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Gauss sums and the number of solutions to the matrix equation $XAX^T = 0$ over $\mathrm{GF}(2^\nu)$

by

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1. Introduction. Let $\mathrm{GF}(q)$ denote a finite field of order $q = p^\nu$, p a prime. Let A and B be symmetric matrices of order n, rank m and order s, rank k, respectively, over $\mathrm{GF}(q)$. Carlitz [2] has determined the number $N_s(A, B)$ of solutions X over $\mathrm{GF}(q)$, for p an odd prime, to the matrix equation

$$XAX^T = B \tag{1.1}$$

of arbitrary rank when $n = m$. Furthermore, Hodges [4] has determined the number $N(A, B, r)$ of $s\times n$ matrices X of rank r over $\mathrm{GF}(q)$, p an odd prime, which satisfy (1.1). Perkins [6], [7] has determined the number $N_s(I_n, 0)$ of solutions X over $\mathrm{GF}(q)$, $q = 2^\nu$, to the matrix equation $XX^T = 0$ and has enumerated the $s\times n$ matrices X of given rank r over $\mathrm{GF}(q)$, $q = 2^\nu$, such that $XX^T = 0$.

The purpose of this paper is to determine the number $N_s(A, 0)$ of solutions X over $\mathrm{GF}(q)$, $q = 2^\nu$, to the matrix equation $XAX^T = 0$. In determining this number, Gauss sums, as developed in Section 2, are used. Also needed are Albert's canonical forms for symmetric matrices over fields of characteristic two ([1]).

Throughout the remainder of this paper, $\mathrm{GF}(q)$ will denote a finite field of order $q = 2^\nu$ and V_n will denote an n-dimensional vector space over $\mathrm{GF}(q)$.

2. Gauss sums and alternating bilinear forms. For α in $\mathrm{GF}(q)$, let t be the mapping from $\mathrm{GF}(q)$ into $\mathrm{GF}(q)$ defined by $t(\alpha) = \alpha+\alpha^2+\ldots \ldots+\alpha^{2^{\nu-1}}$. Then t maps onto the prime subfield of $\mathrm{GF}(q)$. Hence, for each α in $\mathrm{GF}(q)$, $t(\alpha) = m\cdot 1$ where $m = 0$ or 1. Let e be the map from $\mathrm{GF}(q)$ onto the multiplicative subgroup $\{-1, 1\}$ of the reals defined by

$$e(\alpha) = (-1)^m \quad \text{where } t(\alpha) = m\cdot 1. \tag{2.1}$$

Clearly, $t(\alpha+\beta) = t(\alpha)+t(\beta)$ for all α, β in $\mathrm{GF}(q)$. It follows that $e(\alpha+\beta) = e(\alpha)\cdot e(\beta)$ for all α, β in $\mathrm{GF}(q)$ and that

$$\sum_{\beta} e(\alpha\beta) = \begin{cases} q & (\alpha = 0), \\ 0 & (\alpha \neq 0), \end{cases} \tag{2.2}$$

where the summation in (2.2) extends over all β in $\mathrm{GF}(q)$. From (2.2), it follows that

$$\sum_{\alpha,\beta} e(\alpha\beta) = q \tag{2.3}$$

where the summation in (2.3) extends over all α, β in $\mathrm{GF}(q)$.

Perkins [7] has shown that

$$\sum_{B} e\big(\sigma(DB)\big) = \begin{cases} q^{\frac{s(s+1)}{2}} & (D = 0), \\ 0 & (D \neq 0), \end{cases} \tag{2.4}$$

where D is an $s\times s$ symmetric matrix, where the sum extends over all $s\times s$ upper triangular matrices B, and where $\sigma(DB)$ denotes the trace of the matrix DB.

