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Some results on the distribution of additive arithmetic functions, II

by

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Introduction. Let f be a real-valued additive arithmetic function. In this paper we characterize the spectrum of the distribution of $\{f(n)-f(n+1), \ldots, f(n+h-1)-f(n+h)\}$ whenever the above distribution exists, where h is a positive integer. We obtain a theorem of Erdös and A. Schinzel [3] as a corollary of one of our propositions. Under very general conditions we shall show that, for any $m \geqslant 1$, $\{f_1(F_1(m)), \ldots, f_h(F_h(m))\}$ belongs to the spectrum of the distribution of $\{f_1(F_1(n)), \ldots, f_h(F_h(n))\}$ if it exists, where $f_1, \ldots, f_h$ are real additive arithmetic functions and $F_1, \ldots, F_h$ are positive integer-valued polynomials. In the last section we give a sufficient condition for an additive arithmetic function to have a singular distribution. Finally we shall show, under fairly general conditions on F, that if the distributions of $f(n)$ and $f(F(m))$ exist (F is an integer-valued polynomial) and if the distribution of $f(n)$ is absolutely continuous, then the distribution of $f(F(m))$ is also absolutely continuous. At the end we shall give an example to show that this is the best possible result.

Notations and definitions. Define,

$\boldsymbol{P} = \{F\colon F$ is an integer-valued polynomial of degree $\nu_F \geqslant 1$ which is not divisible by the square of any irreducible polynomial and $F(m) > 0$ for $m = 1, 2, \ldots\}$.

Let $r(F, d)$ denote the number of incongruent solutions of the congruence relation $F(m) \equiv 0 \pmod{d}$.

$p, q, \ldots$ denote prime numbers.

$\sum_p$ denote the sum over prime numbers.

Put

$$f'(p) = \begin{cases} f(p) & \text{if } |f(p)| < 1, \\ 1 & \text{otherwise.} \end{cases}$$

Results.

PROPOSITION 1. *Suppose that the series*

$$\sum_p \frac{[f'(p)]^2}{p}$$

is convergent. For any positive integer h,

$$\{f(n)-f(n+1), \ldots, f(n+h-1)-f(n+h)\} \tag{1}$$

has a distribution and for any $n_0 \geqslant 1$, *the vector* $\{f(n_0)-f(n_0+1), \ldots, f(n_0+h-1)-f(n_0+h)\}$ *belongs to the spectrum of the distribution of* (1).

Moreover, if $N_0, N_1, \ldots, N_h$ *are positive integers such that for all* $i = 0, 1, \ldots, h$, $(N_i, (h+1)!) = 1$ *and* $(N_i, N_j) = 1$ $(0 \leqslant i < j \leqslant h)$, *then*

$$\{f(N_0)-f(2N_1), f(2N_1)-f(3N_2), \ldots, f(hN_{h-1})-f((h+1)N_h)\}$$

is in the spectrum of the distribution of (1).

COROLLARY (Erdös and A. Schinzel [3]). *Let* $f(n)$ *be an additive arithmetic function satisfying the following conditions:*

1. $\sum_p \frac{\{f'(p)\}^2}{p} < \infty$;

2. *There is a number* c_1 *such that, for any integer* $M > 0$, *the set of numbers* $f(N)$, *where* $(N, M) = 1$, *is dense in* (c_1, ∞).

Then for any given sequence of h *real numbers* $a_1, a_2, \ldots, a_h$ *and* $\varepsilon > 0$ *the set* $\{n \geqslant 1: |f(n+i)-f(n+i-1)-a_i| < \varepsilon, i = 1, \ldots, h\}$ *has positive natural density.*

PROPOSITION 2. *Suppose that* $F_1, \ldots, F_s$ *belong to* $\boldsymbol{P}$. *Suppose*

$$f_i(p^k) r(F_i, p^k) \to 0$$

as $p \to \infty$ *for* $k = 1, \ldots, \nu_{F_i}-1$ *whenever* $\nu_{F_i} \geqslant 2$. *If, moreover, the distribution of*

$$\{f_1(F_1(m)), \ldots, f_s(F_s(m))\} \tag{2}$$

exists, then one can find a K_0 *such that the spectrum* S *of the distribution of* (2) *is the closure of the set*

$$A = \Big\{\Big(\sum_{\substack{p^t \| F_1(m) \\ p \leqslant k}} f_1(p^t), \sum_{\substack{p^t \| F_2(m) \\ p \leqslant k}} f_2(p^t), \ldots, \sum_{\substack{p^t \| F_s(m) \\ p \leqslant k}} f_s(p^t)\Big): m \geqslant 1, k > k_0\Big\}.$$

Remark 1. Clearly $A \supset B = \{(f_1(F_1(m)), \ldots, f_s(F_s(m))): m \geqslant 1\}$.

