Of course, for $k < 4$, acceptable pairings have appeared in the articles of Klarner and Sebastian.

The construction of acceptable pairings of $[1, 2n]$ when n and 6 are not relatively prime is not really so unnatural as it might appear. For example, in the case when n is of the form $6k+3$, it is natural to ask: Is it possible to delete a pair of the form $(m, 2m+n)$ from the acceptable pairing C_{n+1} of Theorem 2 and form an acceptable pairing of $[1, 2(n+2)]$ by pairing m with $2n+3$ and $2m+n$ with $2n+4$? Examination of this question does lead to the conditions of Theorem 3.

Of course, it would be nice to have a simpler solution for this simple problem of the Shens.

It has recently come to our attention that J. L. Selfridge [5] announced a different solution in 1963.

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UNIVERSITY OF GEORGIA
Athens, Georgia

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Some estimates in the theory of Dedekind Zeta-functions

by

W. Staś and K. Wiertelak (Poznań)

1. Denote by K an algebraic number field, by ν and Δ the degree and the discriminant of the field K respectively and by $\zeta_K(s)$, $s = \sigma+it$, the Dedekind Zeta-function (see [2]).

The function $\zeta_K(s)$ is defined for $\sigma > 1$ by the absolute convergent series

$$\sum_{n=1}^{\infty} F(n) n^{-s},$$

where $F(n)$ denotes the number of ideals of the field K, having the norm equal to n.

The function $\zeta_K(s)$ can be continued over the whole complex plane as a regular function, except $s = 1$, where there is a simple pole.

In the region $\sigma > 1$

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{n=1}^{\infty} G(n) n^{-s} \tag{1.1}$$

where

$$G(n) = \sum_{(N\mathfrak{p})^m = n} \log N\mathfrak{p}$$

and the series in (1.1) is absolutely convergent in this region (see [2], p. 89).

A. Sokolovski [7] proved that $\zeta_K(s) \neq 0$ in the region

$$\sigma > 1 - \frac{c_1}{\log^{\gamma}|t|}, \quad |t| > c_2, \ \gamma > \tfrac{2}{3}, \tag{1.2}$$

where c_1, c_2 are constants depending on the field K. Theorem (1.2) is a deep improvement of E. Landau's classic result, $\gamma = 1$ (see [2], p. 105).

The subject of this note is an investigation of the equivalence between the domain (1.2), in which $\zeta_K(s) \neq 0$, and the estimate of the difference

$$\sum_{n \leqslant x} G(n) - x \overset{\text{def}}{=} \Delta(x, K), \tag{1.3}$$

which is the remainder in the Prime-ideal Theorem (see [4]).

In the case of the Riemann Zeta-function such equivalence was discovered by P. Turán (see [6], p. 150) and was also investigated by the first of the present authors (see [5]).

2. We will prove the following theorems:

THEOREM 1. *Suppose that* $\zeta_K(s)$ *has no zeros in the domain*

$$\sigma > 1 - c_0\eta(|t|), \quad c_0 \leqslant 1, \tag{2.1}$$

where c_0 *is a constant depending on the field* K, *and* $\eta(t)$ *is for* $t \geqslant 0$ *a decreasing function, having a continuous derivative* $\eta'(t)$ *and satisfying the following conditions:*

(a) $$0 < \eta(t) \leqslant \tfrac{1}{2},$$

(b) $$\eta'(t) \to 0 \quad \text{as} \quad t \to \infty,$$

(c) $$\frac{1}{\eta(t)} = O(\log t) \quad \text{as} \quad t \to \infty.$$

Let α *be a fixed number satisfying* $0 < \alpha < 1$. *Then*

$$\Delta(x, K) < c_3\left\{\frac{\nu^2\log^2(|\Delta|+1)}{c_0^2}\, x\exp\left(-\frac{c_0\alpha}{2}\,\omega(x)\right)\right\}, \quad x \to \infty, \tag{2.2}$$

where $\omega(x)$ *is the minimum of* $\eta(t)\log x + \log t$ *for* $t \geqslant 1$, *and* c_3 *depends only on* α *and on the function* $\eta(t)$.

