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On pairings of the first $2n$ natural numbers

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Introduction. In proposing a research problem [2], Mok-Kong Shen and Tsen-Pao Shen noted that the first $2n$ positive integers may be grouped in n pairs, $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$, with $a_i < b_i$ and conjectured that for $n > 2$, there exists a pairing such that the $2n$ numbers $b_i + a_i$ and $b_i - a_i$ are all different. We say that a pairing of any $2n$ distinct positive integers is *acceptable* if these conditions are satisfied.

A program devised by Mr. James C. Fortson for an IBM 360, Model 65, has produced all acceptable pairings of $\{1, 2, \ldots, 2n\}$ for $n < 9$. The printout shows that if $A(n)$ designates the number of acceptable pairings of $\{1, 2, \ldots, 2n\}$, then $A(1) = 1$, $A(2) = 0$, $A(3) = 1$, $A(4) = 8$, $A(5) = 22$, $A(6) = 51$, $A(7) = 342$, and $A(8) = 2669$. This suggests that the difficulty in an existence proof stems from the fact that too many acceptable pairings exist for large values of n and that the problem may be simplified by putting on additional conditions.

M. Slater [4] has suggested that the Shen problem be attacked by requiring that $1 \leqslant a_i \leqslant n$ and conjectured that acceptable pairings satisfying this condition exist except for $n = 2, 3$, or 6. D. A. Klarner [1] noted that the Slater conjecture is related to the "problem of the reflecting queens" and used results of M. Kraitchik to construct all favorable examples for $n = 4, 5, 7$, and 8. J. D. Sebastian [3] used a computer to construct a favorable example in each of the cases $n = 9, 10, 11, \ldots, 27$.

If K_{2n} is a set of $2n$ distinct integers, a *pairing* of K_{2n} is a collection of pairs $\{(a_i, b_i) \mid i \in [1, n]\}$ such that $a_i < b_i$ for all i, $\{a_i, b_i\} \subset K_{2n}$ and each element of K_{2n} occurs in some pair. A pairing such that each of the sets $\{b_i + a_i\}$ and $\{b_i - a_i\}$ is a complete residue system, modulo n, is a good candidate to be acceptable. In this paper the Shen question is given an affirmative answer by studying pairings such that

(∗) each of the sets $\{a_i\}, \{b_i\}, \{b_i + a_i\}, \{b_i - a_i\}$ is a complete residue system, modulo n,

and

(#) $$b_i \equiv 2a_i, \text{ modulo } n.$$

For any n which is relatively prime to 6, there is a unique acceptable pairing of $[1, 2n]$ satisfying (*) and (#). For n sufficiently large, this pairing C_n can be modified slightly to construct acceptable pairings of $[1, 2n]$ for n of the form $6k+2$, $6k+3$, $6k$, and $6k+4$. Descriptions of acceptable pairings for the early cases are provided.

In the first section, two sequences α, β of zeroes and ones are defined and studied. Specific properties of pairings satisfying (*) and (#) are exhibited in the second section to simplify the proof of the main theorem, and its consequences.

In addition to the usual notation: for two positive integers s and t,

$$[s, t] = \{i \mid i \text{ is an integer, } s \leqslant i \leqslant t\};$$

$$\langle e, f\rangle = \{i \mid i \text{ is an integer, } e < i < f\},$$

where e and f are non-integral rational numbers. The symbols $\langle e, t]$ and $[s, f\rangle$ have the obvious meanings.

1. Two sequences of zeroes and ones. The construction of acceptable pairings of $[1, 2n]$ for n and 6 relatively prime depends on the properties of two sequences of zeroes and ones, α and β.

For n and 6 relatively prime, the sequences α and β are defined on $D = [1, n/3\rangle$ by the equations A_i and B_i, where for each $i \in D$, A_i and B_i are given by:

i	*i is odd*	*i is even*
A_i	$\alpha(i) = 0$	$\alpha(i) = 1-\beta(i/2)$

i	$i \in \langle n/6, n/3\rangle$	$i \in \langle n/9, n/6\rangle$
B_i	$\beta(i) = 1$	$\beta(i) = \alpha(i)$

i	$i \in [1, n/9\rangle$
B_i	$\beta(i) = 1+(\alpha(i)-1)(\beta(3i)-\alpha(3i))$

LEMMA 0. *For every $i \in D$, the equations A_i, B_i define $\alpha(i)$ and $\beta(i)$ as integers such that $0 \leqslant \alpha(i) \leqslant \beta(i) \leqslant 1$. Furthermore for $i \in [1, n/9\rangle$, $\beta(i)+\alpha(i) \neq \beta(3i)-\alpha(3i)$.*

The proof is based on a decomposition of D into disjoint subsets: $D = D_0 \cup D_1 \cup D_2 \cup \ldots$, where $D_t = \{i \mid i \in D \text{ and } i = 2^t u, \text{ is odd}\}$.

