslant (m-1)l+t$, are among the ml linearly independent derivatives mentioned above. Further one may construct constants $c_{j,r}$ in F such that each

$$\Phi_r^{(\theta_k)}(0) = \sum_{j=1}^{(m-1)l+t} c_{j,r}(\lambda\varphi_j)^{(\theta_k)}(0) = \delta_r^k$$

for $1\leqslant r\leqslant (m-1)l+t$. Then if $[F: Q(i)] = d < ml(l-t)^{-1}$ we may solve $(d-1)(l-t)$ homogeneous linear equations with coefficients in $Q(i)$ in $(m-1)l+t$ variables and find a non-zero set of A_r in $Q(i)$ such that every derivative of $\sum_{r=1}^{(m-1)l+t} A_r\Phi_r$ at $z = 0$ is in $Q(i)$. This contradiction shows that $d \geq ml(l-t)^{-1}$, in the above case which we shall show is the general case.

Since the $\lambda\varphi_j$ are a fundamental system of solutions of the equation referred to above and since the $\lambda\varphi_j$ are each analytic at $z=0$ our equation must have at worst a regular singularity at $z=0$. None of the roots of the indicial equation at $z=0$ can be other than non-negative integers and in the Frobenius expansions about $z=0$, with coefficients in $Q(i)$, no powers of $\log z$ can occur. Thus there exist ml power series in z, ψ_j, which are a fundamental system of solutions at the above equation and which have coefficients in $Q(i)$. One may write $\sum\limits_{j=1}^{(m-1)l+t} c_j\lambda_j\varphi_j$ as a linear combination of the ψ_j, with coefficients which are linear forms over $Q(i)$ in ml derivatives of $\sum\limits_{j=1}^{(m-1)l+t} c_j\lambda\varphi_j$ at $z=0$. This proves Theorem I.

Proof of Theorem II. We shall establish that if $z_1, \ldots, z_m$ are as in Theorem I, $y_1, \ldots, y_l$ are any fundamental system of solutions of an equation of type (1) satisfied by y, and z_1 and z_2 are as in our present hypotheses, then defining $w_1(z), \ldots, w_m(z)$ by each $w_k(0) = z_k$ and $\prod\limits_{k=1}^{m}(w_k - z_k) = z$ it follows that for sufficiently large m the $y_j(w_k(z))$ are linearly independent. If we establish this then $y(w_1(z)) - y(w_2(z)) \not\equiv 0$ is the difference of two branches at a solution of an equation of type (1) with an ordinary point of $z=0$ (we can always arrange this last by the argument used above in the proof of Theorem I) and Theorem II follows at once by Theorem I of [3].

Possibly renumbering the $w_k(z)$, set each $w_k(z) = \varrho^k z^{1/m} + \ldots$ where $\varrho = \exp(2\pi i m^{-1})$. Recall from the proof of Theorem I that there exist l linearly independent f. s. solutions of our equation of type (1) with each term in a given solution having the same, non-constant, exponential factor. Let us divide the solutions into disjoint sets according to the highest power of z which appears in this exponential factor. If, as in the proof of Theorem I above, we substitute the f. p. s. for each $w_k(z)$ into the above f. s. and do some rewriting we have ml f. s. which, if

$$W\big(y_1(w_1(z)), \ldots, y_l(w_m(z))\big) \equiv 0,$$

satisfy a linear homogeneous differential equation with coefficients in $Q[i, z]$ of order less than ml. If we can show that the ml formal series are linearly independent for some sufficiently large m we will have shown that the $y_j(w_k(z))$ are linearly independent and we will be through. A minimal dependence relation (one involving a minimal number of terms) among the formal series would have to be among the composites of all of the $w_k(z)$ with all of the f. s. in one of the above sets. If one set contains f. s. with exponential factors looking like $\exp(a_{\gamma,j} z^{\gamma^{-1}} + \ldots)$ where no

