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On the uniform ε-distribution of residues within the periods of rational fractions with applications to normal numbers

by

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1. Introduction. Let $Z/m < 1$ be some rational fraction in lowest terms represented in any base g such that $(g, m) = 1$ and consider the distribution on the unit interval $[0, 1]$ of what we shall call the "normalized" power residues r_i/m determined by the fractional parts $\{Zg^i/m\} = r_i/m$ for $i = 0, 1, \ldots, \omega(m)-1$ where $\omega(m) = \mathrm{ord}_m g$ is the number of r_i/m in a complete period. It is clear that we have in some sense a discrete distribution on $[0, 1]$ of a denumerable set of normalized power residues r_i/m. To date, our work in [6 — 9] has proved that there exists broad classes of rational fractions Z/m called Type A, B, and C [6, p. 229] as characterized by the prime decomposition of m which have, over *complete* periods, what we have termed, a uniform ε-distribution [6, p. 223] of the r_i/m on $[0, 1]$. The concept of a uniform ε-distribution is the discrete analog of a uniform distribution defined by H. Weyl [4, p. 22] in 1916. The uniform ε-distribution of the set of r_i/m on $[0, 1]$ means essentially that within arbitrary sub-intervals taken anywhere in the unit interval $[0, 1]$, we have about the same number of distinct points after the complete periodic set of $\omega(m)$ points has been placed on $[0, 1]$. However, the concept in general is not, necessarily, restricted to a periodic set.

In this paper, we will prove that sets of digits slightly greater than the square root of the period length taken *anywhere within* the period of Type A and some of Type B will have a uniform ε-distribution. By means of this result, we can relax considerably the requirements in the construction of transcendental non-Liouville normal numbers which we presented in [7, p. 242, Th. 1]. For example, in contrast to [7, p. 241,

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[(1.1.)], we can now prove that

$$x(g,p) = \sum_{n=1}^{\infty} 1/p^n g^{p^n} \tag{1.0}$$

is a transcendental non-Liouville normal number constructed from any given odd prime and one of its primitive roots mod p^2.

The results here enable us to construct a normal number from only one period or a constant number of periods of $Z_1/m, Z_2/m^2, \ldots$ or even portions of periods placed in juxtaposition slightly greater than a period length. This is in marked contrast to our former need [7, p. 242] for a divergent repetition sequence a_n of the periods of the Z_n/m^n. We can also make a further improvement on the Brouwer question discussed in [6, p. 234–235] related to π and also the same question as defined for rationals. Finally, we offer some refinements on the transcendence conditions as stated in [7, p. 247, Th. 2].

The proofs are based on the estimation of trigonometric sums and a special form of W. LeVeque's bound [4, p. 23, (2)] on the Weyl discrepancy of a finite sequence of fractional parts on [0, 1]. In what follows and according to context, $e(z) = e^{2\pi i z}$, $[x]$ is the greatest integer less than x, and $\{x\}$, the fractional part.

2. Distribution within the period. In [4, p. 23], we find the quantity $B(n)$ depending on some given sequence $x_1, x_2, \ldots, x_n$ given by

$$B(n) = 2\left(6/\pi^2 \sum_{M=1}^{\infty} \left|1/n \sum_{k=1}^{n} e(Mx_k)\right|^2 /M^2\right)^{1/3} \tag{2.0}$$

which LeVeque in 1965 shows by means of characteristic functions in probability theory to be an upper bound on the Weyl discrepancy

$$D_n = \sup_{0\leqslant\alpha<\beta\leqslant 1} \left|\big(N(\beta)-N(\alpha)\big)/n - (\beta-\alpha)\right| \leqslant B(n) \tag{2.1}$$

where $N(\alpha)$ denotes the number of $x_k - [x_k] < \alpha \in [0, 1]$ for all $k \leqslant n$. In [6, p. 223], we defined a uniform ε-distribution for a discrete sequence of fractional parts $\{x_k\}$ for $k \leqslant n$ in terms of D_n in (2.1).