THEOREM 2. *Let* $\eta_1(t)$ *be a function satisfying besides* (a), (b), (c) *also the additional condition*

(d) $$\eta_1(t) \leqslant c_4 \quad \text{for} \quad t > c_5$$

where c_4 *is a sufficiently small positive number and let* $\omega_1(x)$ *be the minimum of* $\eta_1(t)\log x + \log t$ *for* $t \geqslant 1$. *Suppose further the estimate* (2.2). *Then* $\zeta_K(s) \neq 0$ *in the domain*

$$\begin{gathered}\sigma > 1 - \frac{\log t}{400\log\omega_1^{-1}(\log t^{4/\alpha c_0})},\\ t > \max\left\{c_6, \left(\frac{\nu}{c_0}\log(|\Delta|+1)\right)^{10}, |\Delta|+1, \eta^{-1}(e^{-\nu^2})\right\},\end{gathered} \tag{2.3}$$

where $\omega_1^{-1}(x)$ *denotes the function inverse to* $\omega_1(x)$.

Theorem 2 easily implies

THEOREM 3. *Under the conditions of Theorem 2, we have* $\zeta_K(s) \neq 0$ *in the region*

$$\begin{gathered}\sigma > 1 - \frac{\alpha c_0}{(40)^2}\eta_1(t^{4/\alpha c_0}),\\ t > \max\left\{c_6, \left(\frac{\nu}{c_0}\log(|\Delta|+1)\right)^{10}, |\Delta|+1, \eta^{-1}(e^{-\nu^2})\right\}.\end{gathered} \tag{2.4}$$

Choosing $\eta_1(t) = \eta(t) = 1/\log^\gamma t$, $0 < \gamma \leqslant 1$, we obtain from Theorems 1 and 2 the following

THEOREM 4. *If* γ_1 *is the supremum of the numbers* γ *for which*

$$\Delta(x, K) = O\big(x\exp(-c_7\log^\gamma x)\big) \tag{2.5}$$

and γ_2 *is the infimum of the numbers* γ' *for which* $\zeta_K(s) \neq 0$ *in the region*

$$\sigma > 1 - \frac{c_8}{\log^{\gamma'}|t|}, \quad |t| \geqslant c_9, \tag{2.6}$$

then

$$\gamma_1 = \frac{1}{1+\gamma_2}$$

and the constants depend on γ *and the field* K.

3. The proofs of Theorems 1 and 2 will rest on the following lemmas

LEMMA 1. *Let* $z_1, z_2, \ldots, z_h$ *be complex numbers such that*

$$|z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_h|, \quad |z_1| \geqslant 1$$

and let $b_1, b_2, \ldots, b_h$ *be any complex numbers.*

Then, if m *is positive and* $N \geqslant h$, *there exists an integer* r *such that* $m \leqslant r \leqslant m+N$,

$$|b_1z_1^r + b_2z_2^r + \ldots + b_hz_h^r| \geqslant \left(\frac{N}{48e^2(2N+m)}\right)^N \min_{1\leqslant\gamma\leqslant h}|b_1+b_2+\ldots+b_j|. \tag{3.1}$$

This lemma is Turán's second main theorem (see [6], p. 52).

The next lemmas concern the properties of the function $\zeta_K(s)$.

LEMMA 2. *For* $\sigma = 2$,

$$|\zeta_K(s)| > K_1 \tag{3.2}$$

where $K_1 = \left(\frac{6}{\pi^2}\right)^\nu$.