$a_{\gamma,j}$ is zero then let S_γ denote the (finite) set of all $\big(a_{\gamma,j_1}(a_{\gamma,j_2})^{-1}\big)^{\gamma^{-1}}$ with $j_1 \neq j_2$. If we have a minimal dependence relation among the composite series, involving only composites of the above set of f. s. then for each j and k actually appearing in the dependence relation the $a_{\gamma,j}(\varrho^k)^\gamma$ must be equal so the $\varrho^{k_1-k_2} \in S_\gamma$, for any two values of k appearing in a dependence relation. For any two distinct primes m_1 and m_2 the set of m_i-th roots of unity is disjoint except for the root $z=1$. Thus if m is a sufficiently large prime and there is a minimal dependence relation involving composite series for this value of m it must be that $\varrho^{k_1-k_2} = 1$ for all k_1 and k_2 with $w_{k_1}(z)$ and $w_{k_2}(z)$ appearing in the dependence relation. Thus only one $w_k(z)$, say $w_{k_1}(z)$, appears in this dependence relation. It was shown in [1] that a dependence relation among f. s. is equivalent to their Wronskian identically vanishing. Substituting $\varrho^{-k_1}z^{1/m}$ for $z^{1/m}$ in the Wronskian gives us again an identically vanishing Wronskian. Now for some $1 \leq k_2 \leq m$, $w_{k_2}\big(\prod\limits_{j=1}^{m}(z-z_j)\big) = z$. Looking at the expansions about $z=\infty$ we see that here $k_2 = m$. Substituting $\prod\limits_{j=1}^{m}(z-z_j)$ for z in the Wronskian we again get an identically vanishing Wronskian — this time for a collection of linearly independent f. s. solutions to our original equation of type (1). This contradiction proves Theorem II.

Proof of Theorem III. Consider the differential equation

$$D\Big(\prod_{j=1}^{l-1}(z-X_j)D+\theta_j\Big)\,y = 0 \tag{6}$$

where the X_j are distinct, fixed, algebraic numbers such that $\prod\limits_{j=1}^{l-1}(z-X_j) \in Q[i, z]$ and each θ_j is a complex number with real part between 0 and 1. Set $\theta_0 \overset{\text{def}}{=} X_0 \overset{\text{def}}{=} 0$. Then the general solution of (6) looks like

$$\sum_{t=0}^{l-1} c_t\Big(\prod_{j=2}^{l-t}\Big[(z-X_{l-j+1})^{-\theta_{l-j+1}}\int\limits_{X_{l-j+1}}^{z}(z-X_{l-j+1})^{\theta_{l-j+1}}\Big]\Big)(z-X_t)^{-\theta_t}$$

where the c_t are arbitrary constants and where each $\int\limits_{X_j}^{z} f(z) \overset{\text{def}}{=} \int\limits_{X_j}^{z} f(z)\,dz$. Notice that if $f(z)$ is analytic at $z=X_j$ and equals $\sum\limits_{n=1}^{\infty} b_n(z-X_j)^{n-1}$ then

$$(z-X_j)^{-\theta_j}\int\limits_{X_j}^{z}(z-X_j)^{\theta_j}f(z)\,dz = \sum_{n=1}^{\infty} b_n(n+\theta_j)^{-1}(z-X_j)^n$$

is analytic at $z=X_j$ also. Then the $l-1$ functions above with coefficients $c_1, \ldots, c_{l-1}$ are linearly independent, since each has only one singularity and these singularities are at different points. Consider next the differential equation

$$(7) \qquad \gamma y = \sum_{k=0}^{l-1}\Big(\sum_{0\leqslant j<k}\gamma_{j,k}z^j\Big)D^k y + \Big(\prod_{t=1}^{l-1}(z-X_t)\Big)D^l y$$

where the X_t's are as before and the $\gamma_{j,k}$ and γ are parameters taking values in C. If

$$\theta_t \overset{\text{def}}{=} \Big(\sum_{0\leqslant j\leqslant l-2}\gamma_{j,l-1}X_t^j\Big)\Big(\prod_{k\neq t}(X_t-X_k)^{-1}\Big)$$

is not an integer there exists a unique series solution of (7) about the point X_t beginning with the term $(z-X_t)^{-\theta_t}$. These series are well defined for all values of the parameters except when θ_t is a negative integer. If we now restrict our parameters to any simply connected bounded region B of $C^{\frac{l(l+1)}{2}+1}$ we may make each of the series defined (as f. p. s.) on the closure of B by multiplying through by some power of $\prod_{j=1}^{N}(\theta_t+j)$, for some positive integer N. Let us call these new f. p. s., depending on B, $\bar{y}_1, \ldots$ $\ldots, \bar{y}_{l-1}$. It is possible to estimate the absolute values of the coefficients of the series, if the parameters are in B. Thus each series $\bar{y}_k$ converges on $B\times$ (some open set N_k in C) to an analytic function of $l(l+1)/2+2$ variables. Let G denote C minus cuts from the X_j's to $z=\infty$.