Proofs.

Proof of Proposition 1. Let $H_{i-1}(n) = f(n+i-1)-f(n+i)$, $i = 1, \ldots, h$. We extend the functions H_i to the polyadic domain (see Novoselov, [8]) and show that for each i, $H_i \in \mathfrak{H}_0$ ([8]), proceeding as follows.

Let

$$\omega(p^k, x) = \begin{cases} 1 & \text{if } p^k \| x, \\ 0 & \text{otherwise.} \end{cases}$$

For any prime number p define

$$f_{ip}(x) = \sum_{k=1}^{\infty} f(p^k)\,\omega(p^k, x+i), \quad i = 0, 1, \ldots, h-1.$$

Since the random variables $\{f_{ip}(x): p \text{ is a prime}\}$ are mutually independent ([8]) and

$$\sum_p \frac{(f'(p))^2}{p} < \infty,$$

by Kolmogorov's three series theorem, it follows that

$$\sum_p \Big\{f_{ip}(x) - \frac{f'(p)}{p}\Big\}$$

converges almost everywhere for $i = 0, 1, \ldots, h$. Hence

$$\sum_p \{f_{ip}(x) - f_{(i+1)p}(x)\}$$

converges a.e. for $i = 0, 1, \ldots, h-1$. Moreover, it is easy to see that the random variables $[\{f_{ip}(x) - f_{(i+1)p}(x)\}: p \text{ is a prime}]$ are mutually independent random variables for each $i = 0, 1, \ldots, h-1$.

Let

$$g_i(x) = \begin{cases} \sum \{f_{ip}(x) - f_{(i+1)p}(x)\} & \text{if it converges,} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $g_i(x)$ is an extension of $H_i(n)$. By using the Turán–Kubilius inequality ([5]), it is easy to show that $H_i(n) \in \mathfrak{H}_0$ ([8]) and the distribution of $H_i(n)$ is $Q_i(c) = P\{x: g_i(x) < c\}$.

Note that for any h-tuple $(t_0, \ldots, t_{h-1})$ of real numbers the distribution of $\sum_{i=0}^{h-1} t_i H_i(n)$ is given by

$$\mathrm{P}\Big\{x: \sum_{i=0}^{h-1} t_i g_i(x) < c\Big\}.$$

Hence by the Cramer–Wold device ([4]) we find that the distribution of $\{H_0(n), H_1(n), \ldots, H_{n-1}(n)\}$ is given by

$$Q(c_0, \ldots, c_{h-1}) = \mathrm{P}\{x: g_i(x) < c_i, \ i = 0, \ldots, h-1\}.$$

Let $0<\delta<1$. Since

$$\{(f_{0p}(x)-f_{1p}(x),\,\ldots,\,f_{(h-1)p}(x)-f_{hp}(x))\colon\ p \text{ is a prime}\};$$

is a sequence of mutually independent random variables, by using Egoroff's theorem one can find a $H\subset\mathfrak{S}$ such that $P(H)>1-\delta$ and $\sum_{p<r}\{f_{ip}(x)-f_{(i+1)p}(x)\}$ converges uniformly on H for $i=0,1,\ldots,h-1$.

Now fix a positive integer n_0 and a real number $\varepsilon>0$. Let $N=n_0(n_0+1)\ldots(n_0+h)$. Let k be any integer greater than N^2 and such that for $x\in H$

$$\Big|\sum_{p>k}\{f_{ip}(x)-f_{(i+1)p}(x)\}\Big|<\varepsilon \quad\text{for}\quad i=0,\ldots,h-1.$$

Hence

$$\mathrm{P}\Big\{x\colon\Big|\sum_{p>k}[f_{ip}(x)-f_{(i+1)p}(x)]\Big|<\varepsilon \text{ for } i=0,\ldots,h-1\Big\}>1-\delta.$$

Now the density of

$$\{n\geqslant 1\colon\ |f(n+i-1)-f(n+i)-f(n_0+i)|<\varepsilon,\ i=1,2,\ldots,h\}$$

is greater than or equal to

$$\mathrm{P}\Big\{x\colon\Big|\sum_{p>k}[f_{(i-1)p}(x)-f_{ip}(x)]\Big|<\varepsilon \text{ and}$$
$$\sum_{p\leqslant k}[f_{(i-1)p}(x)-f_{ip}(x)]=f(n_0+i-1)-f(n_0+i) \text{ for } i=1,\ldots,h\Big\}\geqslant$$

$$(1-\delta)\mathrm{P}\Big\{x\colon\ \sum_{p\leqslant k}[f_{(i-1)p}(x)-f_{ip}(x)]=f(n_0+i-1)-f(n_0+i);\ i=1,\ldots,h\Big\}.$$

