Of course, for \( k < 4 \), acceptable pairings have appeared in the articles of Klarrer and Sebastian.

The construction of acceptable pairings of \([1, 2n]\) when \( n \) and \( 6 \) are not relatively prime is not really so unnatural as it might appear. For example, in the case when \( n \) is of the form \( 6k + 3 \), it is natural to ask: Is it possible to delete a pair of the form \( (m, 2m + n) \) from the acceptable pairing \( C_\text{ex} \) of Theorem 2 and form an acceptable pairing of \([1, 2(n + 2)]\) by pairing \( m \) with \( 2n + 3 \) and \( 2m + n \) with \( 2n + 4 \)? Examination of this question does lead to the conditions of Theorem 3.

Of course, it would be nice to have a simpler solution for this simple problem of the Shens.

It has recently come to our attention that J. L. Selfridge [5] announced a different solution in 1963.

References


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Some estimates in the theory of Dedekind Zeta-functions

by

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1. Denote by \( K \) an algebraic number field, by \( s \) and \( A \) the degree and the discriminant of the field \( K \) respectively and by \( \zeta_K(s), s = \sigma + it \), the Dedekind Zeta-function (see [2]).

The function \( \zeta_K(s) \) is defined for \( \sigma > 1 \) by the absolute convergent series

\[
\sum_{n=1}^{\infty} \frac{F(n)n^{-s}}{n^{\sigma}}
\]

where \( F(n) \) denotes the number of ideals of the field \( K \), having the norm equal to \( n \).

The function \( \zeta_K(s) \) can be continued over the whole complex plane as a regular function, except \( s = 1 \), where there is a simple pole.

In the region \( \sigma > 1 \)

\[
(1.1) \quad -\frac{\zeta_K'(s)}{\zeta_K(s)} = \sum_{n=1}^{\infty} G(n)n^{-s}
\]

where

\[
G(n) = \sum_{(\mathbb{Z}p)^m = n} \log p
\]

and the series in (1.1) is absolutely convergent in this region (see [2], p. 89).

A. Sokolovski [7] proved that \( \zeta_K(s) \neq 0 \) in the region

\[
(1.2) \quad \sigma > 1 - \frac{c_1}{\log |\mathfrak{d}|}, \quad |t| > c_2, \quad \gamma > \frac{\gamma_K}{2},
\]

where \( c_1, c_2 \) are constants depending on the field \( K \). Theorem (1.2) is a deep improvement of E. Landau’s classic result, \( \gamma = 1 \) (see [2], p. 105).

The subject of this note is an investigation of the equivalence between the domain (1.2), in which \( \zeta_K(s) \neq 0 \), and the estimate of the difference

\[
(1.3) \quad \sum_{n \leq x} G(n) - x^\alpha = \Delta(x, K),
\]

which is the remainder in the Prime-ideal Theorem (see [4]).
In the case of the Riemann Zeta-function such equivalence was discovered by P. Turán (see \cite{6}, p. 150) and was also investigated by the first of the present authors (see \cite{5}).

2. We will prove the following theorems:

**Theorem 1.** Suppose that \( \zeta_K(s) \) has no zeros in the domain
\[
\sigma > 1 - c_0 \eta(|t|), \quad c_0 \leq 1,
\]
where \( c_0 \) is a constant depending on the field \( K \), and \( \eta(t) \) is for \( t \geq 0 \) a decreasing function, having a continuous derivative \( \eta'(t) \) and satisfying the following conditions:
\[
\begin{align*}
& 0 < \eta(t) \leq \frac{1}{2}, \\
& \eta'(t) \to 0 \quad \text{as} \quad t \to \infty, \\
& \frac{1}{\eta(t)} = O(\log t) \quad \text{as} \quad t \to \infty.
\end{align*}
\]

Let \( a \) be a fixed number satisfying \( 0 < a < 1 \). Then
\[
\begin{align*}
\Delta(x, K) &< c_0 \left\{ \frac{x \log x \log t}{\log x} - x \log (x \log x) \right\}, \\
& \quad x \to \infty,
\end{align*}
\]
where \( \omega(x) \) is the minimum of \( \eta(t) \log x + \log t \) for \( t \geq 1 \), and \( c_0 \) depends only on \( a \) and on the function \( \eta(t) \).