Let $z_0\in G$. The Picard Existence Theorem gives l solutions each analytic on B and having I as their Wronskian matrix at z_0. Using this we may continue $\bar{y}_k(z)$ to be analytic on the simply connected region $B\times G$. Alternatively we may define each $\bar{y}_k(z)$ to be a multiple-valued analytic function on $B\times(C-\{X_1, \ldots, X_{l-1}\})$. If we continue $\sum_{t=1}^{l-1} c\bar{y}_t(z)$ around X_k, where the c_t are arbitrary constants, and take the difference of the branches we obtain (for z near X_k and thus in general) a function proportional to $\bar{y}_k(z)$. The coefficients of the arbitrary constants c_t in the proportionality factors (for each value of k) must be functions meromorphic on B. Further, we shall show that these coefficients form a non-singular matrix. Suppose that the rank of this matrix is less than $l-1$. Then we could choose constants c_t (which are functions analytic on B) not all zero so that $\sum_{t=1}^{l-1} c_t\bar{y}_t$ is single valued on $B\times(C-\{X_1, \ldots, X_{l-1}\})$. We have already seen that for the choice of the parameters in (6), it is impossible to choose non-zero constants c_t such that $\sum_{t=1}^{l-1} c_t\bar{y}_t$ is single-valued on $C-$ $-\{X_1, \ldots, X_{l-1}\}$. There must exist a ray out from this value of the parameters on which the coefficients c_t are not identically zero. If s, $0\leqslant s\leqslant 1$, is the parameter of this ray, then dividing by s to an appropriate power and taking the limit as $s\to 0$ we obtain a non-zero function of the form

$\sum_{t=1}^{l-1} b_t\bar{y}_t(z)$, for constants b_t, which is analytic on C. This contradiction shows that the previously mentioned coefficients must have a non-singular matrix.

If we choose $z_0\in Q(i)-\{X_1, \ldots, X_{l-1}\}$, then we shall show that the Wronskian of $\bar{y}_1, \ldots, \bar{y}_{l-1}$ does not vanish identically at $z=z_0$. Assume this for the moment. Then we can construct $Y_1, \ldots, Y_{l-1}$, linear combinations of the $\bar{y}_k$ (with coefficients meromorphic on B) such that $Y_j^{(k)}(z_0) = \delta_j^{k-1}$. Notice also that one could choose coefficients $c_{j,t}$ analytic on B such that, setting $W_j = \sum_{t=1}^{l-1} c_{j,t}\bar{y}_j$, we have that the difference of W_j before and after being continued around c, a simple closed curve which encloses the X_t's, is h_jY_j where h_j is non-zero and analytic on B. Now we show that the Wronskian of $\bar{y}_1, \ldots, \bar{y}_{l-1}$ does not vanish identically. One may construct functions R_j which are solutions of (7) such that the difference of the two branches of each R_j when it is extended around c is $g_j\bar{y}_j$ for some non-zero function g_j analytic on B. Choose our parameters to be in $(Q(i))^{\frac{l(l+1)}{2}+1}$ with $\gamma\neq 0$ and each g_j non-zero. One may choose S, a linear combination of the R_j's such that the differences of its derivatives (on the two different branches) at $z=z_0$ are zero up to the $(l-1)$-st derivative which is either 0 or 1. Now apply Theorem I of [3] to S and c and obtain the contradiction that 0 or 1 is irrational.

All that remains to be shown is that the Y_j are each analytic on $C^{\frac{l(l+1)}{2}+1}\times G$. If we choose $B_1\supset B$ satisfying the same assumptions as B then we will arrive at a new collection of $\bar{y}_k$'s each equal, where they are both defined, to a polynomial in the parameters times the old $\bar{y}_k$. It follows then that the new Y_j's are continuations of the old Y_j's. Thus we need only show above that each Y_j is analytic on $B\times G$. Clearly it is meromorphic on $B\times G$. Further $Y_j(z_0)$ has only one possible analytic continuation to all of B, its constant value. Then, since the closure of B is compact and we may repeat the above argument with $B'\supset\bar{B}$ instead of B, it follows that Y_j is analytic on $B\times N$ where $N\subset G$ is a neighborhood of z_0. It then follows, as before, that one may analytically extend this solution of (7) to all of $B\times G$. This proves Theorem III.

Proof of Theorem IV. Let $W_1, \ldots, W_{l-1}$ be as in the proof of Theorem III.