Put

$$P=\prod_{\substack{p\leqslant k\\ p\nmid N}}p,\qquad Q=N^2P.$$

$$\mathrm{P}\Big\{x\colon\ \sum_{p\leqslant k}[f_{ip}(x)-f_{(i+1)p}(x)]=f(n_0+i)-f(n_0+i+1);\ i=0,\ldots,h-1\Big\}$$
$$=\text{Density of }\Big\{n\geqslant 1\colon\ \sum_{p\leqslant k}[f_{ip}(n)-f_{(i+1)p}(n)]=f(n_0+i)-f(n_0+i+1);$$
$$i=0,1,\ldots,h-1\Big\}\geqslant\frac{1}{Q}>0.$$

In fact, since $(P,N)=1$, we can find an l such that

$$l\equiv n_0\ (\mathrm{mod}\ N^2)\quad\text{and}\quad l\equiv 1\ (\mathrm{mod}\ P).$$

It is easy to show that, for any integer t,

$$\frac{Qt+l+i}{n_0+i},\qquad i=0,1,\ldots,h.$$

is an integer not divisible by any prime $p\leqslant k$. Since $k>N^2$, we have

$$\Big(\frac{Qt+l+i}{n_0+i},\ n_0+i\Big)=1.$$

Hence for any t such that $Qt+l>0$, we get

$$\sum_{p\leqslant k}\{f_{ip}(Qt+l)-f_{(i+1)p}(Qt+l)\}=f(Qt+l+i)-f(Qt+l+i+1),$$
$$i=0,1,\ldots,h-1.$$

But the density of the positive integers of the form $Qs+l$ is equal to $1/Q$. This proves the first part of Proposition 1. The proof of the second part of Proposition 1 is similar to the above proof. So here we only note the following fact:

We put

$$N=N_0N_1\ldots N_h,\qquad P=\prod_{\substack{p\leqslant k\\ p\nmid N}}p\quad\text{and}\quad Q=(h+1)!\,N^2P.$$

Since $(N_i,(h+1)!)=1$ for $i=0,\ldots,h$ and $(N_i,N_j)=1$ $(0\leqslant i\leqslant j\leqslant h)$, it follows from the Chinese Remainder Theorem that there exists a number l satisfying the congruence relations

$$l\equiv 1\ (\mathrm{mod}\ (h+1)!\,P),$$
$$l\equiv -i+N_i\ (\mathrm{mod}\ N_i^2)\qquad(0\leqslant i\leqslant h).$$

It is easy to see that for every integer t the numbers

$$\{(Qt+l+i)/(i+1)N_i\},\qquad i=1,\ldots,h,$$

are integers which are not divisible by any prime $p\leqslant k$. Also the density of the integers $Qt+l$ is $1/Q>0$.

This completes the proof of Proposition 1.

Proof of the Corollary. Let ε be a positive number and let a sequence a_i $(i=1,\ldots,h)$ be given. By condition 2 we can find positive integers $N_0,N_1,\ldots,N_h$ such that

$$(N_i,(h+1)!)=1\ \ (i=0,\ldots,h),\qquad (N_i,N_j)=1\ \ (0\leqslant i<j\leqslant h),$$

$$f(N_0)>c_1+\max_{1\leqslant i\leqslant h}\Big\{f(i+1)-\sum_{j=1}^{i}a_j\Big\}$$

and

$$\Big|f(N_i)-\Big\{f(N_0)-f(i+1)+\sum_{j=1}^{i}a_j\Big\}\Big|<\varepsilon/4\qquad(1\leqslant i\leqslant h).$$

Hence

$$|f((i+1)N_i)-f(iN_{i-1})-a_i|<\varepsilon/2\qquad(1\leqslant i\leqslant h). \tag{3}$$

By Proposition 1, we have

(4) $$\{n \geq 1\colon |f(n)-f(n+1)-f(N_0)+f(2N_1)| < \varepsilon/2, \ldots,$$
$$|f(n+h-1)-f(n+h)-f(hN_{h-1})+f((h+1)N_h)| < \varepsilon/2\}$$

has positive density. Hence the corollary follows from (3) and (4).

Proof of Proposition 2. We need the following two lemmas.

LEMMA 1. *If $h(m)$ and $g(m)$ are integer-valued polynomials having no common factors, then there exists a k_1 such that $p > k_1$ implies that there is no m such that $h(m) \equiv 0 \pmod p$ and $g(m) \equiv 0 \pmod p$.*

LEMMA 2. *If $F \in \boldsymbol{P}$, then there exists a k such that $p > k$ implies*

$$r(F, p^l) = r(F, p) \quad \text{for all } l \geq 1.$$

Also there exists a constant c such that $r(F, p^l) \leq c$ for all p and l.

For proofs of these lemmas see [9].

