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On a restricted type of partitions into parts divisible by the least part

by

S. M. Kerawala (Karachi)

1. $p(n)$ will denote the number of unrestricted partitions of n and $p_c(n)$ those partitions of n, each of which possesses some specified property c. Here will be considered only three variations of c, namely:

(i) the property r will signify that in each partition the g.c.d. of the summands occurs as a summand,

(ii) the property s will imply that in the conjugate of each partition the g.c.d. of the summands presents itself as a summand,

(iii) the property t will denote that each of the two properties r and s is present in each partition.

Recently, Chawla, LeVan and Maxfield [1] have considered the function p_r and have shown that

$$p_r(n) = \sum_{d|n} p(d-1) \tag{1}$$

and that

$$p_r(n) \sim p(n-1), \text{ as } n \to \infty. \tag{2}$$

They have also compiled a table of $\{p_r(n) - p(n-1)\}$ for $n \leqslant 350$.

Evidently a one-one correspondence exists between a partition with the property r and a partition possessing the property s. Consequently,

$$p_r(n) = p_s(n). \tag{3}$$

Here we obtain a relation for $p_t(n)$ similar to (1) and investigate a few simple properties of the function p_t. A table of $p_t(n)$ for $n \leqslant 100$ will also be found at the end of the paper. To facilitate comparision, the relevant values of $p_r(n)$ have also been included in the table.

2. Let $a_1^{l_1} a_2^{l_2} \ldots a_i^{l_i}$ be an arbitrary partition of n dictionary order, i.e., $a_1 > a_2 > \ldots > a_i$ and $l_1, l_2, \ldots, l_i > 0$. The conjugate partition in dictionary order is represented by

$$(l_1 + l_2 + \ldots + l_i)^{a_i} (l_1 + l_2 + \ldots + l_{i-1})^{(a_{i-1} - a_i)} \ldots (l_1 l +_2)^{(a_2 - a_3)} l_1^{(a_1 - a_2)}.$$

Since $(a_1, a_2, \ldots, a_i)$ is a summand, it follows that

$$a_i = (a_1, a_2, \ldots, a_i).$$

Similarly, $l_1 = (l_1, l_1+l_2, \ldots, l_1+l_2+\ldots+l_i) = (l_1, l_2, \ldots, l_i)$. Hence $a_i l_1 | n$, if this partition possesses the property t.

Three cases need to be considered:

(i) $a_i l_1 d = n$, where $d > 2$.

Here $a_i | a_j$ and $l_1 | l_j$ for $j = 1, 2, \ldots, i$.

We divide out each a_j by a_i and each l_j by l_1. The residual partition is a partition of d typified by the property that it has, as well as its conjugate, unity as the least summand. The number of such partitions of d is $p(d-2)$. Further, the number of ways in which each such residual partition could have resulted in this way from the original partition is the number of ways in which n/d can be expressed as an ordered pair (x, y) such that $x \cdot y = n/d$. This number is $I(n/d)$, where $I(k)$ represents the number of divisors of k. As a consequence, with each divisor d (> 2) of n are associated $I(n/d)p(d-2)$ partitions of n with the property t.

(ii) Even if the divisor $d = 2$ of n exists, it has no significance and its contribution to $p_t(n)$ is nil. The reason lies in the fact that there are no partitions of 2 with the property that it has, along with its conjugate, unity as the least summand.