**Theorem 2.** Let \( \eta_1(t) \) be a function satisfying besides (a), (b), (c) also the additional condition
\[
\eta_1(t) \leq c_0 \quad \text{for} \quad t > c_0,
\]
where \( c_0 \) is a sufficiently small positive number and let \( \omega_1(x) \) be the minimum of \( \eta_1(t) \log x + \log t \) for \( t \geq 1 \). Suppose further the estimate (2.2). Then \( \zeta_K(s) \neq 0 \) in the domain
\[
\begin{align*}
\sigma > 1 - \frac{\log t}{400 \log \omega_1(\log t)}
\end{align*}
\]
\begin{align*}
t > \max \left\{ c_0 \left( \frac{\log(|t|+1)}{\log(\log t)} \right)^{10}, |t|+1, \eta_1^{-1}(e^{-r}) \right\},
\end{align*}
where \( \omega_1^{-1}(x) \) denotes the function inverse to \( \omega_1(x) \).

**Theorem 2** easily implies

**Theorem 3.** Under the conditions of Theorem 2, we have \( \zeta_K(s) \neq 0 \) in the region
\[
\begin{align*}
\sigma > 1 - \frac{\alpha_0}{40} \eta_1(\log t)
\end{align*}
\]
\begin{align*}
t > \max \left\{ c_0 \left( \frac{\log(|t|+1)}{\log(\log t)} \right)^{10}, |t|+1, \eta_1^{-1}(e^{-r}) \right\},
\end{align*}
where \( K_1 = \left( \frac{6}{\pi^2} \right)^{1/2} \).

Choosing \( \eta_1(t) = \eta(t) = \frac{1}{\log^2 t}, \quad 0 < \gamma \leq 1 \), we obtain from Theorems 1 and 2 the following

**Theorem 4.** If \( \gamma_3 = \sup \gamma \) for which
\[
\Delta(x, K) = O(x \exp (-c_3 \log^2 x))
\]
and \( \gamma_2 \) is the infimum of the numbers \( \gamma \) for which \( \zeta_K(s) \neq 0 \) in the region
\[
\sigma > 1 - \frac{c_0}{\log^2 |t|}, \quad |t| > c_0,
\]
then
\[
\gamma_1 = \frac{1}{1 + \gamma_2}
\]
and the constants depend on \( \gamma \) and the field \( K \).

3. The proofs of Theorems 1 and 2 will rest on the following lemmas

**Lemma 1.** Let \( z_1, z_2, \ldots, z_n \) be complex numbers such that
\[
|z_1| > |z_2| > \ldots > |z_n|, \quad |z_1| > 1
\]
and let \( b_1, b_2, \ldots, b_n \) be any complex numbers.

Then, if \( m \) is positive and \( N \gg n \), there exists an integer \( r \) such that \( m \leq r \leq m + N \),
\[
|b_1 z_1^r + b_2 z_2^r + \ldots + b_n z_n^r| \geq \left( \frac{N}{18e^2(2N+m)} \right)^{1/2} \min_{1 \leq i < \lambda} |b_i + b_i + \ldots + b_i|.
\]

This lemma is Turán's second main theorem (see \cite{6}, p. 52).

The next lemmas concern the properties of the function \( \zeta_K(s) \).

**Lemma 2.** For \( \sigma = 2 \),
\[
|\zeta_K(s)| > K_4
\]
where \( K_4 = \left( \frac{6}{\pi^2} \right)^{1/2} \).

**Lemma 3.** In the region \(-1 \leq \sigma \leq 4, -\infty < t < \infty, \)
\[
|\zeta_K(s)| \leq K_2(|t|+1)^{2s}
\]
where \( K_2 = c_{10} |\Delta|^{2s}, K_3 = \frac{1}{2} r^2 + 2, \) and \( c_{10} \) is a numerical constant.

**Lemma 4.** For the coefficients \( G(n) \) of (1.1) we have the following estimates:
\[
G(n) \leq K_4 \log^{2n} n,
\]
where \( K_4 = n/\log 2 \).

As regards the proofs of Lemmas 2–4 see \cite{4}. From Lemma 2 and 3 it follows that \( K_2 > K_4 \).
Lemma 5. If \( s_0 = 1 + \mu + it \), \( 0 < \mu \leq 1/40 \), \( t' \geq 10 \) and \( N_1 \) stands for the number of roots of \( \zeta_K(s) \) in the circle \( |s-s_0| \leq 8\mu \), then

\[
N_1 < \frac{c_1 \mu \log((|t|+1)t')}{\log(8\mu)}.
\]

This lemma follows from (3.3) by the use of the Jensen inequality (compare [6], p. 187).