Let $F_i(m) = \prod_{j=1}^{l_i} F_{ij}(m)$, where $\{F_{ij}(m)\colon j = 1, \ldots, l_i\}$ are irreducible and each $F_{ij} \in \boldsymbol{P}$. Such a factorization is possible and is unique.

Let $\{G_1, \ldots, G_h\} = \{F_{ij}\colon j = 1, \ldots, l_i, i = 1, \ldots, s\}$ such that G_i and G_j have no common factors if $i \neq j$. By Lemma 1 choose a k_1 such that $p > k_1$ implies that there is no m such that $G_i(m) \equiv 0 \pmod p$ and $G_j(m) \equiv 0 \pmod p$ $(1 \leq i < j \leq h)$. Let $G_i(x)$ be the continuous extension of $G_i(m)$ to Novoselov's space $\mathfrak{S}$.

It is easy to see that

$$\{(m_i \mid G_i(x);\ i = 1, \ldots, h), (p_1^{t_{1i}} \mid G_i(x), i = 1, \ldots, h), \ldots,$$
$$(p_r^{t_{ri}} \mid G_i(x),\ i = 1, \ldots, h)\}$$

are independent events if t_{ij} are non-negative integers, $r \geq 1$, $p_i > k_1$, $p_i \neq p_j$ if $i \neq j$ and m_i is not divisible by any prime $p > k_1$ $(i = 1, \ldots, h)$. Since either $F_{ij}(m) \equiv F_{rt}(m)$ or $F_{ij}(m)$ and $F_{rt}(m)$ are mutually prime, we infer that

$$\{(m_i \mid F_i(x), i = 1, \ldots, s), (p_1^{t_{1i}} \mid F_i(x), i = 1, \ldots, s), \ldots,$$
$$(p_l^{t_{li}} \mid F_i(x), i = 1, \ldots, s)\}$$

are independent events on Novoselov's space if $l \geq 1$, $t_{ij} \geq 0$, $p_i > k_1$, $p_i \neq p_j$ if $i \neq j$ and m_i is not divisible by any prime $p > k_1$ for any $i = 1, \ldots, s$.

Now choose $k_0 > k_1$ (by using Lemma 2) such that, if $p > k_0$, then

$$r(F_i, p^t) = r(F_i, p) \quad \text{for} \quad t \geq 1, i = 1, \ldots, s$$

and

$$r(F_i, p) < p/2s, \quad i = 1, \ldots, s.$$

We now show that $A \subset S$. Let

$$f_{i0}(x) = \sum_{\substack{p^k \| F_i(x) \\ p \leq k_0}} f_i(p^k), \quad i = 1, \ldots, s.$$

For $p > p_0$ and $i = 1, \ldots, s$, we put

$$f_{ip}(x) = \begin{cases} f_i(p^k) & \text{if } p^k \| F_i(x), k \geq 1, \\ 0 & \text{if either } p \nmid F_i(x) \text{ or } p^k \mid F_i(x) \text{ for all } k \geq 1. \end{cases}$$

By Theorem 2 of [1], we conclude that

$$\sum_p \frac{f_i'(p) r(F_i, p)}{p} \quad \text{and} \quad \sum_p \frac{(f_i'(p))^2 r(F_i, p)}{p}$$

converge. Hence by Kolmogorov's three series theorem $\sum_{p > k_0} f_{ip}(x)$ converges a.e.

Fix a positive real number $\delta < 1/4s$. By Egoroff's theorem choose $H \subset \mathfrak{S}$ such that $\mathrm{P}(H) > 1 - \delta$ and, on H, $\sum_{p > k_0} f_{ip}(x)$ converges uniformly for $i = 1, \ldots, s$.

Now fix $\varepsilon > 0$, $k > k_0$ and $m \geq 1$. Choose $k_2 > k$ such that

$$\mathrm{P}\Big\{x\colon \Big|\sum_{p > k_2} f_{ip}(x)\Big| < \varepsilon;\ i = 1, \ldots, s\Big\} > 1 - \eta \quad \text{where} \quad \eta = \delta s.$$

Let $\mathrm{D}\{\ldots\}$ denote the natural density of integers satisfying the conditions mentioned in $\{\ldots\}$.

$$\mathrm{D}\Big\{\Big|f_i(F_i(n)) - \sum_{k_0 < p \leq k} f_{ip}(m) - f_{i0}(m)\Big| < \varepsilon, i = 1, \ldots, s\Big\}$$
$$\geq \mathrm{P}\Big\{x\colon f_{i0}(x) = f_{i0}(m), \sum_{k_0 < p \leq k} f_{ip}(x) = \sum_{k_0 < p \leq k} f_{ip}(m)$$
$$\text{and } \Big|\sum_{p > k_2} f_{ip}(x)\Big| < \varepsilon,\ i = 1, \ldots, s\Big\}$$
$$\geq (1 - \eta)\mathrm{P}\{x\colon f_{i0}(x) = f_{i0}(m), i = 1, \ldots, s\} \times$$
$$\times \prod_{k_0 < p \leq k} \mathrm{P}\{x\colon f_{ip}(x) = f_{ip}(m), i = 1, \ldots, s\} \times \prod_{k < p \leq k_2} \mathrm{P}\{f_{ip}(x) = 0, i = 1, \ldots, s\}.$$