LEMMA. *We may define, for each $1\leqslant j\leqslant l-1$, a sequence of repeated integral operators $E_j, E_j^2, \ldots$ such that* (i) *given any equation of type* (1) *satisfied by $Y_j = Y_j(\gamma^{-1}, \gamma_{1,1}, \ldots, \gamma_{l-1,l-2}, z)$ we may integrate the equation repeatedly, using integration by parts to differentiate powers of z while integrating D^0Y_j into $D^{0-1}Y_j, \ldots\ Y_j, E_jY_j, E_j^2Y_j, \ldots$ to obtain a valid* *identity and* (ii) $|E_j^N Y_j| \leqslant K^{N+1}(N!)^{-1}$ *for some* $K = K(Y_j) > 0$ *independent of* N. *Further one can extend each* E_j *from* Y_j *to be defined on every* $w_k^\beta(z)\, Y_j^{(\gamma)}(w_k(z))$, *for* $0 \leqslant \beta, \gamma < +\infty$ *where* w_k *is any root of* $\prod_{k=1}^{n}(w - z_k) = z$, *with properties* (i) *and* (ii) *for* $w_k^\beta(z)\, Y_j^{(\gamma)}(w_k(z))$ *instead of* Y_j.

Proof. By what we have seen in the proof of Theorem III we may find such an $E_j^N Y_j$, if h_j does not vanish at this choice of the parameters, by setting it equal to

$$(h_j)^{-1} \int_c (z-t)^{N-1}\big((N-1)!\big)^{-1} W_j(t)\, dt$$

where the path of integration is from z to z along a curve on which the difference of the branches of $W(t)$ equals $h_j Y_j$. The inequality (ii) is easily seen to be satisfied. If one formally integrates an equation of type (1) in Y_j in the manner indicated in (i) one can only differ from having an identity in that, in general, one would have to add a non-homogeneous term which would be a polynomial in z. However, each

$$E_j^N Y_j \equiv (h_j)^{-1} \int_a^z (z-t)^{N-1}\big((N-1)!\big)^{-1} W(t)\, dt - \\ - (h_j)^{-1} \int_a^z (z-t)^{N-1}\big((N-1)!\big)^{-1} W(t)\, dt$$

where $a = c(\tfrac{1}{2})$ on the curve $c = c(t)$ $(0 \leqslant t \leqslant 1)$ and the respective paths of integration are $c(\tfrac{1}{2}+t/2)$ and $c(\tfrac{1}{2}-t/2)$. Each of these last two multiple integrals of Y_j must give rise to the same non-homogeneous term above; hence, the $E_j^N Y_j$ give rise to the non-homogeneous term zero. This defines $E_j^N Y_j$ satisfying (1) except at values where $h_j = 0$.

Suppose that at some value of our parameters we have $h_j = 0$. Then on some ray out from this point in $C^{\frac{l(l+1)}{2}+1}$ parameterized by $0 \leqslant s \leqslant 1$, $h_j(s) \not\equiv 0$. If $h_j(s) = s^a g_j(s)$ where $g_j(0) \neq 0$ then as $s \to 0$ our

$$E_j^N Y_j \to \big(g_j(0)\big)^{-1} \int_c (z-t)^{N-1}\big((N-1)!\big)^{-1} \left(\frac{\partial^a}{\partial s^a} W(0, t)\right) dt$$

uniformly. This latter expression satisfies (i) and (ii) above.

Substituting $z^\beta W_j$ for W_j above in the definition of $E_j^N Y_j$ we may define $E_j^N z^\beta Y_j$ so as to satisfy (i) and (ii). Next we wish to define $E_j^N\big(w_k(z)\big)^\beta Y_j^{(\gamma)}(w_k(z))$ for all non-negative integers β and γ by

$$(h_j)^{-1} \int_{c(z)} \big(z - p(u)\big)^{N-1}\big((N-1)!\big)^{-1} p'(u)\, u^\beta\, Y_j^{(\gamma)}(u)\, du$$

where $p(w) = \prod_{j=1}^{m}(w - z_j)$ and the path of integration is a simple closed curve beginning and ending at the point $w_k(z)$ which is such that the difference of the branches of each $W_j(z)$ around $c(z)$ is $h_j Y_j$. The same proof essentially as in the case of the $E_j^N Y_j$ shows that (i) and (ii) each hold. If $h_j = 0$ then on some ray $E^N\big(w_k(z)\big)^\beta Y^{(\gamma)}(w_k(z))$ approaches

$$\big(g_j(0)\big)^{-1} \int_{c(z)} \big(z - p(u)\big)^{N-1}\big((N-1)!\big)^{-1} p'(u)\, u^\beta \left(\frac{\partial^{\gamma+a}}{\partial s^a \partial u^\gamma} W_j(0, u)\right) du.$$

This proves the lemma.