Let $x_k = \{Zg^{k-1}/m\}$ where $(g, m) = 1$ for $k = 1, 2, \ldots, \omega(m)$, then (2.0) becomes [let us set $k = x$ and $n = h$ in (2.0)]

$$B(h) = 2\left(6/\pi^2 \sum_{M=1}^{\infty} \left|1/h \sum_{x=1}^{h} e(MZg^{x-1}/m)\right|^2 /M^2\right)^{1/3}. \tag{2.2}$$

If we break into residue classes $M = tm + r$ with $r = 1, 2, \ldots, m-1, m$ and $t = 0, 1, \ldots,$ we obtain

$$B(h) = 2\left(6/\pi^2 \sum_{r=1}^{m-1} |S(h, r, Z/m)|^2 \zeta(2, r/m)/m^2 + 1/m^2\right)^{1/3} \tag{2.3}$$

where

$$|S(h, r, Z/m)|^2 = \left|1/h \sum_{x=1}^{h} e(rZg^{x-1}/m)\right|^2 \tag{2.4}$$

and $\zeta(s, w) = \sum_{t=0}^{\infty} 1/(t+w)^s$ is the Hurwitz-zeta function. Now one can show the bounds [5, p. 232] for $s = 2$

$$\zeta(2, r/m)/m^2 < 1/r^2 + 1/mr \tag{2.5}$$

where $r/m < 1$ for $r = 1, 2, \ldots, m-1$ since

$$\zeta(2, r/m)/m^2 = \sum_{t=0}^{\infty} 1/(tm+r)^2 = 1/r^2 + 1/mr - 2/m^2 \int_0^{\infty} \frac{(u-[u])}{(u+r/m)^3}\, du. \tag{2.6}$$

Introducing (2.5) into (2.3), we have now

LEMMA 1. *If $Z/m < 1$ is any rational fraction in lowest terms such that $(g, m) = 1$, then a bound on the discrepancy D_h for the distribution of the fractional parts $\{Zg^{x-1}/m\}$ for $x = 1, 2, \ldots, h$ on [0, 1] is given by*

$$D_h \leqslant B(h) < 2\left(6/\pi^2 \sum_{r=1}^{m-1} |S(h, r, Z/m)|^2 (1/r^2 + 1/mr) + 1/m^2\right)^{1/3}. \tag{2.7}$$

Therefore, knowing a suitable estimate for $S(h, r, Z/m)$, we may (or may not as the case may be!) obtain a uniform ε-distribution of fractional parts $\{Zg^{x-1}/m\}$ for $x = 1, 2, \ldots, h \leqslant \omega(m)$ on [0, 1]. Since a uniform ε-distribution is a necessary and sufficient condition for (j, ε)-normality [6, p. 224], (2.7) can also be used to establish such theorems. This next lemma permits us to estimate partial sums like $S(h, r, Z/m)$ in terms of sums over a full period [see Hua, 3, pp. 9–10].

LEMMA 2. *If Z/m is any rational fraction in lowest terms, $(g, m) = 1$, and $\omega(m) = \mathrm{ord}_m g$, then*

$$|S(h, 1, Z/m| < \left|\frac{1}{\omega(m)} \sum_{x=1}^{\omega(m)} e\left(\frac{Zg^{x-1}}{m}\right)\right| + \frac{(\log \omega(m) - 1)}{h} \left|\sum_{x=1}^{\omega(m)} e(F(x))\right| \tag{2.8}$$

where $F(x) = Zg^{x-1}/m + nx/\omega(m)$ and $\omega(m) \geqslant 60$ with n chosen to maximize the last sum.

Proof. From Hua [3, p. 10], if we set $f(x) = \omega(m)Zg^{x-1}/m$, his "$m$" $= h$, and $q = \omega(m)$, we obtain

$$(2.9)\quad |S(h, 1, Z/m)| \leqslant \frac{1}{\omega(m)}\left|\sum_{x=1}^{\omega(m)} e\left(\frac{Zg^{x-1}}{m}\right)\right| + \frac{1}{h\omega(m)}\sum_{n=1}^{\omega(m)-1}\left|\sum_{t=1}^{h} e\left(\frac{-nt}{\omega(m)}\right)\right|\cdot\left|\sum_{x=1}^{\omega(m)} e\big(F(x)\big)\right|.$$

Using a Vinogradov estimate [11, p. 56] on the quantity before the last sum, we obtain (2.8) for $\omega(m) \geqslant 60$. We may replace the $f(x)$ as above in Hua, since there is no particular restriction on the real function $f(x)$ for $x = 1, 2, \ldots$ at this stage of the argument. Q. E. D.