Lemma 6. Denote by \( V(T) \) the number of zeros of \( \zeta_K(s) \) in the rectangle \( \sqrt{\delta} \leq \sigma \leq 1 \), \( t \leq t < T+1 \) where \( 0 < \delta \leq \left(\frac{2}{t'}\right)^2 \). Then for \( -\infty < T < +\infty \) we have

\[
V(T) \leq \frac{8}{9} \delta^{1/3} \log \frac{K_2}{K_1} \left(T+1+\delta\right)^{K_3}.
\]

Lemma 7. There exists a broken line \( L \) in the vertical strip \( \frac{1}{3} \sqrt{\delta} \leq \sigma \leq \frac{1}{3} \sqrt{\delta} \), \( 0 < \delta \leq \left(\frac{1}{t'}\right)^2 \) consisting of horizontal and vertical segments alternately and having the following property: if we denote by \( T_m \) the ordinates of the horizontal segments, then for each integer \( m \) there exists only one such \( T_m \) that \( m < T_m < m+1 \) and

\[
\left| \frac{\zeta_K'}{\zeta_K}(s) \right| < 17 \delta^{-1/3} \log^2 \frac{K_2}{K_1} \left(|t|+1\right)^{K_3}
\]

holds for \( s \in L \).

If \( \frac{1}{3} \sqrt{\delta} \leq \sigma \leq \frac{1}{3} \sqrt{\delta} \), \( t = T_m \), \( |m| \geq 2 \), then

\[
\left| \frac{\zeta_K'}{\zeta_K}(s) \right| < 15 \delta^{-4/3} \log^2 \frac{K_2}{K_1} \left(|t|+1\right)^{K_3}.
\]

For the proofs of Lemmas 6 and 7 see [4].

Lemma 8. If \( 0 < \delta \leq \left(\frac{1}{t'}\right)^2 \), \( 1 < \sigma \leq \frac{1}{3} \), \( t > 1 \) and \( l \geq 2 \) is a positive integer, then

\[
\left| \left( -1 \right)^l \sum_{n \leq x} \frac{G(n)}{n^s} \frac{\log^l(n)}{(1+s)^{l+1}} \right| < \frac{\varepsilon^{s-1}}{(l+1)!} + \sum_{s} \frac{\varepsilon^{s-1}}{(1-s)^{l+1}} \leq \frac{\varepsilon^{s-1}\log^l \frac{K_2}{K_1} \left(|t|+1\right)^{K_3}}{\min(1, (\sigma-\delta)^{2l+2})}
\]

where \( \sigma_0 = \frac{1}{2} \sqrt{\delta} \) and the sum is taken over all zeros of \( \zeta_K(s) \) lying to the right of the line \( L \).

This lemma can be proved by following mutatis mutandis Appendix V of [3].

4. We pass over to the proof of Theorem 1. As in [1], pp. 60–62, we can prove that

\[
\frac{\zeta_K'}{\zeta_K}(s) + \frac{1}{s-1} = O\left( \frac{1}{c^6} \log^2 \frac{K_2}{K_1} \left(|t|+1\right)^{K_3} \right)
\]

in the region

\[
1 - \alpha_0 t \leq \sigma \leq 1 + \alpha t \text{,} \quad |t| \geq T_2,
\]

\[
1 - \alpha_0 t \leq \sigma \leq 1 + \alpha t \text{,} \quad |t| \leq T_2,
\]

where the constant in (4.1) depends only on \( \eta(t) \) and \( a \); \( T_2 \) depends on \( \eta(t) \).

From Lemma 4 and (4.1)–(4.2) follows the estimate

\[
\sum_{n \leq x} (x-n)G(n) = \frac{x^2}{2} + O\left( \frac{x^{\log^2 \left(|t|+1\right)}}{c^6} \exp(-\alpha_0 t \log(x)) \right).
\]

The constant in (4.3) depends on \( a \) and \( t \) only. Using the relation between \( \sum_{n \leq x} (x-n)G(n) \) and \( \sum_{n \leq x} G(n) - x \), we get, as in [1], p. 64, the estimate (2.2).