LEMMA 3. *In the region* $-1 \leqslant \sigma \leqslant 4$, $-\infty < t < \infty$,

$$|(s-1)\zeta_K(s)| \leqslant K_2(|t|+1)^{K_3} \tag{3.3}$$

where $K_2 = c_{10}^\nu|\Delta|^{3/2}$, $K_3 = \frac{3}{2}\nu + 2$, *and* c_{10} *is a numerical constant.*

LEMMA 4. *For the coefficients* $G(n)$ *of* (1.1) *we have the following estimate:*

$$G(n) \leqslant K_4\log^2 n, \tag{3.4}$$

where $K_4 = \nu/\log 2$.

As regards the proofs of Lemmas 2–4 see [4]. From Lemma 2 and 3 it follows that $K_2 > K_1$.

LEMMA 5. *If $s_0 = 1+\mu+it'$, $0 < \mu \leqslant 1/40$, $t' \geqslant 10$ and N_1 stands for the number of roots of $\zeta_K(s)$ in the circle $|s-s_0| \leqslant 8\mu$, then*

$$N_1 < \frac{c_{11}\nu\log(|\Delta|+1)t'}{\log(8\mu)^{-1}}. \tag{3.5}$$

This lemma follows from (3.3) by the use of the Jensen inequality (compare [6], p. 187).

LEMMA 6. *Denote by $V(T)$ the number of zeros of $\zeta_K(s)$ in the rectangle $\sqrt{\delta} \leqslant \sigma \leqslant 1$, $T \leqslant t \leqslant T+1$ where $0 < \delta \leqslant (\frac{3}{16})^2$. Then for $-\infty < T < +\infty$ we have*

$$V(T) < \tfrac{8}{3}\delta^{-5/6}\log\frac{K_2}{K_1}(|T|+4)^{K_3}. \tag{3.6}$$

LEMMA 7. *There exists a broken line L in the vertical strip $\frac{4}{3}\sqrt{\delta} \leqslant \sigma \leqslant \frac{8}{3}\sqrt{\delta}$, $0 < \delta \leqslant (\frac{3}{16})^2$ consisting of horizontal and vertical segments alternately and having the following property: if we denote by T_m the ordinates of the horizontal segments, then for each integer m there exists only one such T_m that $m < T_m < m+1$ and*

$$\left|\frac{\zeta_K'}{\zeta_K}(s)\right| < 17\delta^{-11/6}\log^2\frac{K_2}{K_1}(|t|+5)^{K_3} \tag{3.7}$$

holds for $s \in L$.

If $\frac{4}{3}\sqrt{\delta} \leqslant \sigma \leqslant 3$, $t = T_m$, $|m| \geqslant 2$, then

$$\left|\frac{\zeta_K'}{\zeta_K}(s)\right| < 15\delta^{-4/3}\log^2\frac{K_2}{K_1}(|T_m|+5)^{K_3}. \tag{3.8}$$

For the proofs of Lemmas 6 and 7 see [4].

LEMMA 8. *If $0 < \delta \leqslant (\frac{3}{16})^2$, $1 < \sigma \leqslant \frac{3}{2}$, $\xi > 1$ and $l \geqslant 2$ is a positive integer, then*

$$\left|(-1)^l\sum_{n\geqslant\xi}\frac{G(n)}{n^s}\,\frac{\log^{l+1}\left(\frac{n}{\xi}\right)}{(l+1)!} - \frac{\xi^{1-s}}{(1-s)^{l+2}} + \sum_{\varrho}\frac{\xi^{\varrho-s}}{(\varrho-s)^{l+2}}\right| < 171\delta^{-11/6}\,\frac{\xi^{\sigma_0-\sigma}\log^2\frac{K_2}{K_1}(|t|+6)^{K_3}}{\min\left(1,\,(\sigma-\sigma_0)^{l+2}\right)} \tag{3.9}$$

where $\sigma_0 = \frac{8}{3}\sqrt{\delta}$ and the sum is taken over all zeros of $\zeta_K(s)$ lying to the right of the line L.

This lemma can be proved by following *mutatis mutandis* Appendix V of [6].