(iii) If $d = 1$, we have $i = 1$ and the original partition is of the form a^l with $al = n$. The contribution of $d = 1$ to $p_t(n)$ is thus $I(n)$. Hence

$$p_t(n) = I(n) + \sum_{d|n,\, d>2} I(n/d)p(d-2). \tag{4}$$

3. There exist $p(n) - p_r(n)$ partitions of n lacking the property r. The conjugate of each such partition may or may not possess the property r. It follows that

$$p_r(n) \geqslant p_t(n) \geqslant p_r(n) - \{p(n) - p_r(n)\} \quad \text{for} \quad n \geqslant 1,$$

or that

$$1 \geqslant p_t(n)/p_r(n) \geqslant 2 - p(n)/p_r(n) \quad \text{for} \quad n \geqslant 1.$$

Since for $k > 2$ each of the partitions $3^k 1, 3^k 2, 3^{k-1}21$ has the property r and lacks the property t, we may write

$$1 > p_t(n)/p_r(n) > 2 - p(n)/p_r(n) \quad \text{for} \quad n > n_0.$$

Also, by (2), $p_r(n) \sim p(n-1)$ and by the Hardy–Ramanujan [2] asymptotic formula,

$$p(n) \sim \frac{1}{4\sqrt{3}n} \exp\{2\pi\sqrt{(n/6)}\}.$$

We conclude that, as $n \to \infty$,

$$1 > p_t(n)/p_r(n) > 1 - \frac{\pi}{\sqrt{(6n)}}.$$

From (4), it follows that

$$p(n-2) < p_t(n) \leqslant p(n) \quad \text{for all } n,$$

which implies that

$$\lim_{n\to\infty} \frac{p_t(n)}{p(n-2)} = 1.$$

If n be composite and q be the smallest odd prime divisor of n, we have from (4)

$$p_t(n) - p(n-2) = I(q)p\left(\frac{n}{q} - 2\right) + \text{smaller terms}.$$

Hence may be deduced that, as $n \to \infty$,

$$p_t(n) - p(n-2) \sim \frac{q}{2\sqrt{3}(n-2q)} \exp\left\{2\pi\sqrt{\left(\frac{n-2q}{6q}\right)}\right\},$$

provided n is neither prime nor twice a prime.

4. We state below a few congruence properties of $p_t(n)$. These are direct consequences of the well-known congruence properties of $p(n)$ given by Ramanujan [3] and of (4).

(i) If $n = 7^k$,

$$p_t(n) \equiv k+1 \pmod 7 \quad \text{and} \quad p_t(n) \equiv 8k+1 \pmod{49}, \quad \text{provided } k \geqslant 1.$$

(ii) If n be a prime of the form $11^i + 8$ or $22i - 3$,

$$p_t(n) \equiv 2 \pmod{11}.$$

(iii) If n be a prime of the form $10k+1$,

$$p_t(n) \equiv 2 \pmod 5.$$

5. A table of $p_r(n)$ and $p_t(n)$ is appended below. It is valid for $n \leqslant 100$. In the table is noticed the property that for $k > 4$,

$$p_t(2k) > p_r(2k-1) \quad \text{and} \quad p_t(2k+1) < p_r(2k).$$

This is easily established for all sufficiently large k with the aid of (1) and (4).

n	$p_r(n)$	$p_t(n)$	n	$p_r(n)$	$p_t(n)$	n	$p_r(n)$	$p_t(n)$
1	1	1	35	12327	10167	69	3088740	2681279
2	2	2	36	15272	12942	70	3566804	3108267
3	3	3	37	17978	14885	71	4087969	3554347
4	5	5	38	22024	18575	72	4713747	4115608
5	6	5	39	26095	21755	73	5392784	4697207
6	11	11	40	31730	26901	74	6203668	5422553
7	12	9	41	37339	31187	75	7091218	6188422
8	20	19	42	45333	38522	76	8140291	7126355
9	25	20	43	53175	44585	77	9289145	8118342
10	37	32	44	64100	54528	78	10647924	9335761
11	43	32	45	75340	63512	79	12132165	10619865
12	70	65	46	90138	76763	80	13880556	12185788
13	78	58	47	105559	89136	81	15798937	13852620
14	114	95	48	126270	108049	82	18041667	15858850
15	143	113	49	147285	124771	83	20506256	18004329
16	196	168	50	175137	149845	84	23386871	20587753
17	232	178	51	204460	173883	85	26543897	23338831
18	330	281	52	241983	207554	86	30220533	26632830
19	386	299	53	281590	239945	87	34266683	30173383
20	530	448	54	332697	286056	88	38951788	34371349
21	641	510	55	386203	330001	89	44108110	38887675
22	836	691	56	454418	391324	90	50076165	44243117
23	1003	794	57	527211	451876	91	56634262	49996055
24	1340	1143	58	617874	532847	92	64202500	56786909
25	1581	1264	59	715221	614156	93	72539414	64121495
26	2037	1691	60	837115	723965	94	82115737	72712079
27	2461	1995	61	966468	831822	95	92670111	82010781
28	3127	2621	62	1127111	975601	96	104784531	92895883
29	3719	3012	63	1300819	1122546	97	118114305	104651421
30	4746	3998	64	1512537	1311830	98	133378317	118364000
31	5605	4567	65	1741713	1505621	99	150206552	133244744
32	7038	5921	66	2021752	1756710	100	169405505	150497338
33	8394	6908	67	2323521	2012560			
34	10376	8705	68	2690068	2340756			

