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Quasiperfect numbers

by

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1. Introduction. In [3], Sierpiński asks whether there exists an integer N the sum of whose positive divisors, excluding 1 and N, equals N. Such a number is called quasiperfect by Cattaneo [1]. If $\sigma(N)$ is the familiar arithmetic function indicating the sum of the positive divisors of N, then the above question is equivalent to the following: does there exist an integer N such that $\sigma(N) = 2N+1$? Much work has been done on perfect numbers ($\sigma(N) = 2N$) or multiperfect numbers ($\sigma(N) = kN$, $k \geqslant 3$), but unlike these classes of numbers, $(\sigma(N), N) = 1$ when N is quasiperfect.

In this paper we continue the study of quasiperfect or QP numbers, determining some of the properties that such numbers possess. In particular, we show that a QP number divisible by 3 (prime to 3) must have at least 5 (at least 8) distinct prime factors. Furthermore, if N is QP, and $5 \nmid N$, then $p|N$, where p is a prime $\equiv 1 \pmod 5$. Finally, we show that if N is QP, then $N > 10^{20}$.

2. Preliminary results. Cattaneo proved the following in [1].

Proposition 0. *If N is a natural number such that $\sigma(N) = 2N+1$, then $N = p_1^{2e_1} \ldots p_r^{2e_r}$ where the p_i are odd primes. Moreover, if $p_i \equiv 1 \pmod 8$, then $e_i \equiv 0$ or $1 \pmod 4$; if $p_i \equiv 3 \pmod 8$, then $e_i \equiv 0 \pmod 2$; if $p_i \equiv 5 \pmod 8$ then $e_i \equiv 0$ or $-1 \pmod 4$. If q is a prime divisor of $\sigma(N)$, then $(-2|q) = 1$, where $(p|q)$ is the familiar Legendre symbol. Finally, if M is a natural number for which $\sigma(M) \geqslant 2M$, then no non-trivial multiple of M is QP.*

Remark. Cattaneo claims to have proved that no QP number is divisible by 3, but Sierpiński pointed out that his proof is erroneous. A. Schinzel proved that if N is QP then N should have at least three distinct prime factors and $N > 11000$. See Sierpiński ([4], pp. 257–258).

Now we apply results on cyclotomy to QP numbers.

Lemma 1. *Let $q = ef+1$ be a prime > 2; let p be a prime different from q.*

(a) *If* $p \not\equiv 1 \pmod q$ *and* p *is an e-th power residue* $\pmod q$, *then* $\sigma(p^{nf-1}) \equiv 0 \pmod q$.

(b) *If* $p \equiv 1 \pmod q$, *then* $\sigma(p^{nq-1}) \equiv 0 \pmod q$.

Proof. (a) Suppose $p \not\equiv 1 \pmod q$, and $x^e \equiv p \pmod q$ has a solution. Then

$$(p-1)\sigma(p^{nf-1}) = p^{nf}-1 \equiv (x^e)^{nf}-1 \pmod q \equiv (x^n)^{q-1}-1 \equiv 0 \pmod q,$$

by the Euler–Fermat theorem. Since $p-1 \not\equiv 0 \pmod q$, we have $\sigma(p^{nf-1}) \equiv 0 \pmod q$.

(b) Suppose $p \equiv 1 \pmod q$. Then

$$\sigma(p^{nq-1}) = \sum_{i=0}^{nq-1} p^i \equiv \sum_{i=0}^{nq-1} 1 = nq \equiv 0 \pmod q.$$

THEOREM 1. *Suppose* $p \neq q$ *are odd primes,* $q = ef+1 \equiv 5$ *or* $7 \pmod 8$, $N = p^k M$, $(M, p) = 1$, *and* $\sigma(N) = 2N+1$.

(a) *If* $p \not\equiv 1 \pmod q$ *and* $x^e \equiv p \pmod q$ *is solvable, then* $k \not\equiv -1 \pmod f$.

(b) *If* $p \equiv 1 \pmod q$, *then* $k \not\equiv -1 \pmod q$.

Proof. This is immediate from Lemma 1 and the statement in Proposition 0 that, if q is a prime divisor of $\sigma(N) = 2N+1$, then $(-2|q) = 1$.