There exists an equation of type (1), with a regular point at $z = 0$, which is satisfied by all of the $\big(w_k(z)\big)^\beta Y_j^{(\gamma)}(w_k(z))$, $1 \leqslant j \leqslant l$ and $0 \leqslant \beta$, $\gamma \leqslant M$ for some positive integer M, as may be seen using the proof of Theorem I. Integrate our equation of type (1) formally many times, using integration by parts to differentiate the powers of z and integrate the $D^k y$ into $D^{k-1}y, \ldots, y, Ey, \ldots$ Replacing our parameters γ^{-1} and $\gamma_{i,j}$ by values in $Q(i)$, setting $z = 0$, and replacing $E^s y$ by $F(s)$ we have a relation of the type appearing in the hypotheses of Lemma II of [5]. From what we have seen above candidates for $F(s)$ which satisfy the conditions of the hypotheses of Lemma III of [5] are certain s-fold integrals of the $\big(w_k(z)\big)^\beta Y_j^{(\gamma)}(w_k(z))$ for $1 \leqslant k \leqslant m$ and $1 \leqslant j \leqslant l-1$. (Since the differential equation was regular at $z = 0$ the condition that $F(s)$ not be zero from some point on is automatically satisfied if $\big(w_k(z)\big)^\beta Y_j^{(\gamma)}(w_k(z)) \not\equiv 0$.) Also there are integrals of the

$$\big(w_k(z)\big)^\beta Y_1^{(\gamma)}(w_k(z)) - \big(w_1(z)\big)^\beta Y_1^{(\gamma)}(w_1(z)),$$

for each $2 \leqslant k \leqslant m$, which satisfy the assumptions on $F(s)$. Therefore there exist such integrals for all linear combinations of the

$$\big(w_k(z)\big)^\beta y_j^{(\gamma)}(w_k(z))$$

for $1 \leqslant k \leqslant t$ and the

$$\big(w_k(z)\big)^\beta y_j^{(\gamma)}(w_k(z)) - \big(w_1(z)\big)^\beta y_j^{(\gamma)}(w_1(z))$$

for $t+1 \leqslant j \leqslant l$ and $j \neq 1$. Following the argument for Theorem I, if M is sufficiently large and $d = [F\colon Q(i)] < m(l-t)^{-1}$, we may construct a non-zero $F(s)$ as a linear combination of these last mentioned s-fold integrals which always has values in $Q(i)$. This contradicts Lemma II of [5]; therefore, we have proven Theorem IV.

Section II

Proof of Theorem V. Suppose $y \in R$. Then y is a p. f. p. s., i.e.

$$y = \sum_{j=1}^{k} \sum_{n=0}^{\infty} a_{n,j} z^{n+c_j}\big(\Gamma(n+c_j-1)\big)^{-1}$$

for some set of complex numbers c_j and $a_{n,j}$, and y satisfies an equation of type (3). Let $\hat{y}$ denote the f. p. s. $\sum_{j=1}^{k}\sum_{n=0}^{\infty} a_{n,j} z^{n+c_j}$. Then $\hat{y}$ satisfies an equation of the form

$$\sum_{j=0}^{l} h_j(zD) z^{-j}\hat{y} = 0. \tag{8}$$

Any linear homogeneous differential equation with coefficients in $Q[i, z]$ may be written in form (8). (Multiply through by a sufficiently high power of z so that one can write the equation as $\sum_{j=l_1}^{l} g_j(zD) z^j y$, for some collection of $g_j(zD)$ in $Q[i, zD]$. Then multiply through by z^{-l_1}, using $z^{-l_1}(zD) = (zD+l_1)z^{-l_1}$.) Thus $\hat{R}$ equals the set of all p. f. p. s. which satisfy a linear homogeneous differential equation with coefficients in $Q[i, z]$ (or $K[z]$ where $[K: Q(i)] < \infty$, since in this latter case one may apply a vector space argument to see that there exists an equation of type (8) with coefficients in $Q[i, z]$ which is satisified by $\hat{y}$). Notice that the correspondence $y \to \hat{y}$ is 1-1 and onto $\hat{R}$. If y_1 and y_2 belong to R then clearly $z\hat{y}_1\hat{y}_2$ belongs to $\hat{R}$, since $\hat{R}$ must be closed under both multiplication of functional values and $+$. Now under our correspondence $z\hat{y}_1\hat{y}_2$ could only have come from $y_1 * y_2$. Thus $y_1 * y_2 \in R$. Also since $\hat{y}_1 + \hat{y}_2 \in \hat{R}$ corresponds to $y_1 + y_2$ we see that $y_1 + y_2 \in R$. This proves Theorem V.