Clearly

$$\mathrm{P}\{f_{ip}(x) = 0, i = 1, \ldots, s\} = 1 - \mathrm{P}\{x\colon f_{ip}(x) \neq 0 \text{ for some } i\}$$
$$= 1 - \sum_{i=1}^{s} \frac{r(F_i, p)}{p} \geq \frac{1}{2} \quad \text{if} \quad p > k_0.$$

Suppose $k_0 < p \leq k$ and $p^{l_{ij}} \| F_{ij}(m)$ for some $l_{ij} \geq 1$ and for some (i, j).

In this case by the definition of k_0, we have clearly

$$\mathrm{P}\{x\colon f_{ip}(x) = f_{ip}(m), i = 1, \ldots, s\} \geqslant \mathrm{P}\{x\colon p^{l_{ij}} \parallel F_{ij}(x)\} = \frac{r(F_{ij}, p^{l_{ij}})}{p^{l_{ij}}} - \frac{r(F_{ij}, p^{l_{ij}+1})}{p^{l_{ij}+1}} > 0.$$

Let $\Phi_i(m) = \prod_{\substack{p^l \| F_i(m) \\ p \leqslant k_0}} p^l$. Note that

$$\mathrm{P}\{x\colon f_{i0}(x) = f_{i0}(m),\ i = 1, \ldots, s\}$$
$$\geqslant \mathrm{D}\{\Phi_i(m) \mid F_i(n) \text{ and } \Phi_i(m)p \nmid F_i(n) \text{ for any } p \leqslant k_0 \text{ and for } i = 1, \ldots, s\}$$
$$> 0 \quad \text{(since } n = m \text{ is a solution of the above relations).}$$

So $A \subset S$. Hence $B \subset A \subset S$. Clearly B is dense in S. This completes the proof of Proposition 2.

Absolute continuity of the distributions of $f(m)$ and $f(F(m))$.

Remark 2. Let f be the strongly additive arithmetic function defined by

$$f(p) = \begin{cases} 0 & \text{if} \quad p \leqslant e^e, \\ \dfrac{1}{(\log\log p)^{3/2}} & \text{if} \quad p > e^e. \end{cases}$$

Let $F(m)$ be any polynomial taking positive integral values for $m \geqslant 1$. From Theorem 1 of [1] we can conclude that $f(F(m))$ has a distribution. Since

$$\sum_{p \leqslant n} \frac{r(F, p)}{p} = r \log\log n + O(1)$$

(see [9]) where r is the number of distinct irreducible factors of F.

Following an argument similar to the argument given in [2] it is not difficult to conclude that the distribution of $f(F(m))$ is absolutely continuous.

Remark 3. Let f be any real-valued additive arithmetic function having a distribution. Suppose that there exist sequences of real numbers g_N, l_N, s_N and a constant b such that $g_N l_N \to 0$, $l_N \to \infty$.

$$\frac{1}{g_N^2}\left\{\sum_{p > s_N} \frac{\{f'(p)\}^2}{p} + \left(\sum_{p > s_N} \frac{f'(p)}{p}\right)^2\right\} \to 0 \quad \text{as} \quad N \to \infty$$

and there exist positive integers $m_1, \ldots, m_{l_N}$ composed of primes $p \leqslant s_N$ such that $\dfrac{1}{\log s_N} \sum_{i=1}^{l_N} \dfrac{1}{m_i} \geqslant b$ for all sufficiently large N. Then the distribution of f is singular.

This fact can be proved as follows. Without loss of generality we can assume that f is strongly additive and $|f(p)| < 1$. We write every positive integer $m = m'm''$, where m' is composed of primes $p \leqslant s_N$ and m'' of primes $p > s_N$. The density of integers $m = m'm''$ such that $m' = m_i$ for some $i = 1, \ldots, l_N$ is

$$(5) \qquad \sum_{i=1}^{l_N} \frac{1}{m_i} \prod_{p \leqslant s_N} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log s_N} \sum_{i=1}^{l_N} \frac{1}{m_i} \geqslant e^{-\gamma} b,$$

where γ is Euler's constant.