5. Proof of Theorem 2 (compare [6], pp. 151–152). Put \( t \geq 2 \). By (1.3) we have for \( n > 1 \)

\[
G(n) = D(n, K) - D(n-1, K) + 1.
\]

Hence

\[
\left| \sum_{N_1 < n \leq N_2} G(n) \exp(-it \log n) \right| \leq \left| \sum_{N_1 < n \leq N_2} \exp(-it \log n) \right| + \left| \sum_{N_1 < n \leq N_2} (D(n, K) - D(n-1, K)) \exp(-it \log n) \right| = I_1 + I_4.
\]

We choose \( N_1, N_2 \) so large that

\[
\omega_1^{-1}(\log^2 \alpha_0) \leq N/2 < N_1 < N_2 < N.
\]

Then by

\[
\omega_1(1+t) < \log(1+t) < \log^2 \alpha_0
\]

we get

\[
I_1 < \frac{c_2 \log^2 \left(|t|+1\right)}{\log t} \frac{N}{t}
\]

(compare [5]).

From the estimate of \( I_1 \) (see [6], p. 158) and (5.3) we get

\[
I_1 < \frac{N}{t}.
\]
Hence by (5.1) it follows that

\[
\sum_{N_1 < n < N_2} G(n) \exp(-it\log n) < c_{16} \frac{s^2 \log^2(|\sigma|+1)}{c_9} \cdot \frac{N^{1-\varepsilon}}{t}.
\]

Suppose that

\[
1 < \sigma < 3/2.
\]

By partial summation and (5.6) we have

\[
\sum_{N_1 < n < N_2} G(n)n^{-it} < c_{16} \frac{s^2 \log^2(|\sigma|+1)}{c_9} \cdot \frac{N^{1-\varepsilon}}{t}.
\]

We choose

\[
\eta > \eta_1, \quad \eta > \eta_3 \log \log n
\]

and apply the inequality (5.8) for

\[
N_1 = \eta \cdot 2^i, \quad N_2 = \eta \cdot 2^{i+1}, \quad i = 0, 1, 2, \ldots
\]

Hence

\[
\sum_{n \leq \eta} G(n)n^{-it} < c_{16} \frac{s^2 \log^2(|\sigma|+1)}{c_9} \cdot \frac{\eta^{1-\varepsilon}}{t(\sigma-1)}.
\]

We choose further

\[
\varepsilon > \eta_1, \quad \log \log n
\]

Denoting by \( l \) a positive integer and following [6], p. 154, we get by (5.10) the estimate

\[
\sum_{n \leq \eta} \frac{G(n)}{n^it\sigma} \log \frac{n}{\varepsilon} < c_{17} \frac{s^2 \log^2(|\sigma|+1) (l+1)! \varepsilon^{1-\varepsilon}}{t(\sigma-1)^{1+\varepsilon}}.
\]

Hence by Lemma 8 with \( \delta = (\delta_0)^2 \) and by (5.3) we get

\[
\sum_{\delta > \delta_0} \xi^{-s} < c_{18} \left( \xi^{1-\varepsilon} \frac{s^2 \log^2(|\sigma|+1) t}{(\sigma-\delta)^{1+\varepsilon}} + \frac{s^2 \log^2(|\sigma|+1) t}{(\sigma-1)^{1+\varepsilon}} \right) < c_{18} \frac{s^2 \log^2(|\sigma|+1) t}{(\sigma-1)^{1+\varepsilon}} \cdot \frac{\varepsilon^{1-\varepsilon}}{t(\sigma-1)^{1+\varepsilon}}.
\]

Let us suppose now that our theorem is not true. Hence there exist such zeros

\[
\sigma^* = \sigma^* + \delta^*, \quad \delta^* \to \infty,
\]

that

\[
\sigma^* > 1 - \frac{\log t^*}{400 \log \omega \log (\log t^*/\log \log \log t^*)},
\]

\[
\varepsilon^* = \max \left( \frac{\varepsilon^*}{\delta_0}, \log (|\sigma|+1) \right), \quad \delta^* = \delta^* + |\delta^*|, \quad \eta^* = \eta^* + |\eta^*|, \quad |\delta^*| + |\eta^*|.
\]