Let f be a symmetric bilinear form on $V_n\times V_n$. Let $V_n^* = \{y \in V_n \mid f(x, y) = 0$ for all x in $V_n\}$. We say that f is *nondegenerate* if $V_n^* = \{0\}$. Clearly, V_n^* is a subspace of V_n. The *rank of* f is defined to be n-dim V_n^*. f is said to be an *alternating bilinear form* if $f(x, x) = 0$ for all x in V_n. An *alternate matrix* over $\mathrm{GF}(q)$ is a symmetric matrix with 0 diagonal. Chevalley [3] has shown that for each nondegenerate alternating bilinear form f on $V_n\times V_n$, there exists a basis for V_n such that, relative to that basis, $f(\xi, \eta) = \xi D\eta^T$ for all ξ, η in V_n, where

$$D = \begin{bmatrix} 0 & I_r \\ I_r & 0 \end{bmatrix},$$

an alternate matrix of rank $2r$. Chevalley [3] has also shown that if f is a bilinear form of rank t on $V_n\times V_n$ and if $f(\xi, \eta) = \xi A\eta^T$ for all ξ, η in V_n, then the matrix rank of A is t. It follows that if f is a degenerate alternating bilinear form of rank p on $V_n\times V_n$, then there exists a basis such that, relative to that basis, $f(\xi, \eta) = \xi D\eta^T$ for all ξ, η in V_n where

$$D = \begin{bmatrix} 0 & I_r & \\ I_r & 0 & \\ & & 0 \end{bmatrix}$$

and, hence, $p = 2r$.

Albert [1] has proved the following theorems:

THEOREM 2.1. *Every matrix congruent to an alternate matrix is an alternate matrix.*

THEOREM 2.2. *Let D be an $s\times s$ nonsingular alternate matrix over $\mathrm{GF}(q)$. Then there is a nonsingular matrix P such that*

$$PAP^T = \begin{bmatrix} 0 & I_r \\ I_r & 0 \end{bmatrix}.$$

THEOREM 2.3. *Let D be an $s\times s$ alternate matrix of rank p over $\mathrm{GF}(q)$. Then there is a nonsingular matrix P such that*

$$PAP^T = \begin{bmatrix} 0 & I_r & \\ I_r & 0 & \\ & & 0 \end{bmatrix} \quad (p = 2r).$$

THEOREM 2.4. *If A is an $s\times s$ symmetric, nonalternate matrix of rank r over $\mathrm{GF}(q)$, then there is a nonsingular matrix P such that*

$$PAP^T = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Let D be an $s\times s$ matrix over $\mathrm{GF}(q)$ and let g_D be the bilinear form defined by $g_D(\xi, \eta) = \xi D\eta^T$. Define

$$T(g_D) = \sum_{\xi,\eta} e[g_D(\xi, \eta)], \tag{2.5}$$

where the summation extends over all ξ, η in V_s.

THEOREM 2.5. *Let D be an $s\times s$ alternate matrix over $\mathrm{GF}(q)$. If $M = PDP^T$ for a nonsingular matrix P, then $T(g_M) = T(g_D)$. Furthermore, if D is of rank $2r$, then $T(g_D) = q^{2(s-r)}$.*

Proof. We have

$$\begin{aligned} T(g_M) &= \sum_{\xi,\eta} e[g_M(\xi, \eta)] \\ &= \sum_{\xi,\eta} e[\xi M\eta^T] = \sum_{\xi,\eta} e[(\xi P)D(\eta P)^T] \\ &= \sum_{\gamma,\delta} e[\gamma D\delta^T] = \sum_{\gamma,\delta} e[g_D(\gamma, \delta)] = T(g_D), \end{aligned}$$

since P is nonsingular.

By Theorem 2.3, if D is of rank $2r$, there is a nonsingular matrix P such that

$$PDP^T = \begin{bmatrix} 0 & I_r & \\ I_r & 0 & \\ & & 0 \end{bmatrix}.$$

Thus, $T(g_D) = T(g_R)$, where

$$R = \begin{bmatrix} 0 & I_r & \\ I_r & 0 & \\ & & 0 \end{bmatrix}.$$

But

$$g_R(\xi, \eta) = \xi R \eta^T = \sum_{i=1}^{r} \xi_i \eta_{r+i} + \sum_{i=1}^{r} \xi_{r+i}\eta_i.$$

