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Reçu le 16. 12. 1971 (246)

ACTA ARITHMETICA
XXIII (1973)

On the problem of odd h-fold perfect numbers

by

M. M. Artuhov (Ordżonikidze)

§ 1. In this paper we generalize a known result, related to the existence problem of odd perfect numbers, which was first obtained by Dickson [3] and then rediscovered by Gradstein [5]. This result consists in Theorem IV of Gradstein's paper [5] and here it is quoted as Theorem A in § 3. All generalizations of this result are here obtained at the expence of an improvement of Gradstein's method. In order to prove his Theorem IV Gradstein first of all described a special class of arithmetical functions, for which he then established a general theorem (Theorem I in [5]). Here (cf. § 10) we shall also use this theorem, but in a more suitable wording for the subject (cf. § 6).

§ 2. It is usual to denote the sum of all different natural divisors of a natural number N by $\sigma(N)$. If $(\sigma(N)-N)/N = h$, where h is a natural number $\neq 1$, then N is called a *multiply perfect number* and in the case $h = 1$ N is simply called a *perfect number*. In this paper a number N with $(\sigma(N)-N)/N = h$, h natural ($= 1$ or $\neq 1$), will be called an *h-fold perfect number* (so the notions "a perfect number" and "a 1-fold perfect number" here coincide).

There are rather vast lists of even h-fold perfect numbers for $h = 1, 2, 3, 4, 5, 6, 7$ (cf. [1], [2], [4], [6], [7], [8]), however, we do not know whether there exists even one odd h-fold perfect number. Further, the question whether infinitely many such numbers exist, is still open. Theorem IV in [5] is just one of the known (at all not numerous) important results connected with the last problem.

Concerning the notation, we make the convention, that in this paper h, n will always be used to denote natural numbers, N will denote an *odd* natural number and the right-hand side of the equality $N = p_1^{\beta_1}p_2^{\beta_2}\dots p_n^{\beta_n}$ will represent the canonical factorization of N ($p_1, p_2, \dots, p_n$ are different odd prime numbers, $\beta_1, \beta_2, \dots, \beta_n$ are natural numbers). The number n occurring in this factorization will be called the *rank of N*, and N itself will be called a *number of rank n*. For any (not necessarily multiply perfect) N the real number $(\sigma(N)-N)/N = \lambda$ will be called the *measure of perfection of N* (so, if $\lambda = h$ is natural, then N is an odd h-fold perfect number).

§ 3. THEOREM A (due to L. E. Dickson and I. S. Gradstein). *For any n, among all odd natural numbers of rank n there cannot be infinitely many 1-fold perfect numbers.*

Gradstein's proof of this theorem is based on the fact that $p_1, p_2, \ldots, p_n$ in the canonical factorization of N are different prime numbers (cf. [5], Ch. II, § § 6, 8), but as can easily be seen, this condition cannot be used in the same way to prove the analogue of this theorem for h-fold perfect numbers with $h \neq 1$. As a matter of fact, a deepened consideration shows us that the above condition on $p_1, p_2, \ldots, p_n$ is unnecessary at all (even for the proof of Theorem A itself). In the sequel we shall avoid this condition, but nevertheless we shall be proving much more than Theorem A.

§ 4. Let $(Q) = \{q_1 < q_2 < \ldots < q_i < \ldots\}$, $(A) = \{a_1 < a_2 < \ldots < a_j < \ldots\}$ be two given infinite sequences of natural numbers. Denote the variable, running over numbers q from (Q), by m, and the variable, running over numbers a from (A), by ν. We shall consider arithmetical functions $f(m, \nu)$, where m, ν are arguments with domains (Q), (A) respectively. We specify a class (f) of such functions by the following special conditions:

(a) $q < q'$ implies $f(q, a) > f(q', a)$ for every a.

(b) $a < a'$ implies $f(q, a) < f(q, a')$ for every q.

(c) For every a there exists $\lim_{m\to\infty} f(m, a)$, where m tends to infinity through the sequence (Q).

(d) For every q there exists $\lim_{\nu\to\infty} f(q, \nu)$, where ν tends to infinity through the sequence (A).

(e) There exists $\lim_{m,\nu\to\infty} f(m, \nu) = \lim_{m\to\infty}\lim_{\nu\to\infty} f(m, \nu) = \lim_{\nu\to\infty}\lim_{m\to\infty} f(m, \nu)$, where m, ν tend to infinity through the sequences (Q), (A) respectively.