(S_v) *If for every $i \in D_t$, $\alpha(i)$ is defined by A_i and $\alpha(i) \in \{0, 1\}$, then for every $i \in D_t \cap \langle n/3^{v+1}, n/3^v\rangle$, $\beta(i)$ is defined by B_i as an integer and $\alpha(i) \leqslant \beta(i) \leqslant 1$.*

The proof is by induction on v.

(S_1) Consider $i \in D_t \cap \langle n/9, n/3\rangle$. If $i \in \langle n/6, n/3\rangle$, then B_i defines $\beta(i) = 1$ and $\alpha(i) \leqslant \beta(i) = 1$ since $\alpha(i) \in \{0, 1\}$. If $i \in \langle n/9, n/6\rangle$, then B_i gives $\beta(i) = \alpha(i)$, $\beta(i)$ is an integer and $\alpha(i) = \beta(i) \leqslant 1$.

(S_k) $\to$ (S_{k+1}) Suppose now that for $i \in D_t \cap \langle n/3^{k+1}, n/3^k\rangle$, $\beta(i)$ is defined by B_i as an integer and $\alpha(i) \leqslant \beta(i) \leqslant 1$, and consider $j \in D_t \cap \langle n/3^{k+2}, n/3^{k+1}\rangle$. It follows that $j \in [1, n/9\rangle$, $3j \in D_t \cap \langle n/3^{k+1}, n/3^k\rangle$ and by the induction assumption, $\beta(3j)$ is defined by B_{3j} as an integer such that $\alpha(3j) \leqslant \beta(3j) \leqslant 1$. Then $\beta(3j)-\alpha(3j) \in \{0, 1\}$ and from B_j for $j \in [1, n/9\rangle$, $\beta(j)$ is defined and $\beta(j) = 1$ or $\beta(j) = \alpha(j)$. In either case, $\beta(j)$ is an integer and $\alpha(j) \leqslant \beta(j) \leqslant 1$.

Since for each $i \in D_t$ there is a v such that $i \in \langle n/3^{v+1}, n/3^v\rangle$, this may be restated:

(0.1) *If for every $i \in D_t$, $\alpha(i)$ is defined by A_i and $\alpha(i) \in \{0, 1\}$, then $\beta(i)$ is defined by B_i for every $i \in D_t$, $\beta(i)$ is an integer and $0 \leqslant \alpha(i) \leqslant \beta(i) \leqslant 1$.*

(0.2) *For any t, the equation A_i defines $\alpha(i)$ for every $i \in D_t$ and $\alpha(i) \in \{0, 1\}$.*

The proof is by induction on t.

For $t = 0$, the equation A_i for i odd defines $\alpha(i) = 0$.

Assume now that for every $i \in D_k$, $\alpha(i)$ is defined by A_i and $\alpha(i) \in \{0, 1\}$. Then by (0.1), for every $i \in D_k$, $\beta(i)$ is defined by B_i and $\beta(i) \in \{0, 1\}$. Consider $j \in D_{k+1}$. Then j is even, $j/2 \in D_k$ and by the induction hypothesis, $\beta(j/2)$ is defined by $B_{j/2}$ and $\beta(j/2) \in \{0, 1\}$. It follows that $\alpha(j)$ is defined by A_j: $\alpha(j) = 1-\beta(j/2)$ and that $\alpha(j) \in \{0, 1\}$.

Since $D = D_0 \cup D_1 \cup D_2 \cup \ldots$, the first assertion in Lemma 0 follows from (0.2) and (0.1). Since for $i \in [1, n/9\rangle$, $\beta(3i)-\alpha(3i) \in \{0, 1\}$, the second assertion follows from B_i by considering these two cases.