Lemma 3. *If $Z/m = Z/p < 1$ where g is a primitive root* mod p^2, *then for $p \geqslant 61$*

$$(2.10)\quad |S(h, r, Z/p)| = \left|\frac{1}{h}\sum_{x=1}^{h} e\left(\frac{rZg^{x-1}}{p}\right)\right| < \frac{p^{1/2}\log p}{h} + \frac{1}{p-1}.\ (^1)$$

Proof. From (2.8), setting $F(x) = rZg^{x-1}/p + nx/(p-1)$ for $p \geqslant 61$ and $\omega(p) = p-1$, we have by a modulus square argument, mod $(p-1)$,

$$(2.11)\quad \left|\sum_{x=1}^{p-1} e\big(F(x)\big)\right| = \left|\sum_{x=0}^{p-2} e\big(F(x+1)\big)\right| = p^{1/2}.$$

The same sum in (2.11) can also be written as a well-known character sum [11, p. 126, 11, a.]

$$(2.12\quad \sum_{x=1}^{p-1} e\big(F(x)\big) = \sum_{x=1}^{p-1} e\left(\frac{rZg^{x-1}}{p} + \frac{n\,\mathrm{ind}\,rZg^{x-1}}{p-1} + \frac{n(1-\mathrm{ind}\,rZ)}{p-1}\right)$$

or

$$(2.13)\quad \left|\sum_{x=1}^{p-1} e\big(F(x)\big)\right| = \left|\sum_{w=1}^{p-1} \chi(w)e(w/p)\right| = p^{1/2}$$

where we define the character $\chi(w)$ over the residue class mod p with $w \equiv rZg^{x-1}$ mod p and $\varphi(p) = p-1$. Since $\sum_{x=1}^{p-1} e(rZg^{x-1}/p) = -1$ and $p^{1/2}\big(\log(p-1)-1\big) < p^{1/2}\log p$, using (2.8), the result in (2.10) follows assuming $\omega(p) = p-1 \geqslant 60 \Rightarrow p \geqslant 61$. Q. E. D.

In the next lemma, we obtain the most general result for the partial sum in (2.7) and (2.8) by means of the residue progressions [6, p. 227] for a rational fraction of Type A [6, p. 229].

(1) A generalization of the estimate in (2.10) by L. J. Mordell is given in Mathematika 19 (1972), pp. 84–87.

Lemma 4. *If $Z/m < 1$ in lowest terms is of Type A where $(g, m) = 1$ and $F(x) = rZg^{x-1}/m + nx/\omega(m)$, then*

$$(2.14)\quad |S| = \left|\sum_{x=1}^{\omega(m)} e\big(F(x)\big)\right| \leqslant (D/C)^{1/2}\big(\omega(m)\big)^{1/2} \leqslant D^{1/2}\big(\omega(m)\big)^{1/2}$$

with C some positive integer and D is defined in [6, Th. 4, p. 227].

Proof. Proceeding on a modulus square argument, we have

$$(2.15)\quad |S|^2 = \sum_{x=1}^{\omega(m)}\sum_{y=1}^{\omega(m)} e\left(\frac{rZg^{y-1}}{m}(g^{x-y}-1) + n(x-y)/\omega(m)\right)$$

and letting x and y run over $1, 2, \ldots, \omega(m)$ for fixed t such that $t \equiv (x-y)$ mod $\omega(m)$, we obtain

$$(2.16)\quad |S|^2 = \sum_{t=1}^{\omega(m)} e\big(nt/\omega(m)\big)S_t$$

where

$$(2.17)\quad S_t = \sum_{y=1}^{\omega(m)} e\left(\frac{rZ}{m}(g^t-1)g^{y-1}\right)$$

for $t = 1, 2, \ldots, \omega(m)$.

In (2.17), we have over residue progressions P_k [see 6, p. 227] where $\omega(m)/\omega(D) = m/D =$ an integer > 1

$$(2.18)\quad S_t = \sum_{(k)}\sum_{(v)} e\left(\frac{rZ}{m}(g^t-1)g^{k+v\omega(D)}\right)$$

with $k = 0, 1, \ldots, \omega(D)-1$ and $v = 0, 1, \ldots, \omega(m)/\omega(D)-1 = m/D-1$. Now due to the fact that we have a residue progression P_k for every fixed k, we obtain

$$(2.19)\quad S_t = \sum_{(k)}\sum_{(s)} e\left(\frac{rZ}{m}(g^t-1)g^k(1+sD)\right)$$

where s runs over $0, 1, \ldots, m/D-1$ since mod m, $g^{v\omega(D)}$ runs over $1, 1+D, 1+2D, \ldots, 1+(m/D-1)D$. Therefore, we obtain

$$(2.20)\quad S_t = \sum_{(k)} e\left(\frac{rZ}{m}(g^t-1)g^k\right)\sum_{s=0}^{m/D-1} e\left(\frac{rZg^k(g^t-1)}{m/D}s\right).$$