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[2] G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. 17 (2) (1918), pp. 75–115.

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Mischungsgeschwindigkeit für Ziffernentwicklungen nach reellen Matrizen

von

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§ 1. Definitionen. $I = (0, 1)^d$ sei der Einheitswürfel des $\boldsymbol{R}^d$, $\mathscr{F}$ die σ-Algebra der Borelschen Teilmengen von I, m das d-dimensionale Lebesgue-Maß auf I. A sei eine nichtsinguläre reelle $(d\times d)$-Matrix, Δ der Betrag ihrer Determinante.

Wir definieren:

(a) $T\colon I\to I$, $Tx = Ax \bmod 1$,

(b) Ist $x\in\boldsymbol{R}$, so sei $[x]$ die größte ganze Zahl $\leqslant x$ und für

$$x = (x_1, \ldots, x_d)\in\boldsymbol{R}^d \quad \text{sei} \quad [x] = ([x_1], \ldots, [x_d])\in\boldsymbol{Z}^d.$$

Weiter sei $M = A\cdot I\cap\boldsymbol{Z}^d$. Für $i = 1, 2, \ldots$ erklären wir Funktionen $k_i\colon I\to M$

$$k_1(x) = [Ax], \qquad k_i(x) = k_1(T^{i-1}x).$$

(c) Sind $a_1, a_2, \ldots, a_s\in M$, so sei

$$I(a_1, \ldots, a_s) = \{x\in I \mid k_i(x) = a_i, 1\leqslant i\leqslant s\},$$

sofern die rechte Menge nicht leer ist. Die Mengen $I(a_1, \ldots, a_s)$ heißen Zylinder s-ter Ordnung und bilden eine Partition von I, die wir mit $\mathscr{I}_s$ bezeichnen wollen.

(d) Wir definieren folgende Teilmengen von $\mathscr{I}_s$:

$$\mathscr{A}_s = \{E\in\mathscr{I}_s \mid T^sE = I\},$$
$$\mathscr{C}_s = \mathscr{I}_s\setminus\mathscr{A}_s,$$
$$\mathscr{D}_s = \{I(a_1, \ldots, a_s)\in\mathscr{I}_s \mid I(a_1)\in\mathscr{C}_1, I(a_1, a_2)\in\mathscr{C}_2, \ldots, I(a_1, \ldots, a_s)\in\mathscr{C}_s\},$$
$$\mathscr{B}_s = \{I(a_1, \ldots, a_s)\in\mathscr{A}_s \mid I(a_1, \ldots, a_{s-1})\in\mathscr{D}_{s-1}\}.$$

(e) Es sei

$$\beta_s = \operatorname{card}\mathscr{B}_s, \qquad \delta_s = \operatorname{card}\mathscr{D}_s, \qquad B_s = \bigcup_{E\in\mathscr{B}_s} E, \qquad D_s = \bigcup_{E\in\mathscr{D}_s} E.$$