We also see from the above that we need only require that the coefficients $h_j(zD)$ in (3) are in $K[zD]$ where $[K: Q(i)] < \infty$ since given any solution y of such an equation we may still form $\hat{y}$ and $\hat{y}$ will satisfy an equation of type (8) with coefficients in $K[zD]$, so $\hat{y} \in \hat{R}$ and $y \in R$.

Proof of Theorem VI. Suppose that we are given two equations of type (3) with coefficients in $K[zD]$ where $[K: Q(i)] < \infty$. If we take two fundamental systems of solutions (f. s. solutions), one for each equation, and consider the set of all products of an element from one system times an element of the other system we obtain a collection of functions (f. s.) which span the space of all solutions (f. s. solutions) to a third equation of type (3) with coefficients in $K[zD]$. (This follows since the Wronskian matrix of the set of products of any two fundamental systems, f.s. or actual functions, will equal the product of a matrix with entries in $K[z]$ and the tensor product of the Wronskian matrices of the two fundamental systems. Thus the Wronskian matrix of the product has rank d if the matrix with entries in $K[z]$ has rank d, and then there is an equation of type (8) with coefficients in $K[zD]$ of order exactly d which has as its solution space (f. s. solution space) the space spanned by the collection of products above.)

If $y_1 \in \hat{R}$ and $y_2 \in \hat{M}$ then $zy_1y_2 \in \hat{R}$. We wish to see when $zy_1y_2 \in \hat{M}$. We see from looking at (3) that if w is a f. p. s. which satisfies an equation of type (8) where $h_0(zD)$ equals some non-zero constant then $w \in \hat{M}$. This last requirement, that $h_0(zD)$ be a constant, is implied by the statement that each f. s. solution of the equation of type (8) has a nonconstant exponential factor. Then with $\hat{y}_1$, and $\hat{y}_2$ as above if $t \in K$ there are at most a finite number of values of t for which $\hat{y}_1(tz)(z\hat{y}_2)$ is not in $\hat{M}$, i.e. those values for which the above products of f. s. solutions corresponding to our equations for $\hat{y}_1(tz)$ and $(z\hat{y}_2)$ contain at least one series with a constant exponential factor. This proves (i). If $\hat{y}_1 \in \hat{R}'$ then there exists some equation of type (8) which is satisfied by $\hat{y}_1$ and is such that each f. s. solution is a f. p. s. in z^{-1} (i.e. each exponential factor is a constant). Thus $\hat{y}_1(z\hat{y}_2)$ is always in $\hat{M}$. This proves (ii).

Proof of Theorem VII. Let us define an integral operator E on p. f. p. s. by

$$E\left(\sum_{n=0}^{\infty} a_n z^{n+a}\big(\Gamma(n+a+1)\big)^{-1}\right) = \sum_{n=0}^{\infty} a_n z^{n+1+a}\big(\Gamma(n+1+a+1)\big)^{-1}$$

and linearity. If y satisfies (3) then, for each positive integer s,

$$\sum_{j=0}^{l} h_j(zD-s)E^s y = 0.$$

If y satisfies an equation of type (1) with a regular point at z_1 then we may rewrite this latter equation as one of type (3) with $h_0(zD) \equiv -1$ which has a regular point at $z = z_1$. Therefore we may assume that

$$E^s y = \sum_{j=1}^{l} h_j(zD-s)E^{s-j} y \tag{9}$$

for all integers s and that (9) has a regular point at $z = z_1$. For some $s_0 \geqslant 0$,

$$E^s y = \int_0^z (z-t)^{s-s_0-1}\big((s-s_0-1)!\big)^{-1}\big(E^{s_0}y(t)\big)dt.$$

Thus

$$|E^s y| \leqslant K^{s+1}(s!)^{-1}$$

for some $K_1 > 0$ independent of s, and this estimate now holds where $z = z_1$ and the path of integration avoids the singularities of $y(z)$. Now if $z_1 \in Q(i)$ one may apply Lemma II of [5] with $F(s) = E^s y(z_1)$. (As remarked in the proof of Theorem IV since z_1 is not a singular point of (9) the condition that $E^s y(z_1)$ does not vanish from some point on is easily seen to be satisfied.) If $z_1 \in K$ where $[K\colon Q(i)] = n < \infty$ then we may apply the a. c. c. of the ring $Q[i, s]$ to see that we may replace (9) (with $z = z_1$) by another equation of the same kind. This proves Theorem VII.