As with sine and cosine, a host of identities may be worked out for α and β. Usually, only the defining properties are needed in this article. However, more specific information is required in the proof of Theorem 5. These properties are listed below.

COROLLARY 0. *For the sequences α and β,*

(0.3) $$i \in \langle 1, n/9\rangle,\ \alpha(i) = \alpha(3i) = 0 \to \beta(i) = 1-\beta(3i),$$

(0.4) $$i \in D_{2s} \to \alpha(i) = 0,$$

(0.5) $$i \in D_{2s} \cap [1, n/9\rangle \to \beta(i) = 1-\beta(3i),$$

(0.6) $$i \in D_{2s-1} \cap [1, n/9\rangle \to \alpha(i) = 1-\alpha(3i),$$

(0.7) $$i \in D_{2s-1} \cap \langle n/9, n/3\rangle \to \alpha(i) = 1,$$

(0.8) $$i \in \langle n/27, n/9\rangle \to \alpha(i) = 0.$$

The non-trivial proofs are indicated below.

The statement (0.4) is proved by induction on s. For $s = 0$, this is simply A_i for i odd.

Suppose then that $\alpha(i) = 0$ for all $i \in D_{2k}$ and consider $j \in D_{2(k+1)}$. Then $j = 4m$, where $\{m, 3m\} \subset D_{2k}$ and $\alpha(m) = \alpha(3m) = 0$ by the induction hypothesis.

If $3m \in \langle n/6, n/3\rangle$, then by B_{3m}, $\beta(3m) = 1$, and from $\alpha(3m) = 0$, $\beta(3m)-\alpha(3m) = 1$ and by B_m, $\beta(m) = \alpha(m)$. Since $\alpha(m) = 0$, $\beta(m) = 0$. Then A_{2m} gives $\alpha(2m) = 1$ and $\alpha(2m) \leqslant \beta(2m)$ gives $\beta(2m) = 1$. Finally, $\beta(2m) = 1$ and A_{4m} gives $\alpha(j) = \alpha(4m) = 0$.

Indeed, even if $m \in [1, n/6\rangle$, $\beta(m) = 0 \to \alpha(4m) = 0$, as above.

There remains the case in which $3m \in [1, n/6\rangle$ and $\beta(m) = 1$. In this case, $6m \in D$. By $\alpha(m) = \alpha(3m) = 0$ and (0.3), $\beta(3m) = 0$ and by A_{6m}, $\alpha(6m) = 1$. Since $\alpha(6m) \leqslant \beta(6m) \leqslant 1$, $\beta(6m) = 1$ and $\beta(6m)-\alpha(6m) = 0$. By B_{2m}, $\beta(2m) = 1$ and from A_{4m}, $\alpha(j) = 1-\beta(2m) = 0$.

The statement (0.7) follows from two observations:

(a) $i \in D_{2s-1} \cap \langle 2n/9, n/3\rangle \to \alpha(i) = 1$.

Proof. $i \in D_{2s-1} \cap \langle 2n/9, n/3\rangle \to i/2 \in D_{2s-2} \cap \langle n/9, n/6\rangle$, $\alpha(i/2) = 0$ by (0.4) and hence $\beta(i/2) = 0$ by $B_{i/2}$. Then $\alpha(i) = 1$ by A_i.

(b) $i \in D_{2s-1} \cap \langle n/9, 2n/9\rangle \to \alpha(i) = 1$.

Proof. $i \in D_{2s-1} \cap \langle n/9, 2n/9\rangle$ implies that $3i/2 \in D_{2s-2} \cap \langle n/6, n/3\rangle$, that $\alpha(3i/2) = 0$ by (0.4) and $\beta(3i/2) = 1$ by $B_{3i/2}$. Thus $\beta(3i/2)-\alpha(3i/2) = 1$, and $\beta(i/2) = \alpha(i/2)$ by $B_{i/2}$. But $i/2 \in D_{2s-2}$ and $\alpha(i/2) = 0$ by (0.4). Hence $\beta(i/2) = 0$ and $\alpha(i) = 1$ by A_i.