Hence,

$$T(g_D) = T(g_R) = \sum_{\xi,\eta} e[g_R(\xi,\eta)]$$
$$= \sum_{\substack{\xi_1,\ldots,\xi_s \in \mathrm{GF}(q)\\ \eta_1,\ldots,\eta_s \in \mathrm{GF}(q)}} e\Big[\sum_{i=1}^{r} \xi_i\eta_{r+i} + \sum_{i=1}^{r}\xi_{r+i}\eta_i\Big]$$
$$= \sum_{\substack{\xi_1,\ldots,\xi_s \in \mathrm{GF}(q)\\ \eta_1,\ldots,\eta_s \in \mathrm{GF}(q)}} \Big[\prod_{i=1}^{r} e(\xi_i\eta_{r+i})\Big]\cdot\Big[\prod_{i=1}^{r} e(\xi_{r+i}\eta_i)\Big]$$
$$= \sum_{\substack{\xi_{2r+1},\ldots,\xi_s\\ \eta_{2r+1},\ldots,\eta_s}} \Big[\prod_{i=1}^{r} \sum_{\xi_i,\eta_{r+i}} e(\xi_i\eta_{r+i})\Big]\cdot\Big[\prod_{i=1}^{r} \sum_{\xi_{r+i},\eta_i} e(\xi_{r+i}\eta_i)\Big]$$
$$= \sum_{\substack{\xi_{2r+1},\ldots,\xi_s\\ \eta_{2r+1},\ldots,\eta_s}} \Big[\prod_{i=1}^{r} q\Big]\cdot\Big[\prod_{i=1}^{r} q\Big] \quad \text{by (2.3).}$$

Thus, $T(g_D) = q^{(s-2r)}\cdot q^{(s-2r)}\cdot q^{2r} = q^{2(s-r)}$. Define

(2.6) $\mathscr{B}_0 = \{B \mid B$ is an $s\times s$ upper triangular matrix with 0 diagonal$\}$

and

(2.7) $\mathscr{A} = \{D \mid D$ is an $s\times s$ alternate matrix$\}$.

Let $M(s, 2r)$ denote the number of $s\times s$ upper triangular matrices B such that rank $(B+B^T) = 2r$. Let $K(s, 2r)$ denote the number of B in $\mathscr{B}_0$ such that rank $(B+B^T) = 2r$. Let $L_0(s, t)$ denote the number of D in $\mathscr{A}$ of rank t.

MacWilliams [5] has found that

$$(2.8)\qquad L_0(s,t) = \begin{cases} 0 & \text{(if } t \text{ is odd)},\\ \displaystyle\prod_{i=1}^{r} \frac{q^{2i-2}}{q^{2i}-1} \prod_{i=0}^{2r-1} (q^{s-i}-1) & \text{(if } t = 2r). \end{cases}$$

THEOREM 2.6. *The mapping τ from $\mathscr{B}_0$ into $\mathscr{A}$ defined by $\tau(B) = B+B^T$ is a one-to-one mapping onto $\mathscr{A}$. For each $r = 0, 1, \ldots, [s/2]$, where $[s/2]$ denotes the largest integer not exceeding $s/2$, define $\mathscr{B}_0(r) = \{B\in\mathscr{B}_0 \mid$ rank $(B+B^T) = 2r\}$ and define $\mathscr{A}(r) = \{D\in\mathscr{A} \mid$ rank $D = 2r\}$. Then τ_r, the restriction of τ to $\mathscr{B}_0(r)$, is a one-to-one mapping onto $\mathscr{A}(r)$ for each $r = 0, 1, \ldots, [s/2]$.*

Proof. Clearly, τ has its range in $\mathscr{A}$ and is onto. If B_1 and B_2 are in $\mathscr{B}_0$ and if $\tau(B_1) = \tau(B_2)$, then $B_1+B_1^T = B_2+B_2^T$. Thus $B_1+B_2 = B_1^T+B_2^T$, from which it follows that B_1+B_2 is upper triangular and lower triangular. Since B_1+B_2 has 0 diagonal, $B_1+B_2 = 0$. Thus $B_1 = B_2$.