The last sum in (2.20) is such that

$$(2.21)\quad \sum_{s=0}^{m/D-1} e\left(\frac{rZg^k(g^t-1)}{m/D}s\right) = \begin{cases} 0, & \text{if } m/D \nmid rZg^k(g^t-1),\\ m/D, & \text{if } m/D \mid rZg^k(g^t-1).\end{cases}$$

But $m/D \mid (g^t-1)$ iff $\omega(m/D) \mid t$, hence we obtain

$$(2.22)\qquad S_t = \begin{cases} 0, & \text{if } \omega(m/D) \nmid t, \\ m/D \sum\limits_{(k)} e\left(\dfrac{rZ}{m}(g^t-1)g^k\right), & \text{if } \omega(m/D) \mid t. \end{cases}$$

Thus, $|S_t| \leqslant m\omega(D)/D$ which implies

$$(2.23)\qquad |S|^2 \leqslant \sum_{\substack{t \leqslant \omega(m) \\ \omega(m/D) \mid t}} m\omega(D)/D \leqslant \frac{\omega(m)}{\omega(m/D)} \cdot m\omega(D)/D$$

and since $\omega(m)/\omega(D) = m/D \Rightarrow m\omega(D)/D = \omega(m)$, we obtain

$$(2.24)\qquad |S|^2 \leqslant \big(\omega(m)\big)^2/\omega(m/D).$$

Finally, we show that $D\omega(m/D) = C\omega(m)$ where C is some positive integer $\geqslant 1$, hence (2.24) becomes

$$(2.25)\qquad |S|^2 \leqslant \big(\omega(m)/\omega(m/D)\big)\omega(m) = (D/C)\,\omega(m) \leqslant D\omega(m)$$

and (2.14) follows. From [6, p. 227, Th. 4], we have $m = 2^n \prod_{(i)} p_i^{n_i}$, $D = 2^n \prod_{(i)} p_i^{t_i}$ where $t_i = \min(n_i, z_i+s_i)$, hence $m/D = \prod_{(i)} p_i^{n_i-t_i}$. Therefore, it follows that

$$(2.26)\qquad D\omega(m/D) = \langle \ldots, Dd_i p_i^{n_i-t_i-z_i} \text{ or } Dd_i\rangle$$

depending whether $n_i > t_i+z_i$ or $n_i \leqslant t_i+z_i$, resp. Clearly, $\omega(m) \mid D\omega(m/D)$ since $\omega(m) = \langle \omega(2^n), \ldots, d_i p_i^{n_i-z_i}$, or $d_i\rangle$ and we have the fact that $\omega(2^n) \mid 2^n$ in the $D = 2^n \prod_{(i)} p_i^{t_i}$, $d_i p_i^{n_i-z_i} \mid Dd_i p_i^{n_i-t_i-z_i}$ or $d_i p_i^{n_i-z_i} \mid 2^n d_i p_i^{n_i-z_i}$, and $d_i \mid Dd_i$ as well. The proof of Lemma 4 is complete.

LEMMA 5. *If $rZ/m < 1$ in lowest terms is of Type A where $(g, m) = 1$, then*

$$(2.27)\qquad \sum_{(x)} e(rZg^{x-1}/m) = 0$$

where $x = 1, 2, \ldots, \omega(m)$.

Proof. Using the residue progressions [6, Th. 4, p. 227], we have

$$(2.28)\qquad \sum_{(x)} e(rZg^{x-1}/m) = \sum_{(k)} \sum_{(r)} e(r_k + rD)/m$$

where $k = 0, 1, \ldots, \omega(D)-1$, and $r = 0, 1, \ldots, m/D-1$. Therefore, (2.27) follows for each fixed k summing over each arithmetic progression $r_k + rD$ in the residue progressions P_k. Q. E. D.