§ 5. Let n be a given natural number and let $\Psi = \prod_{k=1}^{n} f(m_k, \nu_k)$, where $f(m, \nu)$ is some function of the class (f) determined in § 4.

All m_k, ν_k will be considered as independent variables with the above mentioned domains of variation: (Q) — for all m_k, (A) — for all ν_k. For every natural number $l \leqslant 2n$, by a *specialization of the function* Ψ we shall mean each function of l variables, which may be obtained from Ψ by fixing any $2n-l$ of its arguments m, ν on any definite natural numbers q, a from (Q), (A) respectively. The factors in any such specialization will be arranged in the following order:

1. On the first place we put down all factors of the form $f(q, a)$ (i.e. both the arguments m, ν are fixed on definite numbers q, a from (Q), (A) respectively).

2. On the second place we put down all factors of the form $f(q, \nu)$ (i.e. m is fixed on a number q from (Q), while ν persists as a variable).

3. On the third place we put down all factors of the form $f(m, a)$ (i.e. ν is fixed on a number a from (A), while m persists as a variable).

4. On the fourth place we put down all factors of the form $f(m, \nu)$ (i.e. both m, ν persist as variables).

Let any specialization of Ψ be given and let r, s, t, be the numbers of its factors of the first, second and third form, respectively. Every such specialization, independently of the values q, a, on which its $2n-l = 2r+s+t$ arguments m, ν are fixed, will be denoted by $\Psi(r, s, t)$, i.e. $\Psi(r, s, t) = \Pi_1\Pi_2\Pi_3\Pi_4$, where $\Pi_1 = \prod_{k=1}^{r} f(q_k, a_k)$, $\Pi_2 = \prod_{k=r+1}^{r+s} f(q_k, \nu_k)$, $\Pi_3 = \prod_{k=r+s+1}^{r+s+t} f(m_k, a_k)$, $\Pi_4 = \prod_{k=r+s+t+1}^{n} f(m_k, \nu_k)$. If any of these four products is empty, then we replace it by 1. Thus, for instance, the specialization $\Psi(0, 0, 0)$ is the function Ψ itself.

In virtue of special conditions (c), (d), (e) from § 4, any given specialization $\Psi(r, s, t)$ will gave a definite limit when all l its arguments m, ν, which remained as variables, tend in an arbitrary way to infinity. This limit will be called in the sequel the *limitary value* of the given specialization.

On the other hand, if for a given specialization of Ψ, in addition to its $2n-l$ already fixed arguments we fix the remaining l its arguments (on definite numbers q, a respectively), then we obtain a definite numerical value of the given specialization. In the sequel every such value will be called an *ordinary value* of the given specialization.

§ 6. It is easy to verify that the function Ψ with all its specializations satisfies conditions 1, 2, 3 which are inherent to the function $F(\pi_1, \pi_2, \ldots, \pi_m, p_1, p_2, \ldots, p_n)$, described in Gradstein's Theorem I (cf. [5], Ch. I, § 2). This fact allows us to apply that theorem in a corresponding way to the function Ψ and thus we obtain the following

THEOREM B. *Let (Q), (A) be given infinite sequences of natural numbers, n — a natural number, μ — a real number, and let $\Psi = \prod_{k=1}^{n} f(m_k, \nu_k)$ be any function, which is satisfying the conditions of § 5.*

If all specializations of Ψ having μ as their limitary value, cannot have μ as some of their ordinary values, then the equation $\prod_{k=1}^{n} f(m_k, \nu_k) = \mu$ cannot have infinitely many solutions in numbers belonging to (Q) for all m_k and belonging to (A) for all ν_k.

§ 7. Now we are going to consider the (quite definite) function $\Phi = \prod_{k=1}^{n} \frac{m_k^{\nu_k+1}-1}{m_k^{\nu_k}(m_k-1)}$, where all variables $m_1, m_2, \ldots, m_n$ will be supposed to take (independently from one another) values equal to arbitrary *odd* natural numbers $\neq 1$ and all variables $\nu_1, \nu_2, \ldots, \nu_n$ will take (independently from one another and from all m_k), arbitrary natural values. It is clear that Φ is a particular kind of a function Ψ, as introduced in § 5. Here $f(m,\nu) = \frac{m^{\nu+1}-1}{m^\nu(m-1)} = \frac{1+m+\ldots+m^\nu}{m}$, where m, ν are running over the numbers q, α from the sequences $(Q) = \{3, 5, 7, 9, \ldots\}$, $(A) = \{1, 2, 3, 4, \ldots\}$ respectively. As can easily be verified, such a function $f(m,\nu)$ fulfils condition (a)–(e) from § 4, i.e. Φ is in fact just a specific example of a function Ψ. Moreover, let us note that here the functions $f(m,\nu)$ has the following further properties:

$$\lim_{m\to\infty} f(m,\alpha) = 1, \qquad \lim_{\nu\to\infty} f(q,\nu) = \frac{q}{q-1}, \qquad \lim_{\substack{m\to\infty\\ \nu\to\infty}} f(m,\nu) = 1,$$

and that for every q, α (from (Q), (A) respectively) $f(q,\alpha)$ is an improper irreducible rational fraction with an odd denominator.

§ 8. For any specialization of Φ let the occurring factors f be arranged in the before defined order, i.e. let the products Π_j in $\Phi(r,s,t) = \Pi_1\Pi_2\Pi_3\Pi_4$ be formed according to the formulae from § 5. Besides, in order to simplify some notations, assume that $\lim \Pi_j$ denotes the number, to which Π_j converges when its arguments m_k, ν_k, remained as variables, tend to infinity over the sequences (Q), (A) respectively (e.g., $\lim \Pi_3 = \lim_{m_k\to\infty} \prod_{k=r+s+1}^{r+s+t} f(m_k, \alpha_k)$, where all α_k are fixed numbers from (A), while all $m_k \to \infty$ over the sequence (Q)).

In virtue of the properties of $f(m,\nu)$, which are pointed out in § 7, the following statements are true for each specialization $\Phi(r,s,t)$ of the function Φ:

1. Π_1 is a rational number > 1 if $r > 0$, but $\Pi_1 = 1$ if $r = 0$.
2. $\lim \Pi_2 = \prod_{k=r+1}^{r+s} \frac{q_k}{q_k-1}$ if $s > 0$, but $\lim \Pi_2 = \Pi_2 = 1$ if $s = 0$.
3. $\lim \Pi_3 = 1$ for any $t \geq 0$.
4. $\lim \Pi_4 = 1$ for any $r+s+t \leq n$.

§ 9. Lemma. *Neither specialization of the function Φ, introduced in § 7, has any of its ordinary values equal to its limitary value.*

Proof. In the consideration concerning all possible specializations $\Phi(r,s,t)$ of Φ we shall distinguish four cases:

1. $r = s = 0$. In this case, on account of the statements from § 8, the limitary value of $\Phi(0,0,t)$ is equal to 1 (in particular, also when $t = 0$), however any of its ordinary values fulfils $\prod_{k=1}^{n} f(q_k, \alpha_k) > 1$, so in this case the lemma holds.

2. $r \neq 0$, $s = 0$. In this case (by the statements from § 8) the limitary value of $\Phi(r,0,t)$ (where $r \geq 1$, $t \geq 0$) is equal to $\Pi_1 = \prod_{k=1}^{r} f(q_k, \alpha_k)$, while any of its ordinary value fulfils $\Pi_1 \cdot \prod_{k=r+1}^{n} f(q_k, \alpha_k) > \Pi_1$, since in virtue of $l \geq 1$ (see the condition on l in § 5) we have $r < n$ and at the same time $f(q_k, \alpha_k) > 1$ for every k. Thus the lemma also holds.

3. $r = 0$, $s \neq 0$. In this case (again by the statements from § 8) the limitary value of $\Phi(0,s,t)$ (where $s \geq 1$, $t \geq 0$) is equal to $\prod_{k=1}^{s} \frac{q_k}{q_k-1}$, while any ordinary value of such a specialization is equal to $\prod_{k=1}^{n} \frac{1+q_k+\ldots+q_k^{\alpha_k}}{q_k^{\alpha_k}}$. The equality $\prod_{k=1}^{s} \frac{q_k}{q_k-1} = \prod_{k=1}^{n} \frac{1+q_k+\ldots+q_k^{\alpha_k}}{q_k^{\alpha_k}}$ is absurd, since its left-hand side after all reductions persists as a fraction with an even denominator, but at the same time its right-hand side always is a fraction with an odd denominator (in particular, after reductions it may be equal to the unit). Therefore also now the lemma is correct.