For any $r = 0, 1, \ldots, [s/2]$, it is clear that τ_r is one-to-one. Choose any D in $\mathscr{A}(r)$. Since τ is onto, there is a B in $\mathscr{B}_0$ such that $\tau(B) = B+B^T = D$. Since D is in $\mathscr{A}(r)$, rank $(B+B^T) =$ rank $D = 2r$. Thus, $B\in\mathscr{B}_0(r)$, and it follows that τ_r is onto $\mathscr{A}(r)$.

Since $K(s, 2r)$ is the number of elements in $\mathscr{B}_0(r)$ and $L_0(s, 2r)$ is the number of elements in $\mathscr{A}(r)$, Theorem 2.6 yields

$$(2.9)\qquad K(s, 2r) = L_0(s, 2r) \quad \text{for each } r = 0, 1, \ldots, [s/2].$$

LEMMA 2.1. *$M(s, 2r) = q^s L_0(s, 2r)$, for each $r = 0, 1, \ldots, [s/2]$.*

Proof. If B is any matrix from $\mathscr{B}_0(r)$ and if $C = B+D$, where D is any $s\times s$ diagonal matrix, then $B+B^T = C+C^T$ from which it follows that rank $(C+C^T) = 2r$. Thus, $M(s, 2r) = q^s K(s, 2r) = q^s L_0(s, 2r)$ by (2.9).

The following lemma will be needed in Sections 3 and 4.

LEMMA 2.2. *Let A be any $n\times n$ symmetric matrix. If there is a nonsingular matrix P such that $PAP^T = C$, then $N_s(A, 0) = N_s(C, 0)$.*

Proof. Clearly $XCX^T = 0$ if and only if $YAY^T = 0$ where $Y = XP$. Since P is nonsingular, the result follows.

3. Determination of $N_s(A, 0)$, A a nonalternate symmetric matrix.

Perkins [7] has found the number $N_s(I_n, 0)$ of $s\times n$ matrices X over $\mathrm{GF}(q)$ such that $XX^T = 0$.

Let A be any $n\times n$ nonalternate symmetric matrix of rank ϱ. By Theorem 2.4, there is a nonsingular matrix P such that

$$PAP^T = \begin{bmatrix} I_\varrho & 0\\ 0 & 0\end{bmatrix}.$$

By Lemma 2.2, $N_s(A, 0) = N_s(C, 0)$, where

$$C = \begin{bmatrix} I_\varrho & 0\\ 0 & 0\end{bmatrix}.$$

Consider the equation

$$(3.1)\qquad XCX^T = 0.$$

Let $X = [X_1, X_2]$, where X_1 is $s\times\varrho$ and X_2 is $s\times(n-\varrho)$. Then, (3.1) becomes

$$0 = XCX^T = [X_1, X_2]\begin{bmatrix} I_\varrho & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix} = X_1X_1^T.$$

Thus, if $X_1X_1^T = 0$ and X_2 is any $s\times(n-\varrho)$ matrix, then $X = [X_1, X_2]$ satisfies (3.1). The number of ways to choose X_2 is $q^{s(n-\varrho)}$.

This proves the following theorem.

THEOREM 3.1. *Let A be an $n\times n$ nonalternate symmetric matrix of rank ϱ. Then the number of $s\times n$ matrices X over* GF(q) *such that $XAX^T = 0$ is*

$$N_s(A, 0) = q^{s(n-\varrho)}N_s(I_\varrho, 0).$$

4. Determination of $N_s(A, 0)$, A alternate. Let A be an $n\times n$ alternate matrix of rank t. By Theorem 2.3, there is a nonsingular matrix P such that

$$PAP^T = \begin{bmatrix}0 & I_\varrho & \\ I_\varrho & 0 & \\ & & 0\end{bmatrix},\quad t = 2\varrho.$$

By Lemma 2.2, $N_s(A, 0) = N_s(R, 0)$ where

$$R = \begin{bmatrix}0 & I_\varrho & \\ I_\varrho & 0 & \\ & & 0\end{bmatrix}.$$

Thus, it suffices to find $N_s(R, 0)$. Since R is symmetric, XRX^T is symmetric. Hence, by (2.4), $\sum_B e\big(\sigma(XRX^TB)\big) = q^{\frac{s(s+1)}{2}}$ if and only if $XRX^T = 0$, and $\sum_B e\big(\sigma(XRX^TB)\big) = 0$ otherwise, where the summation extends over all $s\times s$ upper triangular matrices B. Thus