For $x \in \mathfrak{S}$ and any prime p, put

$$f_p(x) = \begin{cases} f(p) & \text{if} \quad p \mid x, \\ 0 & \text{otherwise}. \end{cases}$$

Since f has a distribution, $\sum_p f_p(x)$ converges almost everywhere ([8]) and

$$\mathrm{D}\{n\colon f(n) < c\} = \mathrm{P}\Big\{x\colon \sum_p f_p(x) < c\Big\}.$$

Clearly

$$(6) \qquad \mathrm{P}\Big\{x\colon \Big|\sum_{p > s_N} f_p(x)\Big| > g_N\Big\} \leqslant \frac{1}{g_N^2}\left\{\sum_{p > s_N} \frac{f^2(p)}{p} + \left(\sum_{p > s_N} \frac{f(p)}{p}\right)^2\right\} \to 0 \quad \text{as} \quad N \to \infty.$$

Consider the open intervals $(f(m_i) - g_N, f(m_i) + g_N)$, $i = 1, \ldots, l_N$. By (5) and (6)

$$\mathrm{P}\Big\{x\colon \sum_p f_p(x) \in \bigcup_{i=1}^{l_N} \big(f(m_i) - g_N, f(m_i) + g_N\big)\Big\}$$
$$\geqslant be^{-\gamma} - \frac{1}{g_N^2}\left(\sum_{p > s_N} \frac{f(p)^2}{p} + \left(\sum_{p > s_N} \frac{f(p)}{p}\right)^2\right) \geqslant \frac{be^{-\gamma}}{2}$$

for all sufficiently large N.

And the sum of the lengths of these l_N intervals is less than or equal to $2g_N l_N$. Hence it follows that the distribution of $f(m)$ cannot be absolutely continuous. Hence it is singular.

Proposition 3. *Let $F \in \mathbf{P}$. Let f be a real-valued additive arithmetic function such that*

$$f(p^k) r(F, p^k) \to 0 \quad as \quad p \to \infty \quad for \quad k = 1, \ldots, \nu_F - 1,$$

if $\nu_F \geqslant 2$. (This condition can be dropped if F is a product of linear polynomials.)

Let Q be a set of primes such that

(7) $$\sum_{p\in Q}\frac{1}{p}<\infty \text{ and } q\notin Q \text{ implies either } r(F,q)\neq 0$$
$$\text{or } r(F,q)=0 \text{ and } f(q)=0.$$

If $f(m)$ and $f(F(m))$ have distributions, then the distribution of $f(F(m))$ is absolutely continuous if the distribution of $f(m)$ is absolutely continuous.

Proof. By Lemma 2 there exists a constant c such that $r(F,p^k)<c$ for all p and k and

$$r(F,p^k)=r(F,p) \quad \text{for all } k \text{ if } p>c.$$

Without loss of generality we can assume that f is strongly additive.

LEMMA 3. *If $\{X_n\}$ is a sequence of independent discrete random variables and $\{Y_n\}$ is another sequence of independent discrete random variables such that $\sum_n \mathrm{P}\{X_n\neq Y_n\}<\infty$, then $\sum_n X_n$ converges almost everywhere and its distribution function is absolutely continuous iff $\sum Y_n$ converges almost everywhere and its distribution is absolutely continuous.*

The proof of this lemma is well known [10].

LEMMA 4. *Suppose that $0\leqslant s(p)<c$ and $\{a_p\}$ is a sequence of real numbers. Then one can find a sequence of independent random variables $\{Y_p\colon p>2c\}$ defined on a complete probability space $(\Omega,\mathfrak{A},\mathrm{P})$ such that*

$$\mathrm{P}\{Y_p=0\}=1-\frac{s(p)}{p},$$

$$\mathrm{P}\{Y_p=na_p\}=\left(\frac{s(p)}{p}\right)^n\left(1-\frac{s(p)}{p}\right),\quad n=1,2,\ldots$$

and another sequence of independent random variables $\{X_p\colon p>2c\}$ defined on the same probability space $(\Omega,\mathfrak{A},\mathrm{P})$ such that

$$\mathrm{P}\{X_p=0\}=1-\frac{s(p)}{p},\quad \mathrm{P}\{X_p=a_p\}=\frac{s(p)}{p},$$

and

$$\sum_{p>2c}\mathrm{P}\{X_p\neq Y_p\}<\infty.$$

The proof of this lemma is easy and so is omitted.

LEMMA 5. *Suppose that h is the characteristic function of an infinitely divisible distribution with the Levy function M. If the total variation of M is finite and M is discrete, then the distribution corresponding to h is discrete.*

(See [6], p. 124.)

Now we prove Proposition 3. Let $\{X'_p\colon p>2c\}$ be a sequence of independent random variables such that

$$\mathrm{P}\{X'_p=0\}=1-\frac{1}{p}$$

and

$$\mathrm{P}\{X'_p=f(p)\}=\frac{1}{p}.$$

By Lemma 3 and from the results of [1], if f has an absolutely continuous distribution, it follows that $\sum_{p>2c}X'_p$ converges almost everywhere and its distribution function is absolutely continuous.