4. We pass over to the proof of Theorem 1. As in [1], pp. 60–62, we can prove that

$$\frac{\zeta_K'}{\zeta_K}(s) + \frac{1}{s-1} = O\left(\frac{1}{c_0^2}\log^2 K_2(|t|+2)^{K_3}\right) \tag{4.1}$$

in the region

$$\begin{aligned} &1 - ac_0\eta(|t|) \leqslant \sigma \leqslant 1 + a\eta(|t|), \quad |t| \geqslant T_0,\\ &1 - ac_0\eta(T_0) \leqslant \sigma \leqslant 1 + a\eta(|t|), \quad |t| \leqslant T_0, \end{aligned} \tag{4.2}$$

where the constant in (4.1) depends only on $\eta(t)$ and α; T_0 depends on $\eta(t)$.

From Lemma 4 and (4.1)–(4.2) follows the estimate

$$\sum_{n\leqslant x}(x-n)G(n) = \frac{x^2}{2} + O\left(x^2\frac{\nu^2\log^2(|\Delta|+1)}{c_0^2}\exp(-ac_0\omega(x))\right). \tag{4.3}$$

The constant in (4.3) depends on a and $\eta(t)$ only. Using the relation between $\sum_{n\leqslant x}(x-n)G(n)$ and $\sum_{n\leqslant x}G(n)-x$, we get, as in [1], p. 64, the estimate (2.2).

5. Proof of Theorem 2 (compare [6], pp. 151–152). Put $t \geqslant 2$. By (1.3) we have for $n > 1$

$$G(n) = \Delta(n, K) - \Delta(n-1, K) + 1.$$

Hence

$$\begin{aligned} &\Big|\sum_{N_1\leqslant n\leqslant N_2} G(n)\exp(-it\log n)\Big| \\ &\leqslant \Big|\sum_{N_1\leqslant n\leqslant N_2}\exp(-it\log n)\Big| + \Big|\sum_{N_1\leqslant n\leqslant N_2}\big(\Delta(n,K)-\Delta(n-1,K)\big)\exp(-it\log n)\Big| \\ &= I_1 + I_2. \end{aligned} \tag{5.1}$$

We choose N_1, N_2 so large that

$$\omega_1^{-1}(\log t^{4/ac_0}) \leqslant N/2 < N_1 < N_2 \leqslant N. \tag{5.2}$$

Then by

$$\omega_1(1+t^2) < \log(1+t^2) < \log t^{4/ac_0} \tag{5.3}$$

we get

$$I_2 < c_{12}\frac{\nu^2\log^2(|\Delta|+1)}{c_0^2}\cdot\frac{N}{t} \tag{5.4}$$

(compare [5]).

From the estimate of I_1 (see [6], p. 153) and (5.3) we get

$$I_1 < c_{13}\frac{N}{t}. \tag{5.5}$$

Hence by (5.1) it follows that

$$(5.6)\qquad \Big|\sum_{N_1\leqslant n\leqslant N_2} G(n)\exp(-it\log n)\Big| \leqslant c_{14}\frac{\nu^2\log^2(|\Delta|+1)}{c_0^2}\cdot\frac{N}{t}.$$

Suppose that

$$(5.7)\qquad 1<\sigma\leqslant 3/2.$$

By partial summation and (5.6) we have

$$(5.8)\qquad \Big|\sum_{N_1\leqslant n\leqslant N_2} G(n)n^{-s}\Big| < c_{15}\frac{\nu^2\log^2(|\Delta|+1)}{c_0^2}\cdot\frac{N^{1-\sigma}}{t}.$$

We choose

$$(5.9)\qquad \eta\geqslant \omega_1^{-1}(\log t^{4/\alpha c_0})$$

and apply the inequality (5.8) for

$$N_1^i=\eta\cdot 2^i,\qquad N_2^i=\eta\cdot 2^{i+1},\qquad i=0,1,2,\ldots$$

Hence

$$(5.10)\qquad \Big|\sum_{n\geqslant\eta} G(n)n^{-s}\Big| < c_{16}\frac{\nu^2\log^2(|\Delta|+1)}{c_0^2}\cdot\frac{\eta^{1-\sigma}}{t(\sigma-1)}.$$