Proof of Theorem VIII. What we shall show first is that it suffices to prove that $E^N y(z_1)$ is not zero for all sufficiently large integers (even if z_1 is a singular point of our equation of type (9) for y) and to apply the argument used above in the proof of Theorem VII in order to prove Theorem VIII. The functions y_1 and y_2 each satisfy an equation, of their respective types, which has only regular points in $X - \{0\}$. Thus so do $E^{s_1}y_1$ and $E^{s_2}y_2$ for each pair of non-negative integers s_1 and s_2. For large enough values of s_1 and s_2

$$(E^{s_1}y_1)*(E^{s_2}y_2) = \int_0^z \big(E^{s_1}y_1(z-t)\big)\big(E^{s_2}y_2(t)\big)\,dt$$

where the path of integration is, say, the ray from 0 to z. As in [4] we see that $E^{s_1}y_1 * E^{s_2}y_2$ is analytic at each point of $X - \{0\}$. Also near $z = 0$, and hence in general

$$(E^{s_1}y_1)*(E^{s_2}y_2) = E^{s_1+s_2}(y_1 * y_2) = E^{s_1+s_2}y.$$

Thus y and each $E^s y$ are defined on all of $X - \{0\}$. Then we have (9) for all z in $X - \{0\}$ and need only show that if $z_1 \in X - \{0\}$ it is impossible that for some positive integer s_0,

$$\int_0^{z_1} (z_1 - t)^k \big(E^{s_0}y(t)\big)\,dt = 0 \quad \text{for} \quad k = 0, 1, \ldots$$

If the latter integrals were all zero then we would have that

$$\int_0^{z_1} p(t)\big(E^{s_0}y(t)\big)\,dt = 0$$

for all complex polynomials $p(t)$. One may first uniformly approximate, on this ray, the real part of $E^{s_0}y(t)$ and then the imaginary part and then conclude that $E^{s_0}y(z) \equiv 0$. This contradiction proves Theorem VIII.

Addendum. Although the results in this series of papers have shown that the numbers in question can not be approximated too well by rationals (Gaussian rationals) it is possible — using Theorems III and IV along with their proofs — to show that in some cases the order of approximation possible is better than is possible for almost all elements of, in these cases, R^{l-1}. Suppose that $l > 2$ and consider the equation

$$(10) \qquad y = \sum_{j=2}^{l-2} p_j(z) D^j y + D^{l-2}(z^{l-1}+1) D^2 y$$

where each $p_j(z) \in Z[z]$ and has degree at most $j-2$. Let us apply Theorem III with $z_0 = 0$. Each X_t has absolute value one. Each $\bar{y}_t$ (see the proof of Theorem III) is a function defined in a neighborhood of $z = X_t$ by a convergent f. p. s. with coefficients *and exponents* depending on γ and the $\gamma_{j,k}$. Since here, at our value of the parameters γ and $\gamma_{j,k}$, each $\bar{y}_t$ has a zero of order one at $z = X_t$ we see that each $\frac{\partial^a}{\partial s^a} W(0, z)$, see the proof of Theorem IV, must be bounded on $|z| \leqslant 1$. Thus $|E_j^N Y_j(0)| \leqslant K(N!)^{-1}$, for each $1 \leqslant j \leqslant l-1$, where $K > 0$ is independent of N.