COROLLARY 1. *If* $N = p^k M$, $(M, p) = 1$, $\sigma(N) = 2N+1$, *and* $p = 3$, 5, 11, 13, 23 *or* 47, *then* $k \not\equiv 2 \pmod 3$.

Proof. For the stated p, choose q, x, and e as indicated, then apply the theorem.

p	q	x	e
3	13	2	4
5	31	5	10
11	7	2	2
13	61	4	20
23	7	4	2
47	37	3	12

3. Lower bounds on the number of prime divisors of QP numbers.

THEOREM 2. *If* $\sigma(N) = 2N+1$ *then* N *has at least* 5 *distinct prime factors.*

Proof. In accordance with standard terminology, let us call a number N deficient, perfect or abundant if $\sigma(N) < 2N$, $\sigma(N) = 2N$, or $\sigma(N) > 2N$, respectively. Thus, any QP number N is abundant; by Proposition 0, no proper factor of N can be abundant or perfect. In the terminology of Dickson [2], N is primitive abundant, so that a QP number must be an odd primitive abundant square. All such numbers with fewer than 5 distinct prime factors are listed here (see [2]):

$$3^4 5^2 13^2,\ 3^2 5^2 11^4 67^2,\ 3^2 5^4 11^4 137^2,\ 3^2 5^4 13^2 43^2,\ 3^4 5^2 17^2 61^2,$$
$$3^6 5^2 17^2 79^2,\ 3^8 5^4 17^2 223^2,\ 3^8 5^6 17^2 239^2,\ 3^4 5^2 23^2 31^2,\ 3^4 5^4 23^2 41^2.$$

By Proposition 0, if $5|N$ and $\sigma(N) = 2N+1$, then the exponent on 5 is of the form $8k$ or $8k-2$. The only such integer in the above list is $3^8 5^6 17^2 239^2$, but by Corollary 1, the exponent on 3 must be incongruent to $2 \pmod 3$. Hence a QP number has at least 5 distinct prime factors.

Cattaneo [1] showed that if a QP number is prime to 3, then it has at least 7 distinct prime factors. We improve this result as follows.

THEOREM 3. *If a QP number* N *is prime to* 3, *then* N *is divisible by at least* 8 *distinct prime factors, and if by exactly* 8 *prime factors then these include* 5 *and* 7.

Proof. We note that

$$\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{18} < 2$$

from which Cattaneo concluded that N must be divisible by at least 7 distinct primes. Since

$$\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{23}{22}\cdot\frac{29}{28} < 2,$$

if N is divisible by exactly 7 distinct primes, 6 of them must be 5, 7, 11, 13, 17 and 19. Furthermore

$$\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot\frac{41}{40} < 2,$$

so the seventh prime must be 23, 29, 31 or 37. By Proposition 0, the least exponent allowable on 5, 7, 11, 13, 17 and 19 are 6, 2, 4, 6, 2 and 4, respectively. Since $19|\sigma(7^2)$, the least exponent on 7 is 4. By Theorem 1 (b) and Corollary 1, 4 and 8 are impossible as exponents on 11, so 11 must be raised to at least the twelfth power.

Let $N = 5^6 7^4 11^{12} 13^6 17^2 19^4 37^2$; then $\sigma(N) > 2N+1$. If we replace 37 by a smaller prime or increase the exponents on any of the primes, we obtain an integer M which satisfies the inequalities

$$\frac{\sigma(M)}{2M+1} \geqslant \frac{\sigma(N)}{2N+1} > 1.$$

Hence M is not QP. Finally, we notice that

$$\frac{5}{4}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot\frac{23}{22}\cdot\frac{29}{28}\cdot\frac{31}{30} < 2,$$

so that N must be divisible by both 5 and 7, if there are exactly 8 distinct prime factors.

4. Lower bounds on the size of QP numbers. With the help of several technical lemmas, we will show that if N is QP, then $N > 10^{20}$.