By Lemmas 3 and 4 one can find a sequence $\{X_p\}$ of independent random variables such that

$$\mathrm{P}\{X_p=0\}=1-\frac{1}{p},$$

$$\mathrm{P}\{X_p=nf(p)\}=\frac{1}{p^n}\left(1-\frac{1}{p}\right),\quad n=1,2,\ldots$$

$\sum_{p>2c}X_p$ converges almost everywhere and its distribution is absolutely continuous. If $h(t)$ is the characteristic function of $\sum_{p>2c}X_p$, then clearly

$$\log h(t)=i\gamma' t+\sum_{p>2c}\sum_{k=1}^{\infty}\left(e^{itkf(p)}-1-\frac{itkf(p)}{1+k^2f^2(p)}\right)\frac{1}{kp^k}$$

for some γ'. Since

$$\sum_{p\in Q}\frac{1}{p}+\sum_{p>2c}\sum_{k=2}^{\infty}\frac{1}{kp^k}<\infty,$$

by Lemma 5 we infer that the distribution function corresponding to the characteristic function

$$\varphi(t)=\exp\left\{\sum_{\substack{p\notin Q\\ p>2c}}\left(e^{itf(p)}-1-\frac{itf(p)}{1+(f(p))^2}\right)\frac{1}{p}\right\}$$

is absolutely continuous. From now on we write $r(p)$ for $r(p,F)$.

Now suppose that $\{Y'_p\colon p>2c\}$ is a sequence of independent random variables such that

$$\mathrm{P}\{Y'_p=0\}=1-\frac{r(p)}{p}\quad\text{and}\quad \mathrm{P}\{Y'_p=f(p)\}=\frac{r(p)}{p}.$$

Since $f(F(m))$ has a distribution, $\sum_{p>2c} Y'_p$ converges almost everywhere [1] and the distribution function of $f(F(m))$ is absolutely continuous if the distribution function of $\sum_{p>2c} Y'_p$ is absolutely continuous. Again, by Lemmas 3, 4 and 5 as above, we conclude that the distribution function of $\sum_{p>2c} Y'_p$ is absolutely continuous if the distribution function corresponding to the characteristic function $g(t)$ given by

$$g(t) = \exp\Big\{\sum_{\substack{p>2c\\ p\notin Q}}\Big(e^{itf(p)}-1-\frac{itf(p)}{1+(f(p))^2}\Big)\frac{r(p)}{p}\Big\}$$

is absolutely continuous.

Since

$$\sum_{\substack{p>2c\\ p\notin Q}}\Big(e^{itf(p)}-1-\frac{itf(p)}{1+(f(p))^2}\Big)\frac{1}{p}$$

and

$$\sum_{\substack{p>2c\\ p\notin Q}}\Big(e^{itf(p)}-1-\frac{itf(p)}{1+(f(p))^2}\Big)\frac{r(p)}{p}$$

converge absolutely and uniformly in every compact interval of the real line,

$$\sum_{\substack{p>2c\\ p\notin Q}}\Big(e^{itf(p)}-1-\frac{itf(p)}{1+(f(p))^2}\Big)\frac{(r(p)-1)}{p}$$

converges absolutely and uniformly in every compact interval of the real line. Since $r(p)\geqslant 1$ or $f(p)=0$ if $p\notin Q$, it follows that

$$l(t) = \exp\Big\{\sum_{\substack{p>2c\\ p\notin Q}}\Big(e^{itf(p)}-1-\frac{itf(p)}{1+f^2(p)}\Big)\frac{(r(p)-1)}{p}\Big\}$$

is a characteristic function. We note that $g(t)=\varphi(t)\cdot l(t)$.

Since $\varphi(t)$ is a characteristic function of an absolutely continuous distribution, $g(t)$ is also a characteristic function of an absolutely continuous distribution. This completes the proof of Proposition 3.

(7) holds for many polynomials. In fact, if F has a linear factor, then condition (7) obviously holds. (7) is not a necessary condition, as is evident from Remark 2. But Proposition 3 is the best possible in the sense that if condition (7) is omitted then the conclusion of the proposition is not necessarily true.

EXAMPLE. Let f be the strongly additive arithmetic function defined by

$$f(p) = \begin{cases}\dfrac{1}{(\log\log p)^{3/2}} & \text{if } p>e^e \text{ and } p\equiv 3 \pmod 4,\\ 0 & \text{otherwise.}\end{cases}$$

Let $F(m)$ be the polynomial m^2+1.

The following lemma shows that $f(F(m))=0$ for all m and hence $f(F(m))$ has a degenerate distribution.