Upon applying E_j^N to (10) and setting $z = 0$ we obtain, using the notation of (7) now, that

$$(11) \qquad E_j^N Y_j(0) = (-1)^{l-1} N \ldots (N-l+2) E_j^{N-1} Y_j(0) + \\ + \sum_{t=2}^{l-2} \Big(\sum_{k \geqslant t}^{l-2} \gamma_{k,k-t}\Big)(-1)^{k-t}(N-t-1)\ldots(N-k) E_j^{N-t} Y_j(0) + E_j^{N-l} Y_j(0).$$

If we express $E_j^N Y_j(0)$ as a linear combination over the integers of $Y_j(0), \ldots, Y_j^{(l-1)}(0)$ we see that the term arising from choosing $t = 1$ in each case in (11) has absolute value less than

$$I_N = K_1 (N!)^{l-1} N^{\frac{-(l-1)(l-2)}{2}},$$

for some $K_1 > 0$ independent of N. There exists some $K_2 > 0$ such that the substitution in this above proceedure of some term corresponding to $k > 1$ at and only at the point where we have n substituted for N yields a term with absolute value less than $I_N K_2 n^{-2}$ for $n = 1, 2, \ldots$ Then the sum of the absolute values of all of the coefficients is less than

$$I_N \Big(1 + K_2 \frac{\pi^2}{6} + K_2^2 \frac{\pi^2}{6}\Big(\frac{\pi^2}{6} - 1\Big) + \ldots\Big) < K_3 I_N,$$

for some $K_3 > 0$ independent of N.

Recall that each $Y_j^{(a)}(0) = \delta_j^{a+1}$ for $1 \leqslant j,\ a+1 \leqslant l-2$. Thus for each positive integer N we have $l-1$ forms $q_N Y_j^{(l-1)}(0) - P_{N,j}$ where q_N and $P_{N,j}$ are integers, each

$$|q_N Y_j^{(l-1)}(0) - P_{N,j}| < K(N!)^{-1}$$

and

$$|q_N| < K_1 K_4 (N!)^{l-1} N^{\frac{-(l-1)(l-2)}{2}}$$

for some $K_4 > 0$ independent of N. Then for each $0 < \varepsilon < 1$ there exists a $c(\varepsilon) > 0$ such that for $N = 1, 2, \ldots$,

$$\max_{1 \leqslant j \leqslant l-1} \{|q_N Y_j^{(l-1)}(0) - P_{N,j}|\} < \big(c(\varepsilon)\,|q_N|^{(l-1)^{-1}} (\log |q_N|)^{\frac{l-2}{2} - \varepsilon}\big)^{-1}.$$

If $l > 4$ our $l-1$ numbers $Y_j^{(l-1)}(0)$ may be approximated better than almost all $(l-1)$-tuples in R^{l-1}. It is not difficult to show, by the methods of these papers, that one can never approximate much better in the above case, i.e. with a somewhat larger exponent on the $\log|q_N|$ the last inequality could only be satisfied finitely often for any choice of q_N and $P_{N,j}$.

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MATHEMATICS RESEARCH CENTER
NAVAL RESEARCH LABORATORY
Washington, D.C.

Received on 22. 4. 1972 (272)

ACTA ARITHMETICA
XXIII (1973)

On the difference of consecutive terms of sequences defined by divisibility properties, II

by

E. Szemerédi (Budapest)

In a paper of the same title P. Erdös proved the following theorem: Let $b_1 < b_2 < \ldots$ be an infinite sequence of integers satisfying

$$\sum \frac{1}{b_i} < \infty \qquad (b_i, b_j) = 1.$$

Denote by $a_1, a_2, \ldots$ the sequence of integers not divisible by any b. Then there is an absolute constant c, independent of our sequence $b_1 < b_2 < \ldots$ so that for all sufficiently large x the interval $(x, x+x^{1-c})$ contains a's.

P. Erdös conjectured that perhaps $a_{i+1} - a_i = o(a_i)^\varepsilon$ holds for every $\varepsilon > 0$. We are unable to prove this at present, but we are going to prove the following sharpening of the result of P. Erdös.

THEOREM. *Let $B = \{b_1 < b_2 < \ldots\}$ be an increasing sequence of positive integers such that*

$$\text{(i)} \qquad \sum_{i=1}^{\infty} \frac{1}{b_i} < \infty$$

and

$$\text{(ii)} \qquad (b_i, b_j) = 1 \quad \text{if} \quad i \neq j.$$

Then for every $\varepsilon > 0$, if x is large enough, the interval $(x, x+x^{1/2+\varepsilon})$ contains a number a which is divisible by no b_j.

Proof. We can assume $b_1 > 1$. Let us define ε_1 and a so that

$$\varepsilon_1 = \min\left\{\prod_{j=1}^{\infty}\left(1-\frac{1}{b_j}\right), \varepsilon^2\right\} \tag{1}$$

and

$$\sum_{j=a}^{\infty} \frac{1}{b_j} < \varepsilon_1^2 < \varepsilon/8. \tag{2}$$

We shall assume that x is greater than a suitable function of ε, ε_1 and a.