2. Special pairings of $[1, 2n]$.

LEMMA 1. *If n is odd and $Q_n = \{(a_i, b_i)|\ i \in [1, n]\}$ is a collection of pairs such that $\{a_i, b_i\} \subset [1, 2n]$ and $a_i \equiv i$, $b_i \equiv 2i$, modulo n, then Q_n is a pairing of $[1, 2n]$ if and only if*

(1.1) $$i \in \langle n/2, n] \to (a_i, b_i) = (i, 2i),$$

(1.2) $$i \text{ is odd} \to a_i = i,$$

(1.3) $$i \in [1, n/2\rangle \to a_i < b_i \quad \text{and} \quad b_i + a_{2i} = 4i+n.$$

If Q_n is a pairing of $[1, 2n]$, then $a_i < b_i$ for all i and no a_i is equal to a b_j.

For $i \in \langle n/2, n]$, $a_i \equiv i$ and $b_i \equiv 2i$, modulo n and $\{a_i, b_i\} \subset [1, 2n]$ imply that $a_i \in \{i, i+n\}$ and $b_i \in \{2i-n, 2i\}$. For $i \leqslant n$, $2i-n \leqslant i$ and $i+n \geqslant 2i$. Hence $a_i < b_i$ implies that $(a_i, b_i) = (i, 2i)$.

Thus for any pairing of $[1, 2n]$ satisfying the property $a_i \equiv i$ and $b_i \equiv 2i$, modulo n, it follows that $(a_i, b_i) \in \{(i, 2i), (i, 2i+n), (i+n, 2i+n)\}$, that $b_i - a_i \in \{i, i+n\}$ and that $b_i + a_i \in \{3i, 3i+n, 3i+2n\}$.

If i is odd, then $(i+n)/2 \in \langle n/2, n]$ and by (1.1) $b_{(i+n)/2} = i+n$. Since $a_i \in \{i, i+n\}$ and Q_n is a pairing of $[1, 2n]$, $a_i \neq i+n$ and $a_i = i$.

For any pairing, $a_i < b_i$ and this holds in particular for $i \in [1, n/2\rangle$. Also for $i \in [1, n/2\rangle$, $a_{2i} \in \{2i, 2i+n\}$ and $b_i \in \{2i, 2i+n\}$ and $a_{2i} \neq b_i$ gives $a_{2i} + b_i = 4i+n$.

Conversely, for any collection Q_n with $a_i \equiv i$ and $b_i \equiv 2i$, modulo n, and $\{a_i, b_i\} \subset [1, 2n]$ which satisfies (1.1) and (1.3), $a_i < b_i$. If $m \in [1, 2n]$ and $m \equiv i$, modulo n, where i is odd, then $m \in \{i, i+n\}$. If $m = i$, then $m = a_i$ by (1.2) and if $m = i+n$, $m = b_{(i+n)/2}$, by (1.1). If $m \equiv i$, modulo n, where i is even, then by (1.3) $a_i + b_{i/2} = 2i+n$ and $m = a_i$ or $b_{i/2}$. Since each $m \in [1, 2n]$ occurs in a pair of Q_n, Q_n is a pairing of $[1, 2n]$. For any such pairing,

(1.4) $$\text{If } i \in \langle n/4, n/2\rangle \to b_i = 2i+n.$$

If $i \in \langle n/4, n/2\rangle$, then $2i \in \langle n/2, n\rangle$ and, $a_{2i} = 2i$ and $b_i = 2i+n$ from (1.3).

For these pairings, $b_i - a_i \equiv i$ and $b_i + a_i \equiv 3i$, modulo n. If n and three are relatively prime, then no two sums or differences are congruent, modulo n, and the sums are distinct and the differences are distinct. We might well ask what additional conditions are imposed by requiring that Q_n be acceptable. Instead, it turns out that an apparently milder condition yields enough information.

LEMMA 2. *For n and 6 relatively prime and $P_n = \{(a_i, b_i)\}$ a pairing of $[1, 2n]$ such that $a_i \equiv i$ and $b_i \equiv 2i$, modulo n, and $b_i + a_i \neq b_{3i} - a_{3i}$ for $i \in [1, n/3\rangle$, then*

(2.1) $$(b_{3i} - a_{3i} - 3i)(b_i - a_i - i) = (b_i - 2i - n)(b_{3i} - a_{3i} - 3i - n),$$

(2.2) $$i \in \langle n/6, n/3\rangle \to b_i = 2i+n,$$

(2.3) $$i \in \langle n/3, n/2\rangle \to a_i = i,$$

(2.4) $$i \in \langle n/9, n/6\rangle \to b_i = a_i + i.$$

Suppose that $b_i + a_i \neq b_{3i} - a_{3i}$ for all $i \in [1, n/3\rangle$. If $b_{3i} - a_{3i} = 3i$, then $b_i + a_i \in \{3i+n, 3i+2n\}$, in either case, $b_i = 2i+n$ and (2.1) is true. If $b_{3i} - a_{3i} = 3i+n$, then $b_i + a_i \in \{3i, 3i+2n\}$, in either case $b_i = a_i + i$, and (2.1) is true.