LEMMA 4A. *Suppose N is QP, and let $N = p^k M$, where $(M, p) = 1$ and p is a prime.*

(a) *If $p \equiv 5 \pmod 8$, then $k \geqslant 6$.*

(b) *If $p \equiv 3 \pmod 8$, then $k \geqslant 4$.*

(c) *If $p = 23$ or 47, then $k \geqslant 4$.*

(d) *If $p = 11$, then $k \geqslant 12$.*

(e) *If $p \equiv 1 \pmod 5$, then $k \neq 4$.*

(f) *If $N \equiv 0 \pmod 3$ and $p \equiv 1 \pmod 3$, then $k \geqslant 4$; furthermore, if $N \equiv 0 \pmod 3$ and $p = 19$, then $k \geqslant 12$.*

Proof. (a) and (b) follow from Proposition 0, (c) follows from Corollary 1, (d) was provided in the proof of Theorem 3, and (e) follows from Theorem 1(b). As for (f), if $p \equiv 1 \pmod 3$, then $\sigma(p^2) \equiv \sigma(N) \equiv 0 \pmod 3 \equiv \sigma(19^8)$, contrary to the fact that $(\sigma(N), N) = 1$. Finally, $19 \equiv 3 \pmod 8$, so $k \equiv 0 \pmod 4$, and since $\sigma(19^4) \equiv 0 \pmod{151}$ and $151 \equiv 7 \pmod 8$, we must have $k \geqslant 12$. Similarly, if $p \equiv 2$ or $4 \pmod 7$ or $p \equiv 3$ or $9 \pmod{13}$, then $k > 2$. Except for 7, 17, 31, 41, 71, 73, 89, 97, 101 or 103, if $p < 120$ then $k > 2$. If in addition $N \equiv 0 \pmod 3$, then except for 17, 41, 71, 89, 167, 239, 257, 281 and 311, if $p < 350$ then $k > 2$.

LEMMA 4B. *No QP number is divisible by $3 \cdot 5 \cdot 7$, $3 \cdot 5 \cdot 11$, $3 \cdot 5 \cdot 13$, $3 \cdot 5 \cdot 17 \cdot 19$, $3 \cdot 5 \cdot 17 \cdot 23$, $3 \cdot 5 \cdot 17 \cdot 29$, or $3 \cdot 5 \cdot 17 \cdot 31$.*

Proof. By Lemma 4A the exponents on 3 and 5 are at least 4 and 6, respectively. But if $M = 3^4 5^6 13^2$ or $3^4 5^6 17^2 31$, then $\sigma(M) > 2M+1$.

LEMMA 4C. *If N is QP and divisible by at least 7 distinct primes, then $N > 10^{20}$.*

Proof. The number $N = 3^4 7^2 17^2 31^2 41^2 71^2 73^2 89^2$ satisfies $N > 10^{20}$, is not QP, and any QP number divisible by at least 8 distinct primes is greater than N. By Theorem 3, a QP number having exactly 7 distinct prime factors is divisible by 3. Let

$$M = 3^4 7^4 17^2 41^2 71^2 89^2 167^2.$$

Using Lemmas 4A, 4B and the proof of Lemma 4A, the inequalities

$$3^4 < 17^2 < 41^2 < 7^4 < 71^2 < 89^2 < 5^6 < 167^2 < p^4$$

where p is any prime other than 3 or 7 requiring at least a fourth power, and the fact that $M > 10^{20}$, we conclude that a QP number divisible by exactly 7 primes is greater than 10^{20}.

LEMMA 4D. *If a QP number N is divisible by exactly 5 or 6 distinct primes, then one of the primes must be 5, 7 or 11. If $(N, 35) = 1$, then $N > 10^{20}$.*

Proof. If m and n are natural numbers, then $m > n$ implies $\frac{m+1}{m} < \frac{n+1}{n}$. Since

$$\frac{3}{2} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \frac{29}{28} < 2,$$

the 5 or 6 distinct primes must include 5, 7 or 11. If $(N, 35) = 1$ then $33 | N$; since $\sigma(3^4) \equiv 0 \pmod{11}$ and $\sigma(3^8) \equiv 0 \pmod{13}$, the exponent on 3 is at least 12. By Theorem 1 (b) and Corollary 1, the exponent on 11 is at least 12. Hence,

$$N \geqslant 3^{12} 11^{12} 13^2 17^2 19^2 > 10^{20},$$

and the lemma is proved in case N has exactly 5 prime factors. If N has exactly 6 prime factors, then

$$N \geqslant 3^{12} 11^{12} 13^2 17^2 19^2 23^2 > 10^{20}.$$

LEMMA 4E. *No QP number containing 5 distinct primes can be of the form $3^4 5^6 p^2 q^2 r^2$.*