Putting in the estimate (5.13)

\[
\sum_{\delta > \delta_0} \xi^{-s} = \eta^* + |\delta^*| = \delta^* + |\delta^*| + |\eta^*|,
\]

where

\[
\xi = \exp \left( |t| + \frac{1}{2} \right),
\]

\[
\log t^* = 1 < 2 \leq \frac{x}{400 \log \omega \log (\log t^*/\log \log \log t^*)},
\]

\[
\log \omega \log (\log t^*/\log \log \log t^*)
\]

we can verify without difficulty that (5.7) and (5.11) are satisfied. Multiplying both sides of (5.13) by

\[
|\xi^* - \xi^*| = |\xi^* - \xi^*| = \xi^* - \xi^* = |\xi^* - \xi^*| + |\xi^* - \xi^*|,
\]

we have

\[
\sum_{\delta > \delta_0} \left( \xi^* - \xi^* \right)^{1+\varepsilon} = \xi^* - \xi^* = \xi^* - \xi^* = \xi^* - \xi^* + |\delta^*|.
\]

In virtue of (5.14), (5.16) and (5.18) it follows that

\[
\left( \frac{\sigma^* - \sigma}{\sigma - 1} \right)^{1+\varepsilon} = \left( \frac{1 - e - \sigma}{e - 1} \right) \leq \frac{1}{2} \log x.
\]

If the conditions (5.13) and \( \delta^* > \delta_0(c_2) \) are satisfied, we get by the estimate (5.20) the following inequality:

\[
\left| \sum_{\delta > \delta_0} \xi^{-s} \left( \frac{\sigma^* - \sigma}{\sigma - 1} \right)^{1+\varepsilon} \right| < e^{-x/2} \xi^{1-\varepsilon}.
\]

By virtue of Lemma 6 we get, similarly to [6], p. 156, the estimates

\[
\left| \sum_{\delta > \delta_0} \xi^{-s} \left( \frac{\sigma^* - \sigma}{\sigma - 1} \right)^{1+\varepsilon} \right| < c_{19} \xi^{1-\varepsilon},
\]

\[
\left| \sum_{\delta > \delta_0} \xi^{-s} \left( \frac{\sigma^* - \sigma}{\sigma - 1} \right)^{1+\varepsilon} \right| < c_{20} \xi^{1-\varepsilon},
\]

\[
\left| \sum_{\delta > \delta_0} \xi^{-s} \left( \frac{\sigma^* - \sigma}{\sigma - 1} \right)^{1+\varepsilon} \right| < c_{21} \xi^{1-\varepsilon}.
\]
From the above estimates and (5.21) it follows, for \( t^* > c_{34}(\epsilon), \) that

\[
V = \left| \sum_{1 < \epsilon_1 < \ldots < \epsilon_k < 1} \left( e^{it(\epsilon - \epsilon^*)} \frac{s_1 - \epsilon^*}{s_1 - \epsilon} \right)^{k+2} \right| < \frac{c_{1}^{1 - \epsilon^*}}{t^{1/3}}.
\]

We estimate the sum \( V \) from below by the use of Lemma 1. We choose
\[
\epsilon_j = \frac{s_1 - \epsilon^*}{s_1 - \epsilon} \exp\{1(\epsilon - \epsilon^*)\},
\]
and

\[
m = \log t^*.
\]

The region
\[
1 - 3(s_2 - s_1) \leq \sigma \leq 1, \quad |t - t^*| \leq 6(s_2 - s_1)
\]
is contained in the circle \(|s - s_1| \leq 8(s_2 - s_1)|. Hence denoting by \( N_1 \) the number of roots of \( \zeta_N(s) \) in this circle, and using Lemma 5 and the definition of \( \eta_1(t) \) with \( \epsilon_4 = \exp(-50g_{11}) \), we have, for \( s_0 = s_1 = s_1 + it^* \),

\[
\frac{\log t^*}{10 \log \omega_1^{-3}(\log t^{4/29})} \leq \mu.
\]

and under (5.15) and \( t^* > c_{34} \) the estimate

\[
N_1 < \frac{\log t^*}{14}.
\]

In virtue of Lemma 1 there exists an exponent \( l + 2 \) such that

\[
V > \left( \frac{1}{48s^2} \frac{\log t^*}{28 \log t^*} \right)^{l+2} > \frac{1}{t^{2.66}}.
\]

From (5.22) and (5.24) it follows that

\[
1 - \sigma^* > \frac{1}{400} \frac{\log t^*}{\log \omega_1^{-1}(\log t^{4/29})}
\]

and this contradicts (5.14). This completes the proof of Theorem 2.

References