For (2.2), note that $3i \in \langle n/2, n\rangle$ and apply (1.1) and (2.1). For (2.3) and i even, note that $i/2 \in \langle n/6, n/4\rangle$ and apply (2.2) and (1.3). For (2.4), note that $3i \in \langle n/3, n/2\rangle$ and apply (1.4), (2.3), and (2.1).

The essential usefulness of Lemmas 1 and 2 in the sequel is that statements that integers are not equal are replaced by equations. These equations simplify the arguments to be made below.

3. The construction of acceptable pairings of $[1, 2n]$ for $n > 2$.

THEOREM 1. *If n and 6 are relatively prime and $P_n = \{(a_i, b_i)\}$ is a pairing of $[1, 2n]$ such that $a_i \equiv i$, $b_i \equiv 2i$, modulo n, and $b_i + a_i \neq b_{3i} - a_{3i}$, then P_n is the union of*

$$\{(i, 2i+n)|\ i \in \langle n/3, n/2\rangle\} \cup \{(i, 2i)|\ i \in \langle n/2, n]\}$$

and

$$\{(i + n\alpha(i), 2i + n\beta(i))|\ i \in [1, n/3\rangle\},$$

where α and β are the sequences defined above. Moreover, the above pairing is an acceptable pairing of $[1, 2n]$.

From (1.1), it follows that for $i \in \langle n/2, n]$, $(a_i, b_i) = (i, 2i)$ and from (1.4), (2.2), and (2.3) that $(a_i, b_i) = (i, 2i+n)$ for $i \in \langle n/3, n/2\rangle$.

For $i \in [1, n/3\rangle$, $(a_i, b_i) = (i+n\lambda(i), 2i+n\mu(i))$, where $\lambda(i)$ and $\mu(i)$ are integers and $0 \leqslant \lambda(i) \leqslant \mu(i) \leqslant 1$. Note the following implications of Lemmas 1 and 2.

(1.2) $\to$ If i is odd, then $\lambda(i) = 0$.

(1.3) $\to$ If i is even, then $\mu(i/2)+\lambda(i) = 1$.

(2.2) $\to$ If $i \in \langle n/6, n/3\rangle$, then $\mu(i) = 1$.

(2.4) $\to$ If $i \in \langle n/9, n/6\rangle$, then $\mu(i) = \lambda(i)$.

(2.1) $\to$ If $i \in [1, n/9\rangle$, then $\mu(i) = 1+(\lambda(i)-1)(\mu(3i)-\lambda(3i))$.

Thus λ and μ satisfy the definitions of α and β. For $i \in [1, n/3\rangle$,

$$(a_i, b_i) = (i+n\alpha(i), 2i+n\beta(i)).$$

This pairing will be designated by C_n. A check show that C_n satisfies the conditions of Lemma 1, and is thus a pairing of $[1, 2n]$.

For all $j \in [1, n]$, it is easy to see that $b_j - a_j < 3n/2$ and that for $i \in \langle n/3, n]$, $b_i + a_i > 3n/2$. Hence, if there is an i and a j such that $b_i + a_i = b_j - a_j$, $i \in [1, n/3\rangle$. Also $b_i + a_i = b_j - a_j$ implies that $3i \equiv j$, modulo n and since $3i < n$, that $3i = j$.

Now consider $i \in \langle n/6, n/3\rangle$ so that $b_i = 2i+n$ and $b_i + a_i \geqslant 3i+n$. In this case, $3i \in \langle n/2, n\rangle$ and $b_{3i} - a_{3i} = 3i$. Hence $b_i + a_i \neq b_{3i} - a_{3i}$.

Consider $i \in \langle n/9, n/6\rangle$ so that $b_i + a_i \in \{3i, 3i+2n\}$. In this case $3i \in \langle n/3, n/2\rangle$ and $b_{3i} - a_{3i} = 3i+n$. Hence, $b_i + a_i \neq b_{3i} - a_{3i}$.