Therefore, introducing Lemma 4 and 5 into (2.8), we obtain Lemma 6 since

$$\big(\log\omega(m) - 1\big) < \log\omega(m) \quad \text{for} \quad \omega(m) \geqslant 60.$$

LEMMA 6. *If $Z/m < 1$ in lowest terms is of Type A where $(g, m) = 1$, then for $\omega(m) \geqslant 60$*

$$(2.29)\qquad |S(h, r, Z/m)| < \frac{\sqrt{D}}{h}\big(\omega(m)\big)^{1/2}\log\omega(m).$$

We are now prepared to prove

THEOREM 1. *The sequence of normalized residues $r_i/p = \{Zg^i/p\}$ where $p \geqslant 61$ is an odd prime and g is a primitive root* mod p^2 *has a uniform ε-distribution on $[0, 1]$ for $i = 0, 1, \ldots, h > p^{1/2+\delta}$ where $\delta > 0$, $\lambda = \frac{2}{3}\delta$, and $\varepsilon = O(\log^{2/3}p/p^{\lambda})$.*

Proof. In (2.7) for $m = p$, set $|S(h, r, Z/p)| = O\big((p^{1/2}\log p)/h\big)$ based on the estimate in (2.10), and we obtain

$$(2.30)\qquad B(h) < 2\Big(6/\pi^2 O(p\log^2 p/h^2) \sum_{r=1}^{p-1}(1/r^2 + 1/pr) + 1/p^2\Big)^{1/3}.$$

Since

$$\sum_{r=1}^{p-1} 1/r^2 = \pi^2/6 + O(1/p)$$

and

$$\sum_{r=1}^{p-1} 1/pr = \big(\log(p-1)\big)/p + \gamma/p + O\big(1/p(p-1)\big),$$

it is clear that we have

$$(2.31)\qquad B(h) = O\big((p\log^2 p)/h^2\big)^{1/3}.$$

Also, we want $D_h < \varepsilon$ for a uniform ε-distribution [6, p. 223, (1.3)] where here we have $D_h \leqslant B(h) < \varepsilon$; therefore, we set $\varepsilon = O(\log^{2/3}p/p^{\lambda})$ for $h > p^{\frac12+\delta}$ where $\delta > 0$ and $\lambda = \frac{2}{3}\delta$. Theorem 1 is now proved.

By a similar proof, using (2.29) in (2.7), we have as well

THEOREM 2. *The sequence of normalized residues $r_i/m = \{Zg^i/m\}$ where $Z/m < 1$ in lowest terms is a rational fraction of Type A with $(g, m) = 1$ and $\omega(m) = \mathrm{ord}_m g \geqslant 60$ has a uniform ε-distribution on $[0, 1]$ for $i = 0, 1, \ldots, h > \big(\omega(m)\big)^{\frac12+\delta}$ where $\delta > 0$, $\lambda = \frac{2}{3}\delta$, and $\varepsilon = O\big(\log^{2/3}\omega(m)/(\omega(m))^{\lambda}\big)$ with $D^{1/2}$ contained in "O".*

Based on Theorems 1 and 2, we may view these results as statements [6, p. 222, (1.1)] about the distribution or relative frequencies of blocks within the period of Z/p or Z/m measured over the set of $h > p^{\frac12+\delta}$ or $\big(\omega(m)\big)^{\frac12+\delta}$ digits. Since [6, p. 223] with $\alpha = B_j/g^j$ and $\beta = (B_j+1)/g^j$ for the discrepancy implies $1/g^j - \varepsilon < N(B_j, g)/h < 1/g^j + \varepsilon$, we require $1/g^j - \varepsilon > 0$ so that the relative frequency $N(B_j, g)/h > 0$. Thus, we have based on Theorem 1, two theorems which state the (j, ε)-normality in the sense of Besicovitch [8, p. 202, (2.1)] for the sets of h digits within the period.

THEOREM 3. *If $Z/p<1$ is in lowest terms where $p\geqslant 61$ is a prime and g is a primitive root* mod p^2, *then for $\lambda=\frac{2}{3}\delta>0$ and $h>p^{\frac12+\delta}$, we have*

$$|N(B_j,g)/h-1/g^j|=O(\log^{2/3}p/p^{\lambda}) \tag{2.32}$$

where $j\leqslant[\log_g J]$ with $J=O(p^{\lambda}/\log^{2/3}p)$.

Also, we have based on Theorem 2,

THEOREM 4. *If $Z/m<1$ is in lowest terms and of Type A where $\omega(m)\geqslant 30$ and $(g,m)=1$, then for $\lambda=\frac{2}{3}\delta>0$ and $h>(\omega(m))^{\frac12+\delta}$*

$$|N(B_j,g)/h-1/g^j|=O\left(\log^{2/3}\omega(m)/(\omega(m))^{\lambda}\right) \tag{2.33}$$

where $j\leqslant[\log_g J]$ with $J=O\left((\omega(m))^{\lambda}/\log^{2/3}\omega(m)\right)$.