(4.1) $$\sum_B\sum_X e\big(\sigma(XRX^TB)\big) = \sum_X\sum_B e\big(\sigma(XRX^TB)\big) = N_s(R, 0)\,q^{\frac{s(s+1)}{2}}.$$

A simple calculation shows that

(4.2) $$\sigma(XRX^TB) = \sum_{k=1}^{\varrho}\Big[\sum_{j=1}^{s}\sum_{i=1}^{s}x_{i,\varrho+k}b_{ji}x_{jk}\Big] + \sum_{k=\varrho+1}^{2\varrho}\Big[\sum_{j=1}^{s}\sum_{i=1}^{s}x_{i,k-\varrho}b_{ji}x_{jk}\Big].$$

Let g_B be the bilinear form defined by $g_B(x, y) = xBy^T$ for all x, y in V_s. Then, $g_B(x, y) = \sum_{i=1}^{s}\sum_{j=1}^{s}x_ib_{ij}y_j$. Thus, (4.2) becomes

(4.3) $$\sigma(XRX^TB) = \sum_{k=1}^{\varrho}g_B(x_k, x_{\varrho+k}) + \sum_{k=\varrho+1}^{2\varrho}g_B(x_k, x_{k-\varrho}).$$

Hence,

$$\sum_B\sum_X e\big(\sigma(XRX^TB)\big) = \sum_B\sum_X e\Big[\sum_{k=1}^{\varrho}g_B(x_k, x_{\varrho+k}) + \sum_{k=\varrho+1}^{2\varrho}g_B(x_k, x_{k-\varrho})\Big]$$
$$= \sum_B\sum_X\Big[\prod_{k=1}^{\varrho}e\big(g_B(x_k, x_{\varrho+k})\big)\Big]\Big[\prod_{k=\varrho+1}^{2\varrho}e\big(g_B(x_k, x_{k-\varrho})\big)\Big]$$
$$= \sum_B\sum_X\prod_{k=1}^{\varrho}e\big(g_B(x_k, x_{\varrho+k})\big)\cdot e\big(g_B(x_{\varrho+k}, x_k)\big).$$

Thus, (4.1) becomes

(4.4) $$N_s(R, 0)\,q^{\frac{s(s+1)}{2}} = \sum_B\sum_X\prod_{k=1}^{\varrho}e\big(g_B(x_k, x_{\varrho+k})\big)\cdot e\big(g_B(x_{\varrho+k}, x_k)\big).$$

Let $X = [x_1^T, \dots, x_n^T]$, where $x_k = (x_{1k}, \dots, x_{sk})$, $1\leqslant k\leqslant n$. Furthermore, let $\sum_{x_k^T}$ indicate a sum extending over all vectors x_k in V_s. Then (4.4) becomes

(4.5) $$N_s(R, 0)\,q^{\frac{s(s+1)}{2}}$$
$$= \sum_B\sum_{x_1^T}\cdots\sum_{x_n^T}\Big[\prod_{k=1}^{\varrho}e\big(g_B(x_k, x_{\varrho+k})\big)\cdot e\big(g_B(x_{\varrho+k}, x_k)\big)\Big]$$
$$= \sum_B\sum_{x_{2\varrho+1}^T}\cdots\sum_{x_n^T}\Big[\prod_{k=1}^{\varrho}\sum_{x_k^T}\sum_{x_{\varrho+k}^T}e\big(g_B(x_k, x_{\varrho+k})\big)e\big(g_B(x_{\varrho+k}, x_k)\big)\Big].$$

Next, consider

$$\sum_\xi\sum_\eta e\big(g_B(\xi, \eta)\big)e\big(g_B(\eta, \xi)\big)$$
$$= \sum_{\xi,\eta}e[\xi B\eta^T + \eta B\xi^T] = \sum_{\xi,\eta}e[\xi B\eta^T + \xi B^T\eta^T]$$
$$= \sum_{\xi,\eta}e[\xi(B+B^T)\eta^T] = \sum_{\xi,\eta}e[g_{B+B^T}(\xi, \eta)]$$
$$= T(g_{B+B^T}),$$ where $T(g_D)$ is as defined in (2.5).