LEMMA 6. *If p is a prime $\equiv 1 \pmod 4$, the congruence*

$$x^2 \equiv -1 \pmod p \tag{8}$$

has exactly two incongruent solutions. The congruence (8) *has no solution when p is a prime $\equiv 3 \pmod 4$.*

See [7], p. 99, Theorem 58.

Now we shall show that the distribution of $f(m)$ exists and is absolutely continuous.

We need the following

LEMMA 7. *If $F\in \boldsymbol{P}$ and the number of distinct factors of F is k, then*

$$\sum_{p\leqslant x}\frac{r(F,p)}{p} = k\log\log x+O(1).$$

See [9].

The characteristic function of the distribution function of $f(m)$ is given by

$$L(u) = \prod_p\Big(1-\frac{1-e^{iuf(p)}}{p}\Big).$$

Now as in [2] for $u\neq 0$

$$|L(u)| \leqslant \prod{}'\Big|1-\frac{1-\exp\big(iu(\log\log p)^{-3/2}\big)}{p}\Big| \tag{9}$$

where the product $\prod'$ for each fixed $u\neq 0$, is taken over those primes which satisfy the following conditions:

$$p>e^e,\quad p\equiv 3 \pmod 4\quad \text{and}\quad 3\pi<4|u(\log\log p)^{-3/2}|<5\pi. \tag{10}$$

Now each factor of the product on the right of (9) is less than $1-\dfrac{1}{p}$; so that

$$|L(u)| \leqslant \prod{}'\Big(1-\frac{1}{p}\Big).$$

Hence

$$|L(u)| = O\left(\exp\left(-\sum{}' 1/p\right)\right),$$

where, for each fixed $u \neq 0$, $\sum'$ denotes the sum over those primes which satisfy (10). By Lemmas 6 and 7 we get

$$\sum_{\substack{p\equiv 3 (\mathrm{mod}\, 4)\\ p\leqslant x}} 2/p = \log\log x + O(1).$$

Hence

$$|L(u)| = O\left(\{\exp(-c|u|^{2/3})\}\right),$$

where

$$c = \frac{1}{2}\left(\frac{4}{\pi}\right)^{2/3}\left(\frac{1}{3^{2/3}} - \frac{1}{5^{2/3}}\right) > 0.$$

So $L(u)$ is integrable and hence $L(u)$ is the characteristic function of an absolutely continuous distribution function.

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A simplification of the formula for $L(1, \chi)$ where χ is a totally imaginary Dirichlet character of a real quadratic field

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1. Introduction. Let $k = Q(\sqrt{\Delta})$ be a real quadratic field, and let χ be a totally imaginary Dirichlet character of k. The Dirichlet L-series $L(s, \chi)$ evaluated at $s = 1$ can be written in the following form:

$$L(1, \chi) = -\pi^2 W(\chi)^{-1} N(D_k \mathfrak{b}_\chi)^{-1/2}\Big(\sum_{A} \overline{\chi(A)}G(A)\Big)$$

where $\mathfrak{b}_\chi$ is the conductor of χ, the summation is over integral ideal representatives of the ray class group $\mathrm{Id}(\mathfrak{b}_\chi)/R(\mathfrak{b}_\chi)$, the bar denotes complex conjugation, $W(\chi)$ is a constant of absolute value 1 (see [1], page 300), $G(A)$ is a rational number with denominator at most $12\mathfrak{b}$ where $\mathfrak{b}$ is the smallest rational integer divisible by $\mathfrak{b}_\chi$, and D_k is the different of k. The rational number $G(A)$ does not depend on the class of A modulo $R(\mathfrak{b}_\chi)$. For details see [4], p. 171.

Of interest here is the rational number $G(A)$ for any given integral ideal A. We begin by defining $G(A)$ explicitly.

For rational numbers u, v with $u, v \in [0, 1)$ and $(u, v) \neq (0, 0)$ we introduce the following modified theta function:

$$\theta\left(z \,\middle|\, \begin{matrix} u \\ v \end{matrix}\right) = q^{\frac{1}{2}\left(\frac{1}{6} - v + v^2\right)} \prod_{m=0}^{\infty}(1 - q^m t)\prod_{m=1}^{\infty}(1 - q^m t^{-1})$$

where $q = e^{2\pi i z}$ and $t = e^{2\pi i (vz - u)}$.

If u' and v' are any rational numbers, we denote by $\overline{\begin{pmatrix} u' \\ v' \end{pmatrix}}$ the *normalized* pair $\begin{pmatrix} u \\ v \end{pmatrix}$ with $0 \leqslant u, v < 1$ and $u' \equiv u$, $v' \equiv v \pmod 1$.

Let $\mathrm{SL}(2, Z)$ denote the special linear group of two by two matrices with entries in Z and determinant $+1$. For any matrix M in $\mathrm{SL}(2, Z)$

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