We choose further

$$(5.11)\qquad \xi>\omega_1^{-1}(\log t^{4/\alpha c_0}).$$

Denoting by l a positive integer and following [6], p. 154, we get by (5.10) the estimate

$$(5.12)\qquad \Big|\sum_{n\geqslant\xi}\frac{G(n)}{n^s}\log^{l+1}\frac{n}{\xi}\Big| < c_{17}\frac{\nu^2\log^2(|\Delta|+1)}{c_0^2}\cdot\frac{(l+1)!\,\xi^{1-\sigma}}{t(\sigma-1)^{l+2}}.$$

Hence by Lemma 8 with $\delta=(\frac{3}{16})^2$ and by (5.3) we get

(5.13)

$$\Big|\sum_{\varrho}\frac{\xi^{\varrho-s}}{(\varrho-s)^{l+2}}\Big| < c_{18}\Big(\xi^{\frac{1}{2}-\sigma}\frac{\nu^2\log^2(|\Delta|+1)t}{(\sigma-\frac{1}{2})^{l+2}}+\frac{\nu^2\log^2(|\Delta|+1)}{c_0^2}\cdot\frac{\xi^{1-\sigma}}{t(\sigma-1)^{l+2}}\Big)$$
$$< c_{18}\frac{\nu^2\log^2(|\Delta|+1)t}{c_0^2}\cdot\frac{\xi^{1-\sigma}}{t(\sigma-1)^{l+2}}.$$

Let us suppose now that our theorem is not true. Hence there exist such zeros

$$\varrho^*=\sigma^*+it^*,\qquad t^*\to\infty,$$

that

$$(5.14)\qquad \sigma^*>1-\frac{\log t^*}{400\log\omega_1^{-1}(\log t^{*4/\alpha c_0})},$$

$$(5.15)\qquad t^*\geqslant\max\Big\{\Big(\frac{\nu}{c_0}\log(|\Delta|+1)\Big)^{10},\ e^{2\cdot 14},\ \inf\{t'\colon\ \eta(t^*)=e^{-\nu^2}\},\ |\Delta|+1\Big\}.$$

Putting in the estimate (5.13)

$$(5.16)\qquad s=s_1=1+\frac{1}{10}\frac{\log t^*}{\log\omega_1^{-1}(\log t^{*4/\alpha c_0})}+it^*=\sigma_1+it^*,$$

$$(5.17)\qquad \xi=\exp\big((l+2)\lambda\big),$$

where

$$(5.18)\qquad \log t^*\leqslant l+2\leqslant\tfrac{5}{4}\log t^*,$$

$$(5.19)\qquad \lambda=\frac{\log\omega_1^{-1}(\log t^{*4/\alpha c_0})}{\log t^*},$$

we can verify without difficulty that (5.7) and (5.11) are satisfied. Multiplying both sides of (5.13) by

$$|\xi^{s_1-\varrho^*}(s_1-\varrho^*)^{l+2}|=\xi^{\sigma_1-\sigma^*}(\sigma_1-\sigma^*)^{l+2},$$

we have

$$(5.20)\qquad \Big|\sum_{\varrho}\xi^{\varrho-\varrho^*}\Big(\frac{s_1-\varrho^*}{s_1-\varrho}\Big)^{l+2}\Big| < c_{18}\frac{\nu^2\log^2(|\Delta|+1)t^*}{c_0^2}\cdot\frac{\xi^{1-\sigma^*}}{t^*}\Big(\frac{\sigma_1-\sigma^*}{\sigma_1-1}\Big)^{l+2}.$$

In virtue of (5.14), (5.16) and (5.18) it follows that

$$\Big(\frac{\sigma_1-\sigma^*}{\sigma_1-1}\Big)^{l+2}=\Big(1+\frac{1-\sigma^*}{\sigma_1-1}\Big)^{l+2}\leqslant t^{*1/32}.$$