Proof. Let $M = 3^4 5^6 17^2$; then

$$\frac{\sigma(M)}{M} = \frac{725518057}{365765625} < 2.$$

In order that

$$725518057\,(x+1) = 2(365765625x),$$

we must have $120 < x < 121$. Thus if $3^4 5^6 17^2 q^2 r^2$ is QP, both q and r must be greater than 120. For if $q < 120$, then

$$\frac{\sigma(N)}{N} > \frac{\sigma(M)}{M} \frac{q+1}{q} > 2 + \frac{1}{N}$$

and N would not be QP. On the other hand, both primes cannot exceed 242; if so, then

$$\frac{\sigma(N)}{N} < \frac{\sigma(M)}{M} \frac{q}{q-1} \cdot \frac{r}{r-1} < \frac{\sigma(M)}{M} \cdot \frac{244}{243} \cdot \frac{243}{242} = \frac{\sigma(M)}{M} \cdot \frac{122}{121} < 2.$$

Hence one of q and r must be a prime between 121 and 242. By the proof of Lemma 4A, this prime must be 167 or 239. But $\sigma(239^2) \equiv 0 \pmod{29}$ and $29 \equiv 5 \pmod 8$, which is disallowed by Proposition 0. If $N = 3^4 5^6 17^2 167^2 r^2$, then $\sigma(N) \not\equiv 1 \pmod 5$, but $2N+1 \equiv 1 \pmod 5$; hence

$3^4 5^6 p^2 q^2 r^2$ cannot be QP. We use the proof of Lemma 4A and observe that

$$\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{41}{40}\cdot\frac{71}{70}\cdot\frac{89}{88} < 2,$$

from which we conclude that $3^4 5^6 p^2 q^2 r^2$ cannot be QP.

LEMMA 4F. *If N is QP and $N \equiv 0 \pmod{3\cdot 5\cdot 17}$, then $N > 10^{20}$.*

Proof. Let $N = 3^a 5^b 17^c p^d q^e$. By the proof of the previous lemma, it is clear that p and q are greater than 121, and one of p or q is greater than 242. The last remark follows from the inequality

$$\frac{\sigma(M)}{M}\cdot\frac{240}{239}\cdot\frac{242}{241} > 2+\frac{1}{M}$$

where $M = 3^4 5^6 17^2$. If $a \neq 4$, $a \geqslant 12$; if $b \neq 6$, $b \geqslant 16$; if $c \neq 2$, $c \geqslant 8$. If $d = e = 2$ and $121 < p < 242 < q$, then $p \geqslant 167$ and $q \geqslant 257$ by the proof of Lemma 4A; if, in addition, either $a > 4$, $b > 6$ or $c > 2$, then

$$N \geqslant 3^{12} 5^6 17^2 167^2 257^2 > 10^{20}.$$

If $a = 4$, $b = 6$ and $c = 2$, then $d \geqslant 4$ or $e \geqslant 4$ by Lemma 4E. Let $N = 3^4 5^6 17^2 p^4 q^2$. It is clear that $p > 121$ and $p \equiv 2 \pmod 3$, for otherwise $\sigma(N) \equiv 2 \pmod 3$. For $q \geqslant 167$, we have

$$3^4 5^6 17^2 131^4 q^2 > 10^{20}.$$

Hence, if $a = 4$, $b = 6$ and $c = 2$, then

$$3^4 5^6 17^2 p^d q^e \geqslant 3^4 5^6 17^2 131^4 167^2 > 10^{20}.$$

If N has exactly 6 distinct prime factors, then

$$N = 3^a 5^b 17^c p^d q^e r^f \geqslant 3^4 5^6 17^2 167^2 257^2 281^2 > 10^{20}.$$

Finally, by Lemma 4C, if N has more than 6 distinct prime factors, then $N > 10^{20}$.