Finally, if $i \in [1, n/9\rangle$, $b_i + a_i = b_{3i} - a_{3i}$ implies that $\beta(i)+\alpha(i) = \beta(3i)-\alpha(3i)$ and this contradicts Lemma 0.

Thus no sum of a pair in C_n is equal to the difference of a pair in C_n. Since the set of sums [differences] is a complete residue system, modulo n, C_n is acceptable.

This is an illustration of the familiar experience that an existence proof is most easily constructed when there is a unique affirmative example.

It is possible to give explicit formulas for the sequences α and β, but it is neither interesting or useful. For n less than a hundred, α and β are quickly computed from the defining equations and these latter are all that is needed in the later arguments. (Except in the proof of Theorem 5, which does require Corollary 0.)

Acceptable pairings of $[1, 2(6k+2)]$ for $k > 0$.

THEOREM 2. *For $n \equiv 1$, modulo 6, and $n \geqslant 7$ and C_n the pairing of Theorem* 1, *let*

$$C_{n-1} = C_n \setminus \{((n+1)/2, n+1)\}$$

and let

$$Q = \{((n+1)/2, n+1+n\alpha(2)), (n+1+n\beta(1), 2n+2)\}.$$

Then $C_{n-1} \cup Q$ is an acceptable pairing of $[1, 2(n+1)]$.

Since $\alpha(2)+\beta(1) = 1$, $C_{n-1} \cup Q$ is a pairing of $[1, 2(n+1)]$.

As a subset of the acceptable pairing C_n, C_{n-1} is acceptable and it may be checked that Q is acceptable for $n \geqslant 7$. It remains to show that no sum or difference in Q is a sum or difference in C_{n-1}. Since no two sums [differences] in C_{n-1} are congruent, modulo n, to show that an integer m is not equal to a sum [difference] in C_{n-1}, it is sufficient to exhibit a sum [difference] in C_{n-1} which is congruent, modulo n, to m and is not equal to m.

The sum $3n+3+n\beta(1)$ is not equal to a sum or difference in C_{n-1}. Indeed, the greatest sum in C_{n-1} is $3n$.

The difference $n+1-n\beta(1)$ is not equal to a sum or difference in C_{n-1}. In C_{n-1}, $b_1 - a_1 = n\beta(1)+1$ and this is not equal to $n+1-n\beta(1)$. Since $n \equiv 1$, modulo 6, $(2n+1)/3$ is an integer and

$$b_{(2n+1)/3} + a_{(2n+1)/3} = 2n+1 > n+1-n\beta(1).$$

The difference $(n+1)/2+n\alpha(2)$ is not equal to a sum or difference in C_{n-1}.

Since C_{n-1} is constructed by deleting $((n+1)/2, n+1)$ from C_n, there is no pair in C_{n-1} for which the difference is congruent, modulo n, to $(n+1)/2+n\alpha(2)$. Since $n \equiv 1$, modulo 6, $(5n+1)/6$ is an integer and

$$b_{(5n+1)/6} + a_{(5n+1)/6} = (5n+1)/2 > (n+1)/2+n\alpha(2).$$

The sum $(3n+3)/2+n\alpha(2)$ is not equal to a sum or difference in C_{n-1}. Since n is odd,

$$(n+3)/2 \in \langle n/2, n] \quad \text{and} \quad b_{(n+3)/2} - a_{(n+3)/2} = (n+3)/2,$$

which is less than and congruent to $(3n+3)/2+n\alpha(2)$. Since C_{n-1} was constructed by deleting $((n+1)/2, n+1)$ from C_n, there is no pair in C_{n-1} whose sum is congruent, modulo n, to $(3n+3)/2+n\alpha(2)$.

Acceptable pairings of $[1, 2(6k+3)]$.

THEOREM 3. *For $n \equiv 1$, modulo 6, let C_n and C_{n+1} be the acceptable pairings of Theorems* 1 *and* 2. *For $n \geqslant 13$, there exists an integer m such that*

(3.0) $$m \equiv 0, \text{ modulo } 3;$$

(3.1) $$a_m = m, \quad a_{(m+3)/3} = (m+3)/3 \quad \text{in } C_n;$$

(3.2) $$m \in \langle (n+8)/4, (5n+24)/12\rangle.$$

For any such m,

$$H_{n+2} = (C_{n+1} \setminus \{(m, 2m+n)\}) \cup \{(m, 2n+3), (2m+n, 2n+4)\}$$

is an acceptable pairing of $[1, 2(n+2)]$.