It is clear from the above results that, for example, for increasing p, given some particular g in Theorems 1 and 3, that we have increasing uniformity of the distribution (or "density") of the normalized residues r_i/p on the unit interval for sets of digits within the period approximately greater than the square root of a period which increases as p if g is a primitive root mod p^2. Similarly, the relative frequencies $N(B_j,g)/h$ over these sets of digits within the period are non-zero for larger and larger blocks B_j for any combination of j digits as p increases. Furthermore, let us emphasize that we have shown that there is a uniform ε-distribution in sets of digits taken over *any* subportion of the period $h>p^{\frac12+\delta}$ in length, not necessarily over the first h digits of Z/p for some given Z. This follows, since the results were independent of Z, i.e. we have results valid over the first h digits greater than the square root of the periods of $1/p, 2/p, \ldots, (p-1)/p$.

Finally, all the above statements can be made for the distributions within sub-portions of the periods of Z/m independent of the choice of Z of length $h>(\omega(m))^{\frac12+\delta}$ for Type A.

3. Normal numbers. We prove the following theorem:

THEOREM 5. *Let b complete periods from each of the fractions Z_1/m, $Z_2/m^2,\ldots$ represented in any base g such that $(g,m)=1$ where the Z_i/m^i are in lowest terms and m is any positive integer, be placed in juxtaposition. The resulting real number $x(g,m)$ which can be written*

$$x(g,m)=\sum_{n=0}^{\infty}(Z_{n+1}/m^{n+1}-Z_n/m^n)/g^{S(n,m)} \tag{3.0}$$

where

$$S(n,m)=b\sum_{i=1}^{n}\omega(m^i),\qquad S(0,m)=0,\qquad \textit{and}\qquad \omega(y)=\mathrm{ord}_y g,$$

is normal in the base g^t for each integer $t>0$.

Let us first prove the construction for one period, $b=1$; then below (3.31), we will complete the proof for $b>1$ periods in a general discussion of these kind of constructions.

Proof. Consider

$$x(g,m,n)=.E_1E_2\ldots E_nB_r \tag{3.1}$$

where E_i is the set of digits in one period of Z_i/m^i represented in a base g such that $(g,m)=1$ and B_r represents a block of r digits into the period of E_{n+1}. Define the relative frequency of B_j

$$N(t,x,B_j)/t=\Big(\sum_{i=1}^{n}N(B_j,E_i)+N(B_j,r)\Big)/(S(n,m)+r) \tag{3.2}$$

in the first $t=S(n,m)+r$ digits of (3.1) where $N(B_j,E_i)$ or $N(B_j,r)$ denote the number of occurrences of a block B_j consisting of any combination of j digits taken from $0,1,\ldots,g-1$ in one period E_i or in the first r digits of the period E_{n+1}, respectively. First, we have some preliminaries. As described in [7, p. 243], we must account for the possibility of anomalous blocks, i.e. in the count of some B_j, there may be some that extend across the $n-1$ junctures of $\widehat{E_iE_{i+1}}$, $i=1,2,\ldots,n-1$ and an additional $j-1$ across $\widehat{E_nB_r}$. Therefore, we are lead to consider as in [7, p. 243, (2.4), (2.5)]

$$|N(t,x,B_j)/t-I|\leqslant n(j-1)/t \tag{3.3}$$

where I is the right hand side of (3.2).

Since $S(n,m)\geqslant C_0+C_1m^{n-k}$ [7, p. 247, (2.31)], we have

$$n(j-1)/t\leqslant n(j-1)/(r+C_0+C_1m^{n-k}) \tag{3.4}$$

and clearly, $\lim\limits_{n\to\infty}n(j-1)/t=0$ for any fixed choice of j for given r, C_0, C_1, m, and k. Hence, we have

$$\lim_{t\to\infty}N(t,x,B_j)/t=\lim_{n\to\infty}I \tag{3.5}$$

or

$$\lim_{n\to\infty}I=\lim_{n\to\infty}(P_n+Q_n)/(1+R_n) \tag{3.6}$$

where

$$P_n=\Big(\sum_{i=1}^{n}N(B_j,E_i)\Big)/S(n,m),\qquad Q_n=N(B_j,r)/S(n,m),\qquad \text{and } R_n=r/S(n,m). \tag{3.7}$$

We distinguish 2 cases, i.e. $r\gtrless(\omega(m^{n+1}))^{\frac12+\delta}$. In case 1 where $r<(\omega(m^{n+1}))^{\frac12+\delta}$, we find that the effect of the B_r set is negligible in the limit. If $r>(\omega(m^{n+1}))^{\frac12+\delta}$ for case 2, then such an additional block participates in the normality.