If the conditions (5.15) and $t^*>c_{19}(c_3)$ are satisfied, we get by the estimate (5.20) the following inequality:

$$(5.21)\qquad \Big|\sum_{\varrho}\xi^{\varrho-\varrho^*}\Big(\frac{s_1-\varrho^*}{s_1-\varrho}\Big)^{l+2}\Big| < t^{*-2/3}\xi^{1-\sigma^*}.$$

By virtue of Lemma 6 we get, similarly to [6], p. 156, the estimates

$$\Big|\sum_{t_\varrho\geqslant t^*+6(\sigma_1-\sigma^*)}\xi^{\varrho-\varrho^*}\Big(\frac{s_1-\varrho^*}{s_1-\varrho}\Big)^{l+2}\Big| < \frac{c_{20}}{t^*}\xi^{1-\sigma^*},$$

$$\Big|\sum_{t_\varrho\geqslant t^*-6(\sigma_1-\sigma^*)}\xi^{\varrho-\varrho^*}\Big(\frac{s_1-\varrho^*}{s_1-\varrho}\Big)^{l+2}\Big| < \frac{c_{21}}{t^*}\xi^{1-\sigma^*},$$

$$\Big|\sum_{\substack{|t_\varrho-t^*|\leqslant 6(\sigma_1-\sigma^*)\\ t_\varrho<1-3(\sigma_1-\sigma^*)}}\xi^{\varrho-\varrho^*}\Big(\frac{s_1-\varrho^*}{s_1-\varrho}\Big)^{l+2}\Big| < \frac{c_{22}}{t^*}\xi^{1-\sigma^*}.$$

From the above estimates and (5.21) it follows, for $t^* > c_{23}(c_3)$, that

$$V = \Bigg|\sum_{\substack{|t_\varrho - t^*| \leqslant 6(\sigma_1-\sigma^*)\\ \sigma_\varrho \geqslant 1-3(\sigma_1-\sigma^*)}} \left(e^{\lambda(\varrho-\varrho^*)}\frac{s_1-\varrho^*}{s_1-\varrho}\right)^{l+2}\Bigg| < \frac{\xi^{1-\sigma^*}}{t^{*2/3}}. \tag{5.22}$$

We estimate the sum V from below by the use of Lemma 1. We choose

$$z_j = \frac{s_1-\varrho^*}{s_1-\varrho}\exp\big(\lambda(\varrho-\varrho^*)\big),$$

and

$$m = \log t^*. \tag{5.23}$$

The region

$$1-3(\sigma_1-\sigma^*) \leqslant \sigma \leqslant 1, \quad |t-t^*| \leqslant 6(\sigma_1-\sigma^*)$$

is contained in the circle $|s-s_1| \leqslant 8(\sigma_1-1)$. Hence denoting by N_1 the number of roots of $\zeta_K(s)$ in this circle, and using Lemma 5 and the definition of $\eta_1(t)$ with $c_4 = \exp(-56c_{11})$, we have, for $s_0 = s_1 = \sigma_1+it^*$,

$$\mu = \frac{\log t^*}{10\log\omega_1^{-1}(\log t^{*4/ac_0})}$$

and under (5.15) and $t^* > c_{24}$ the estimate

$$N_1 < \frac{\log t^*}{14}.$$

In virtue of Lemma 1 there exists an exponent $l+2$ such that

$$V > \left(\frac{1}{48e^2}\cdot\frac{\log t^*}{28\log t^*}\right)^{\log t^*/14} > \frac{1}{t^{*0.66}}. \tag{5.24}$$

From (5.22) and (5.24) it follows that

$$1-\sigma^* > \frac{1}{400}\cdot\frac{\log t^*}{\log\omega_1^{-1}(\log t^{*4/ac_0})}$$

and this contradicts (5.14). This completes the proof of Theorem 2.

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INSTITUTE OF MATHEMATICS
OF THE ADAM MICKIEWICZ UNIVERSITY, Poznań

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