For $n = 13, 19, 25, 31, 37$, and 43; $m = 6, 9, 9, 12, 12$, and 18, respectively satisfy (3.0), (3.1), and (3.2). For $n \geq 49$, $(5n+24)/12 - n/3 > 6$, and there is an m such that $m \equiv 0$, modulo 6 and $m \in \langle n/3, n/2\rangle \cap \cap \langle (n+8)/4, (5n+24)/12\rangle$. Such an m satisfies (3.0), (3.1), and (3.2).

It may be checked as in the proof of Theorem 2 that H_{n+2} is a pairing of $[1, 2(n+2)]$ and is acceptable. Theorem 3 furnishes an acceptable pairing of $[1, 2(6k+3)]$ for $k \geq 2$.

Finally, for $[1, 2\cdot 3]$ and $[1, 2\cdot 9]$, acceptable pairings are given by

$$\{(2,3), (1,5), (4,6)\}$$

and

$$\{(1,12), (2,18), (3,15), (4,11), (5,14), (6,16), (7,10), (8,13), (9,16)\}.$$

Acceptable pairings of $[1, 2(6k)]$ for all k. The pairing

$$\{(1,10), (2,8), (3,4), (5,7), (6,11), (9,12)\}$$

is acceptable for $[1, 2\cdot 6]$.

For $k \in [2, 8]$, there is a unique acceptable pairing of $[1, 2(6k)]$ of the form

$$\{(a_i, b_i) |\ i \in [1, 6k],\ a_i \equiv i,\ b_i \equiv 2i-1, \text{ modulo } (6k+1)\}.$$

This may be verified in each of the seven cases by using the ideas of the proofs of Lemmas 1 and 2.

For $k \geq 9$, acceptable pairings of $[1, 2(6k)]$ are given by the following:

THEOREM 4. *Let $n \equiv 5$, modulo 6. If $n \geq 53$, there is an m such that $m \equiv 2$, modulo 3, and $m \in \langle (5n+12)/6, (8n+18)/9\rangle$. For such an n and m and C_n the pairing of Theorem 1,*

$$H_{n+1} = (C_n \setminus \{(m, 2m)\}) \cup \{(m, 2n+2), (2m, 2n+1)\}$$

is an acceptable pairing of $[1, 2(n+1)]$.

For $n = 53$, $m = 47$ satisfies the conditions of the theorem. For $n \geq 59$, $(8n+18)/9 - (5n+12)/6 > 3$ and there exists an m which is congruent to 2, modulo 3 and satisfies the required inequality.

Using the methods of Theorem 2, it may be checked that H_{n+1} is a pairing of $[1, 2(n+1)]$ and is acceptable.

Acceptable pairings of $[1, 2(6k+4)]$ for all k. For $k \geq 74$, acceptable pairings of $[1, 2(6k+4)]$ are given by the following:

THEOREM 5. *If $n \equiv 5$, modulo 6, and $n \geq 449$, there is an odd j such that $6j \in \langle (11n+9)/36, (4n-3)/12\rangle$. For every such j and C_n the pairing of Theorem 1, the pairing H_{n-1},*

$$(C_n \setminus \{(6j+n, 12j+n), (\overline{n-1}/2, 2n-1), (n, 2n)\}) \cup \{(n, 6j+n), (\overline{n-1}/2, 12j+n)\}$$

is an acceptable pairing of $[1, 2(n-1)]$.

For $n = 449$, $6\cdot 23 \in \langle (11\cdot 449+9)/36, (4\cdot 449-3)/12\rangle$. For $n \geq 455$, $(4n-3)/12 - (11n+9)/36 > 12$ and there is a q such that $q \in \langle (11n+9)/36, (4n-3)/12\rangle$ and $q = 6(2s+1)$.

As in Theorem 2, it may be checked that H_{n-1} is a pairing of $[1, 2(n-1)]$ and is acceptable.

