On a restricted type of partitions into parts divisible by the least part

by

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1. $p_0(n)$ will denote the number of unrestricted partitions of $n$ and $p_r(n)$ those partitions of $n$, each of which possesses some specified property $c$. Here will be considered only three variations of $c$, namely:

(i) the property $r$ will signify that in each partition the g.c.d. of the summands occurs as a summand,

(ii) the property $s$ will imply that in the conjugate of each partition the g.c.d. of the summands presents itself as a summand,

(iii) the property $t$ will denote that each of the two properties $r$ and $s$ is present in each partition.

Recently, Chawla, LeVan and Maxfield [1] have considered the function $p_r$ and have shown that

$$(1) \quad p_r(n) = \sum_{d|n} p(d-1)$$

and that

$$(2) \quad p_r(n) \sim p(n-1), \quad \text{as} \quad n \to \infty.$$ 

They have also compiled a table of $(p_r(n) - p(n-1))$ for $n < 350$.

Evidently a one-one correspondence exists between a partition with the property $r$ and a partition possessing the property $s$. Consequently,

$$(3) \quad p_s(n) = p_r(n).$$

Here we obtain a relation for $p_t(n)$ similar to (1) and investigate a few simple properties of the function $p_t$. A table of $p_t(n)$ for $n \leq 100$ will also be found at the end of the paper. To facilitate comparison, the relevant values of $p_t(n)$ have also been included in the table.

2. Let $a_1^1 a_2^2 \ldots a_k^k$ be an arbitrary partition of a dictionary order, i.e., $a_1 > a_2 > \ldots > a_k$ and $l_1, l_2, \ldots, l_k > 0$. The conjugate partition in dictionary order is represented by

$$(l_1 + l_2 + \ldots + l_k)^{a_1^1} (l_1 + l_2 + \ldots + l_k)^{a_2^2} \ldots (l_1 + l_2 + \ldots + l_k)^{a_k^k}.$$
Since \((a_1, a_2, \ldots, a_i)\) is a summand, it follows that
\[a_i = (a_1, a_2, \ldots, a_i)\]
Similarly, \(I_i = (l_1, l_1+l_2, \ldots, l_1+l_2+\ldots+l_i) = (l_1, l_2, \ldots, l_i)\). Hence \(a_i I_i = a_i I_i\) if this partition possesses the property \(r\).

Three cases need to be considered:

(i) \(a_i I_i = n\), where \(a_i \geq 2\).

Here \(a_i I_i = a_i I_i\) for \(j = 1, 2, \ldots, i\).

We divide each \(a_i\) by \(a_i\) and each \(l_j\) by \(l_j\). The residual partition is a partition of \(d\) typified by the property that it has, as well as its conjugate, unity as the least summand. The number of such partitions of \(d\) is \(p(d-2)\). Further, the number of ways in which each such residual partition could have resulted in this way from the original partition is the number of ways in which \(a_i I_i\) can be expressed as an ordered pair \((a, b)\) such that \(a \cdot b = a_i I_i\). This number is \(I(a/b)\), where \(I(k)\) represents the number of divisors of \(k\). As a consequence, with each divisor \(d (\geq 2)\) of \(n\) are associated \(I(a/b) p(d-2)\) partitions of \(n\) with the property \(r\).

(ii) Even if the divisor \(d = 2\) of \(n\) exists, it has no significance and its contribution to \(p_i(n)\) is nil. The reason lies in the fact that there are no partitions of 2 with the property that it has, along with its conjugate, unity as the least summand.

(iii) If \(d = 1\), we have \(i = 1\) and the original partition is of the form \(a_i\) with \(a_i I_i = n\). The contribution of \(d = 1\) to \(p_i(n)\) is thus \(I(n)\).

Hence
\[
p_i(n) = I(n) + \sum_{d|n, d > 2} I(n/d) p(d-2).
\]  

3. There exist \(p(n) - p_i(n)\) partitions of \(n\) lacking the property \(r\). The conjugate of each such partition may or may not possess the property \(r\). It follows that
\[
p_i(n) = p_i(n) + (p(n) - p_i(n)) \quad \text{for} \quad n \geq 1,
\]
or that
\[
1 \geq p_i(n)/p_r(n) \geq 2 - p(n)/p_r(n) \quad \text{for} \quad n \geq 1.
\]

Since for \(k > 2\) each of the partitions \(3^k, 3^k, 3^k-1, 3^k+1\) has the property \(r\) and lacks the property \(i\), we may write
\[
1 > p_i(n)/p_r(n) > 2 - p(n)/p_r(n) \quad \text{for} \quad n > n_0.
\]