Thus, (4.5) becomes

(4.6) $$N_s(R, 0)\,q^{\frac{s(s+1)}{2}} = \sum_B\sum_{x_{2\varrho+1}^T}\cdots\sum_{x_n^T}\Big[\prod_{k=1}^{\varrho}T(g_{B+B^T})\Big]$$
$$= q^{s(n-2\varrho)}\sum_B[T(g_{B+B^T})]^{\varrho}.$$

Since $M(s, 2r)$ denotes the number of $s\times s$ upper triangular matrices such that rank $(B+B^T) = 2r$, it follows from Theorem 2.5 that

$$N_s(R, 0)q^{\frac{s(s+1)}{2}} = q^{s(n-2\varrho)}\sum_{r=0}^{[s/2]} M(s, 2r)\,(q^{2(s-r)})^{\varrho}. \tag{4.7}$$

From Lemma 2.1, it follows that

$$N_s(R, 0)q^{\frac{s(s+1)}{2}} = q^{s(n-2\varrho)}\sum_{r=0}^{[s/2]} q^s L_0(s, 2r)\,(q^{2(s-r)})^{\varrho}. \tag{4.8}$$

This completes the proof of the following theorem.

THEOREM 4.1. *Let A be an $n\times n$ alternate matrix of rank 2ϱ over* GF(q). *The number of $s\times n$ matrices X over* GF(q) *such that $XAX^T = 0$ is*

$$N_s(A, 0) = \frac{q^{s(n+1)}}{q^{\frac{s(s+1)}{2}}}\sum_{r=0}^{[s/2]} L_0(s, 2r)q^{-2\varrho r}$$

where $L_0(s, 2r)$ is given by (2.8).

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Slowly growing sequences and discrepancy modulo one

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§ 1. Introduction. Let $y_1, y_2, \ldots, y_k \ldots$ be numbers in the interval

$$[0, 1) = \{x\colon 0\leqslant x < 1\}.$$

We say that $y_1, y_2, \ldots$ is a *uniformly distributed sequence* if for any $[a, b)$ $(0\leqslant a < b\leqslant 1)$, the number k' of $y_1, \ldots, y_k$ falling in $[a, b)$ satisfies

$$k' = (b-a)k + o(k) \quad \text{as } k\to\infty. \tag{1.1}$$

One can prove [3] that if (1.1) is true for all a and b $(0\leqslant a < b\leqslant 1)$, it holds uniformly in a and b: that is, the *discrepancy* $D(k)$ of the sequence $(y_k)_{k=1}^{\infty}$, defined by

$$D(k) = \sup_{0\leqslant a<b\leqslant 1}\left|\frac{k'}{k} - (b-a)\right|, \tag{1.2}$$

satisfies $\lim\limits_{k\to\infty} D(k) = 0$.

The behaviour of $D(k)$ is closely related to that of the exponential sums

$$s(k, h) = \left|\sum_{j=1}^{k} e^{2\pi i y_j h}\right| \quad (k\geqslant 1, h\geqslant 1). \tag{1.3}$$

It can be shown that

$$\lim_{k\to\infty} D(k) = 0 \quad \text{iff} \lim_{k\to\infty}\frac{s(k, h)}{h} = 0 \quad \text{for all } h\geqslant 1 \tag{1.4}$$

and, more precisely,

$$\frac{1}{2\pi}\sup_{h\geqslant 1}\frac{s(k, h)}{h} \leqslant kD(k) \leqslant 150\left(\frac{k}{m+1} + \sum_{h=1}^{m}\frac{s(k, h)}{h}\right) \tag{1.5}$$

for all integers $m\geqslant 1$ ([7], Theorem III and [1], p. 14).

Now suppose that

$$\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_n\leqslant\ldots \tag{1.6}$$