LEMMA 4G. *If N is QP and $N \equiv 0 \pmod{15}$, then $N > 10^{20}$.*

Proof. In the previous lemma, we investigated the case $3\cdot 5\cdot 17 \mid N$. If $3\cdot 5\cdot 19 \mid N$, then $N > 10^{20}$ since by Lemma 4A, 19 must be raised to at least the twelfth power. By Lemma 4B, $(N, 7\cdot 11\cdot 13) = 1$. Let

$$N = 3^4 5^6 p^a q^b r^c.$$

By Lemma 4E, one of a, b and c is greater than 2: suppose it is a. Let

$$N = 3^4 5^6 23^4 q^2 r^2.$$

By the proof of Lemma 4A, the values allowable for q and r less than 350 are 41, 71, 89, 167, 239, 257, 281 and 311. Since

$$\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{23}{22}\cdot\frac{q}{q-1}\cdot\frac{r}{r-1} < 2$$

in case both q and r exceed 105, we must have q or $r = 41, 71, 89$. In order that $\sigma(N) \equiv 2N+1 \pmod 7$, we cannot have q or $r \equiv \pm 1 \pmod 7$. This condition eliminates 41, 71, 167, 239, and 281 as either of the primes leaving 89, 257 and 311 from the above list. But

$$\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{23}{22}\cdot\frac{89}{88}\cdot\frac{q}{q-1} < 2 \quad \text{for} \quad q \geqslant 257.$$

Thus $3^4 5^6 23^4 q^2 r^2$ is not QP. By Lemma 4A, it is impossible that 29, 31, 37 or 41 be raised to the fourth power in a QP number. If

$$N = 3^4 5^6 43^4 q^2 r^2,$$

then $\sigma(N) \equiv 2 \pmod 3$. We then consider

$$N = 3^4 5^6 p^4 q^2 r^2, \quad p \geqslant 47,\ q \geqslant 41,\ r \geqslant 71.$$

If $q = 41$ or 71, then $r \geqslant 167$ in order that $\sigma(N) \equiv 1 \pmod 5$; if $q = 89$, $r \geqslant 239$ in order that $\sigma(N) \equiv 1 \pmod 5$. But

$$\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{41}{40}\cdot\frac{47}{46}\cdot\frac{167}{166} < 2.$$

If

$$N = 3^4 5^6 p^6 q^2 r^2, \quad p \geqslant 23,\ q \geqslant 41,\ r \geqslant 71,$$

then $N \geqslant 10^{20}$. It is then clear that a QP number having exactly 5 distinct prime divisors and divisible by 15 is greater than 10^{20}. If

$$N = 3^4 5^6 41^2 71^2 89^2 167^2,$$

then $N > 10^{20}$. If

$$N = 3^a 5^b p^c q^d r^e,$$

where either $a > 4$ or $b > 6$, then the inequalities

$$N \geqslant 3^{12} 5^6 41^2 71^2 89^2 > 10^{20}, \qquad 23^4 > 89^2$$

using Lemma 4A and Lemma 4F establish the result in this case. Lemmas 4A, 4B, 4F and the inequality $167^2 < 23^4$ establish that a QP number divisible by 15 and by more than 5 distinct primes must exceed 10^{20}.

LEMMA 4H. *Let N be QP and $N \not\equiv 0 \pmod 5$. Then N must have a prime divisor congruent to $1 \pmod 5$.*

Proof. Suppose $N \not\equiv 0 \pmod 5$ and N has no prime divisor congruent to $1 \pmod 5$. Let s and t denote the number of primes dividing N which are raised to twice an odd power and are congruent to 2 and 3 (mod 5), respectively. Then $\sigma(N) \equiv 2^s 3^t \pmod 5$; if $s+t$ is even, $\sigma(N) \equiv 1$ or 4 and $2N+1 \equiv 3 \pmod 5$, and if $s+t$ is odd, $\sigma(N) \equiv 2$ or 3 and $2N+1 \equiv 4 \pmod 5$. Thus the lemma is established.