Also, by (3), \(p_i(n) \sim p(n-1)\) and by the Hardy-Ramanujan [2] asymptotic formula,
\[
p(n) \sim \frac{1}{4\sqrt{3}n} \exp\left(2\pi\sqrt{n/3}\right).
\]

We conclude that, as \(n \to \infty,\)
\[
1 > \frac{p_i(n)}{p_r(n)} \geq 1 - \frac{\pi}{\sqrt{6n}}.
\]
From (4), it follows that
\[
p(n-2) < p_i(n) \leq p(n) \quad \text{for all} \quad n,
\]
which implies that
\[
\lim_{n \to \infty} \frac{p_i(n)}{p(n-2)} = 1.
\]

If \(n\) be composite and \(q\) be the smallest odd prime divisor of \(n\), we have from (4)
\[
p_i(n) - p(n-2) = I(q) p\left(\frac{n}{q} - 2\right) + \text{smaller terms}.
\]

Hence may be deduced that, as \(n \to \infty\),
\[
p_i(n) - p(n-2) \sim \frac{q}{2\sqrt{3} \left(n/2q\right)} \exp\left(2\pi\sqrt{\left(n-2q\right)/6q}\right),
\]
provided \(n\) is neither prime nor twice a prime.

4. We state below a few congruence properties of \(p_i(n)\). These are direct consequences of the well-known congruence properties of \(p(n)\) given by Ramanujan [3] and of (4).

(i) If \(n = 7^k\),
\[
p_i(n) = k+1 \pmod{7} \quad \text{and} \quad p_i(n) = 8k+1 \pmod{49}, \quad \text{provided} \quad k \geq 1.
\]

(ii) If \(n\) be a prime of the form \(11^k + 8\) or \(23^k - 3\),
\[
p_i(n) = 2 \pmod{11}.
\]

(iii) If \(n\) be a prime of the form \(10^k + 1\),
\[
p_i(n) = 2 \pmod{5}.
\]

5. A table of \(p_i(n)\) and \(p_r(n)\) is appended below. It is valid for \(n \leq 100\). In the table is noticed the property that for \(k > 4\),
\[
p_i(2k) > p_r(2k-1) \quad \text{and} \quad p_r(2k+1) < p_r(2k).
\]
This is easily established for all sufficiently large \(k\) with the aid of (1) and (4).
Mischungsgeschwindigkeit für Ziffernentwicklungen nach reellen Matrizen

von

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§ 1. Definitionen. $I = (0, 1)^d$ sei der Einheitswürfel des $R^d$, $\mathcal{S}$ die $\sigma$-Algebra der Borelschen Teilmengen von $I$, $\nu$ das dimensonale Lebesgue-Maß auf $I$. $A$ sei eine nichtsingular reelle $(d \times d)$-Matrix, $\Delta$ der Betrag ihrer Determinante.

Wir definieren:
(a) $T : I \to I$, $T \nu = Ax \mod 1$,
(b) Ist $x \in \mathbb{R}$, so sei $[x]$ die größte ganze Zahl $\leq x$ und für
$$x = (a_1, \ldots, a_d) \in R^d \quad \text{sei} \quad [x] = ([a_1], \ldots, [a_d]) \in Z^d.$$

Weiter sei $M = A \cdot I \cap Z^d$. Für $i = 1, 2, \ldots$ erklären wir Funktionen $i_k : I \to M$

$$k_i(x) = [a_i], \quad k_i(x) = k_i(T^{i-1}x).$$

(c) Sind $a_1, a_2, \ldots, a_d \in M$, so sei
$$I(a_1, \ldots, a_d) = \{x \in I \mid k_i(x) = a_i, 1 \leq i \leq d\},$$
sofern die rechte Menge nicht leer ist. Die Mengen $I(a_1, \ldots, a_d)$ heißen Zylinder $\ast$-Ter von Ordnung und bilden eine Partition von $I$, die wir mit $\mathcal{S}$ bezeichnen wollen.

(d) Wir definieren folgende Teilmengen von $\mathcal{S}$:
$$\mathcal{F}_\delta = \{E \in \mathcal{S} : T^\delta E = I\}, \quad \mathcal{F}_\mathcal{S} = \mathcal{S} \setminus \mathcal{F}_\delta,$$
$$\mathcal{B}_\mathcal{S} = I(a_1, \ldots, a_d) \in \mathcal{F}_\mathcal{S}, I(a_1) \in \mathcal{F}_\mathcal{S}, \ldots, I(a_1, \ldots, a_d) \in \mathcal{F}_\mathcal{S},$$
$$\mathcal{B}_\mathcal{S} = I(a_1, \ldots, a_d) \in \mathcal{F}_\mathcal{S}, I(a_1, \ldots, a_{d-1}) \in \mathcal{F}_\mathcal{S}.$$