Using a program devised by B. J. Williams, pairings of $[1, 2(6k+4)]$ of the form

$$\{(a_i, b_i) |\ a_i \equiv i,\ b_i \equiv 2i-1, \text{ modulo } (6k+5),\ i \in [1, 6k+4]\}$$

were checked for $k \in [4, 73]$ by the IBM360, Model 65. The printout shows that for each k there exists one and only one acceptable pairing of this type, except for $k \in [9, 13] \cup [55, 61]$ and that in these latter cases there is none.

Theorem 5 provides an acceptable pairing of $[1, 2(6k+4)]$ for $k \in [57, 61]$ since $6\cdot 19 \in \langle (11n+9)/36, (4n-3)/12\rangle$ for $n \in \{347, 353, 359, 365, 371\}$.

Favorable examples for $k \in [9, 13] \cup \{55, 56\}$ are listed below:

H_{58}: $(C_{55} \setminus \{(57, 59), (28, 56)\})$ and $\{(28, 111), (56, 115), (57, 113), (59, 116), (112, 114)\}$.

H_{64}: $(C_{61} \setminus \{(63, 65), (31, 62)\})$ and $\{(31, 123), (62, 127), (63, 125), (65, 128), (124, 126)\}$.

H_{70}: $(C_{67} \setminus \{(69, 71), (34, 68)\})$ and $\{(34, 135), (68, 139), (69, 137), (71, 140), (136, 138)\}$.

H_{76}: $(C_{73} \setminus \{(75, 77)\})$ and $\{(148, 150), (75, 149), (77, 152), (147, 151)\}$.

H_{82}: $(C_{79} \setminus \{(81, 83)\})$ and $\{(160, 162), (81, 161), (83, 164), (159, 163)\}$.

H_{334}: $(C_{331} \setminus \{(166, 332)\})$ and $\{(166, 663), (332, 667), (664, 665), (666, 668)\}$.

H_{340}: $(C_{337} \setminus \{(169, 338)\})$ and $\{(169, 675), (338, 679), (676, 677), (678, 680)\}$.

Of course, for $k < 4$, acceptable pairings have appeared in the articles of Klarner and Sebastian.

The construction of acceptable pairings of $[1, 2n]$ when n and 6 are not relatively prime is not really so unnatural as it might appear. For example, in the case when n is of the form $6k+3$, it is natural to ask: Is it possible to delete a pair of the form $(m, 2m+n)$ from the acceptable pairing C_{n+1} of Theorem 2 and form an acceptable pairing of $[1, 2(n+2)]$ by pairing m with $2n+3$ and $2m+n$ with $2n+4$? Examination of this question does lead to the conditions of Theorem 3.

Of course, it would be nice to have a simpler solution for this simple problem of the Shens.

It has recently come to our attention that J. L. Selfridge [5] announced a different solution in 1963.

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Some estimates in the theory of Dedekind Zeta-functions

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1. Denote by K an algebraic number field, by ν and Δ the degree and the discriminant of the field K respectively and by $\zeta_K(s)$, $s = \sigma+it$, the Dedekind Zeta-function (see [2]).

The function $\zeta_K(s)$ is defined for $\sigma > 1$ by the absolute convergent series

$$\sum_{n=1}^{\infty} F(n)n^{-s},$$

where $F(n)$ denotes the number of ideals of the field K, having the norm equal to n.

The function $\zeta_K(s)$ can be continued over the whole complex plane as a regular function, except $s = 1$, where there is a simple pole.

In the region $\sigma > 1$

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{n=1}^{\infty} G(n)n^{-s} \tag{1.1}$$

where

$$G(n) = \sum_{(N\mathfrak{p})^m = n} \log N\mathfrak{p}$$

and the series in (1.1) is absolutely convergent in this region (see [2], p. 89).

A. Sokolovski [7] proved that $\zeta_K(s) \neq 0$ in the region

$$\sigma > 1 - \frac{c_1}{\log^{\gamma}|t|}, \quad |t| > c_2, \ \gamma > \tfrac{2}{3}, \tag{1.2}$$

where c_1, c_2 are constants depending on the field K. Theorem (1.2) is a deep improvement of E. Landau's classic result, $\gamma = 1$ (see [2], p. 105).

The subject of this note is an investigation of the equivalence between the domain (1.2), in which $\zeta_K(s) \neq 0$, and the estimate of the difference

$$\sum_{n \leqslant x} G(n) - x \stackrel{\text{def}}{=} \Delta(x, K), \tag{1.3}$$

which is the remainder in the Prime-ideal Theorem (see [4]).