Case 1. $r < (\omega(m^{n+1}))^{\frac{1}{2}+\delta}$. Since $S(n, m) > \omega(m^n)$, we have

$$(3.8)\qquad N(B_j, r)/S(n, m) \leqslant r/S(n, m) < (\omega(m^{n+1}))^{\frac{1}{2}+\delta}/\omega(m^n).$$

For n sufficiently large, we have for the upper bound in (3.8)

$$(3.9)\qquad (\omega(m^{n+1}))^{\frac{1}{2}+\delta}/\omega(m^n) = (m\omega(m^n))^{\frac{1}{2}+\delta}/\omega(m^n) = m^{\frac{1}{2}+\delta}/(\omega(m^n))^{\frac{1}{2}-\delta}.$$

Therefore, for any δ such that $0 < \delta < \frac{1}{2}$, it follows that $\lim_{n\to\infty} m^{\frac{1}{2}+\delta}/(\omega(m^n))^{\frac{1}{2}-\delta} = 0$. Consequently, from (3.7), we have $\lim_{n\to\infty}(Q_n, R_n) = 0$, and therefore (3.5) becomes

$$(3.10)\qquad \lim_{t\to\infty} N(t, x, B_j)/t = \lim_{n\to\infty} P_n = \lim_{n\to\infty} N(B_j, E_n)/\omega(m^n).$$

From this point on, to complete Case 1, the argument is precisely [7, p. 245, (2.18)–(2.22)] and we are led to

$$(3.11)\qquad \lim_{t\to\infty} N(t, x, B_j)/t = \lim_{n\to\infty} N(B_j, E_n)/\omega(m^n) = 1/g^j$$

for all j and $g \geqslant 2$ such that $(g, m) = 1$.

It is interesting to note in this construction that m can be any positive integer, no matter what its composite structure, since from some point on in the sequence $Z_1/m, Z_2/m^2, \ldots$ there exists an N such that all Z_n/m^n for $n > N$ become and remain Type A. This is the condition necessary for the (j, ε)-normality which leads to (3.11).

Case 2. $r > (\omega(m^{n+1}))^{\frac{1}{2}+\delta}$. We first prove a useful lemma related to the basic concept of (j, ε)-normality defined by Besicovitch in 1934 [8, p. 202, (2.1)].

Consider 2 sets of digits H and P (or integers!) consisting of h and p digits, respectively, represented in a base $g \geqslant 2$ chosen from $0, 1, \ldots, g-1$ which are (j, ε)-normal, i.e.

$$|N(B_j, H)/h - 1/g^j| < \varepsilon_h \quad\text{and}\quad |N(B_j, P)/p - 1/g^j| < \varepsilon_p$$

where $N(B_j, H)$ and $N(B_j, P)$ designate the number of occurrences of a block B_j contained in H and P, respectively. Consider as well the set of digits formed by placing the sets H and P in juxtaposition, yielding a total of $h+p$ digits. We have

LEMMA 7. *If two sets of (j, ε)-normal digits H and P are placed in juxtaposition, then the resulting set of $h+p$ digits with*

$$\varepsilon = (h\varepsilon_h + p\varepsilon_p)/(h+p)$$

is (j, ε)-normal assuming $(h+p)/(h\varepsilon_h + p\varepsilon_p) \geqslant g \geqslant 2$ so that

$$j \leqslant [\log_g (h+p)/(h\varepsilon_h + p\varepsilon_p)] \geqslant 1.$$

Proof. From the given (j, ε)-normal properties of the H and P sets, we have

$$(3.12)\qquad |(N(B_j, H) + N(B_j, P))/(h+p) - 1/g^j| < (h\varepsilon_h + p\varepsilon_p)/(h+p)$$

and we require that $1/g^j - (h\varepsilon_h + p\varepsilon_p)/(h+p) > 0$ to insure that

$$(N(B_j, H) + N(B_j, P))/(h+p) > 0$$

which implies

$$j \leqslant [\log_g (h+p)/(h\varepsilon_h + p\varepsilon_p)].$$

If $(h+p)/(h\varepsilon_h + p\varepsilon_p) \geqslant g$, then $j \geqslant 1$. Q. E. D.