LEMMA 4I. *If N is QP and $N \equiv 0 \pmod{21}$, then $N > 10^{20}$.*

Proof. Since $N \equiv 0 \pmod 3$, the exponent on 7 is $\geqslant 4$. We first consider the case when the exponent on 7 is 4. Write

$$N = 3^m 7^4 p^a q^b r^c, \quad m \geqslant 4.$$

Now one of p, q and r (say r) must be congruent to $1 \pmod 3$ (and then $c \equiv 4 \pmod 6$) in order that $\sigma(N) \equiv 2N+1 \pmod 3$. By Lemma 4A, $r \geqslant 43$ or $c > 4$. By Lemma 4H, p, q, or $r \equiv 1 \pmod 5$. If $r = 43$, $c \geqslant 4$, $p = 11$ and $a \geqslant 12$, then $N \geqslant 10^{20}$. If $r = 43$, $c \geqslant 4$, $p = 31$ and $a \geqslant 6$, then $N \geqslant 10^{20}$. If $p = 41$, then $q = 11$ in order that $\sigma(N) > 2N$, since

$$\frac{3}{2}\cdot\frac{7}{6}\cdot\frac{13}{12}\cdot\frac{41}{40}\cdot\frac{43}{42} < 2.$$

67 is the smallest prime r greater than 43 such that $r \equiv 1 \pmod 3$. Since

$$\frac{3}{2}\cdot\frac{7}{6}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot\frac{67}{66} < 2,$$

N must be divisible by 11 or 13. But

$$3^m 7^4 11^{12} 17^2 67^4 > 3^m 7^4 13^6 17^2 67^4 > 10^{20}.$$

If $c > 4$ then $c \geqslant 10$ and then $N > 10^{20}$. This follows from Lemma 4A and the fact that

$$3^4 7^4 11^{16} > 3^4 7^4 31^{10} > 10^{20}.$$

Let

$$N = 3^4 7^k p^a q^b r^c, \quad \text{where} \quad k > 4.$$

Since $\sigma(7^6) \equiv 0 \pmod{29}$, $\sigma(7^8) \equiv 0 \pmod 3$, and $\sigma(7^{10}) \equiv \sigma(7^4) \pmod 3$, either $k \geqslant 12$ or $k = 10$ and we have the above situation all over again. Hence $k \geqslant 12$, and

$$3^m 7^k p^a q^b r^c \geqslant 3^4 7^{12} 17^2 41^2 71^2 > 10^{20}.$$

In case N has exactly 6 distinct prime factors, the following values of $N > 10^{20}$ establish the result:

$$N = 3^4 7^4 17^2 41^2 43^4 71^2, \quad N = 3^4 7^{12} 17^2 41^2 71^2 89^2.$$

The lemma is established by applying Lemma 4C in case N has more than 6 distinct prime factors.

THEOREM 4. *If N is quasiperfect then $N > 10^{20}$.*

Proof. The theorem is established, since we have treated the cases for N divisible by at least 7 distinct primes (Lemma 4C), and by exactly 5 or 6 distinct primes (Lemmas 4D, 4F, 4G and 4I).

In the case $3 \nmid N$ and $5 \nmid N$ where N is QP we can give a considerable improvement on the bound.

THEOREM 5. *Any QP number N such that $3 \nmid N$ and $5 \nmid N$ is divisible by at least 10 distinct prime factors and $N > 10^{32}$.*

Proof. In ([5], Theorem 1 (B)), Suryanarayana proved that if P is the smallest prime factor of an odd integer N not divisible by 3, then

$$\frac{\sigma(N)}{N} < \left(\frac{2P+4\omega(N)+8}{2P+1}\right)^{1/2}$$

where $\omega(N)$ is the number of distinct primes dividing N. Since N is QP

$$2 < \left(\frac{2P+4\omega(N)+8}{2P+1}\right)^{1/2}.$$

Hence $\omega(N) > \frac{3P}{2} - 1$. Since $3 \nmid N$ and $5 \nmid N$, we must have $P \geqslant 7$ so that $\omega(N) \geqslant 10$.

It follows that $N \geqslant (7\cdot 17\cdot 31\cdot 41\cdot 71\cdot 73\cdot 89\cdot 97\cdot 101\cdot 103)^2 > 10^{32}$. If N were divisible by a prime $\leqslant 103$ other than those appearing in the above formula, then by proof of Lemma 4A, such a prime would appear with exponent $\geqslant 4$ and hence the lower bound would be larger.

The question of Sierpiński as to the existence of quasiperfect numbers remains unanswered. Just as with odd perfect numbers, there are many necessary conditions that a QP number must satisfy, but no such number has been found.

We note in conclusion that the techniques of this paper are useful in obtaining results about pseudoperfect numbers, i.e. numbers N for which $\sigma(N) = 2N-1$.

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