4. $r \neq 0$, $s \neq 0$. In this last case (once more by the statements from § 8) the limitary value of $\Phi(r,s,t)$ (where $r \geq 1$, $s \geq 1$, $t \geq 0$) is equal to $\Pi_1 \cdot \prod_{k=r+1}^{r+s} \frac{q_k}{q_k-1}$, while any ordinary value of such a specialization is equal to $\Pi_1 \cdot \prod_{k=r+1}^{n} \frac{1+q_k+\ldots+q_k^{\alpha_k}}{q_k^{\alpha_k}}$. But $\prod_{k=r+1}^{r+s} \frac{q_k}{q_k-1} = \prod_{k=r+1}^{n} \frac{1+q_k+\ldots+q_k^{\alpha_k}}{q_k^{\alpha_k}}$ is false by the very same reasons as in the preceding case. This completes the proof of the lemma.

§ 10. Theorem C. *For any given rational number $\lambda > 0$ and any given n there cannot be infinitely many odd natural numbers of rank n with measure of perfection equal λ.*

Proof. Let $N = p_1^{\beta_1} p_2^{\beta_2} \ldots p_n^{\beta_n}$ be any odd number with measure of perfection equal λ. Then we have $\lambda+1 = \frac{\sigma(N)}{N} = \prod_{k=1}^{n} f(p_k, \beta_k)$, where $f(p_k, \beta_k)$ are some ordinary value of the function $f(m,\nu) = \frac{m^{\nu+1}-1}{m^\nu(m-1)}$, introduced in § 7. By the equalities $\prod_{k=1}^{n} f(p_k, \beta_k) = \lambda+1$, the numbers p_k, β_k $(k = 1, 2, \ldots, n)$ form a definite solution of the equation $\prod_{k=1}^{n} f(m_k, \nu_k) = \lambda+1$ in numbers p, β, belonging respectively to the sets (Q), (A), introduced in § 7. The left-hand side of this equation is the function Φ, defined in § 7, so by the Lemma (§ 9), for this function Theorem B is fulfilled (with $\mu = \lambda+1$ and $\Psi = \Phi$). Consequently, the assertion really holds.

We remark that in the above theorem λ may always be assumed to be a rational fraction with an odd denominator (which, in particular, may be equal to 1) since numbers of this form are the only possible ordinary values of $\Phi-1$.

Besides, we note that in this theorem the words "natural numbers of rank n" may be replaced by the words "natural numbers of rank l with any $l \leqslant n$".

From Theorem C we can immediately conclude the following

COROLLARY 1. *For any given natural numbers h, n there cannot be infinitely many odd h-fold perfect numbers of rank n.*

This Corollary includes Theorem A as a particular case.

COROLLARY 2. *For any given natural number n there cannot be infinitely many odd n-rank's multiply perfect numbers.*

This result follows from Corollary 1 owing to the obvious inequality $\frac{\sigma(N)}{N} < \prod_{k=1}^{n} \frac{P_k}{P_k-1}$, where $N = p_1^{\beta_1} p_2^{\beta_2} \dots p_n^{\beta_n}$ is any one odd n-rank's number, while $P_1 = 3$, $P_2 = 5, \dots, P_n$ are n initial successive odd prime numbers.

§ 11. In our proof of Theorem C the sequence (Q) (all odd natural numbers except 1) was used only partially. In order to form the canonical factorization of some number N of rank n we were taking out from (Q) only different prime numbers $p_1, p_2, \dots, p_n$. However, with the same effect one can take out from (Q) arbitrary collections of any odd numbers $q_1, q_2, \dots, q_n$ (some or even all of them may be identical); this possibility gives us a more general result:

THEOREM D. *Let for a given n each of $q_1, q_2, \dots, q_n$ be an odd natural number $\neq 1$, each of $\alpha_1, \alpha_2, \dots, \alpha_n$ be a non-negative integer, and let S_n be the set of all products $q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n}$. Then for any given rational number $\mu > 1$ there cannot be infinitely many products $q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n}$ for which*

$$\prod_{k=1}^{n} \frac{1+q_k+\dots+q_k^{\alpha_k}}{q_k^{\alpha_k}} = \mu .$$

In connection therewith we remark in conclusion that it is unknown till now whether there exists even one product $q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n} \neq 1$ (where all q_k are odd numbers $\neq 1$), for which $\prod_{k=1}^{n} \frac{1+q_k+\dots+q_k^{\alpha_k}}{q_k^{\alpha_k}}$ is equal to some natural number.

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Received on 10. 2. 1972 (254)