Therefore, if we consider that each set of $\omega(m^i)$ digits in $N(t, x, B_j)$ are of Type A and are, therefore, (j, ε)-normal with each ε_i prescribed, and also by Theorem 4; the set of $r > (\omega(m^{n+1}))^{\frac{1}{2}+\delta}$ digits constituting B_r contained in E_{n+1} are (j, ε)-normal as well; it follows, using Lemma 7, that in the combined set $N(t, x, B_j)/t$, we have

$$(3.13)\qquad |N(t, x, B_j)/t - 1/g^j| < \Big(\sum_{i=1}^{n} \varepsilon_i \omega(m^i) + r\varepsilon_r\Big)/(S(n, m) + r)$$

where according to Theorem 4, $\varepsilon_r = O\big(\log^{2/3}\omega(m^{n+1})/(\omega(m^{n+1}))^{\lambda}\big)$.

Applying Cauchy's limit theorem to (3.13), we obtain

$$(3.14)\qquad \lim_{t\to\infty} |N(t, x, B_j)/t - 1/g^j| < \lim_{n\to\infty} (\varepsilon_n \omega(m^n) + r\varepsilon_r)/(\omega(m^n) + r) = \varepsilon.$$

The right hand side of (3.14) can be written

$$(3.15)\qquad \varepsilon = \lim_{n\to\infty} (\varepsilon_n + (\varepsilon_r - \varepsilon_n))/(1 + \omega(m^n)/r)$$

where $\varepsilon_n = O(\log^{2/3}\omega(m^n))/(\omega(m^n))^{\lambda}$ and therefore, $\varepsilon \to 0$ as $n \to \infty$ in (3.15) since $\varepsilon_n \to 0$ and $\varepsilon_r \to 0$ independent of the behavior of $\omega(m^n)/r$ for $\lambda > 0$. By Lemma 7, $j \leqslant \lim_{n\to\infty} [\log_g 1/\varepsilon_n]$, therefore, $\lim_{n\to\infty} x(g, m, n)$ is normal in all g such that $(g, m) = 1$ for all j. Q. E. D.

Let us now prove a theorem which shows that we may form a normal number by placing in juxtaposition, one set of any portion of the periods of $Z_1/m, Z_2/m^2, \ldots$ greater than the square root of the successive lengths $\omega(m^i)$.

After the proof of Theorem 6 below, we will discuss the validity of Theorems 5 and 6 for the case of $b > 1$ integral periods of Z_i/m^i for $i = 1, 2, \ldots$ or portions of periods.

THEOREM 6. *Let sets of $h_i > (\omega(m^i))^{\frac{1}{2}+\delta_i}$ digits be chosen successively anywhere within the periods of the fractions $Z_1/m, Z_2/m^2, \ldots$ of Type A in lowest terms represented in a base g such that $(g, m) = 1$ where $\delta_i \geqslant \delta > 0$ for all i and any fixed $\delta > 0$, and place these sets in juxtaposition. The resulting real number which can be written*

$$(3.16) \qquad \theta(g, m) = \sum_{n=1}^{\infty} [Z_n g^{h_n}/m^n]/g^{S(h)},$$

where $S(h) = \sum_{i=1}^{n} h_i$ and $[y]$ denotes the greatest integer less than y, is normal in the base g^t for each positive integer t.

Proof. Let E_i' designate any set of $h_i > (\omega(m^i))^{\frac{1}{2}+\delta_i}$ digits where $\delta_i \geqslant \delta > 0$ for all i taken anywhere within the complete periods E_i of Z_i/m^i for $i = 1, 2, \ldots, n$ and B_r, a set of $h_{n+1} > (\omega(m^{n+1}))^{\frac{1}{2}+\delta_{n+1}}$ digits within the period E_{n+1} of Z_{n+1}/m^{n+1}. Let, as usual, $N(B_j, E_i')$ and $N(B_j, r)$ designate the number of blocks B_j contained in the E_i' sets and the block B_r of r digits into E_{n+1}', respectively.

We form the juxtaposition construction

$$(3.17) \qquad \theta(g, m, n) = .\, E_1' E_2' \ldots E_n' B_r$$

where

$$(3.18) \qquad N(t, x\ B_j)/t = \Big(\sum_{i=1}^{n} N(B_j, E_i') + N(B_j, r)\Big)\Big/\big(S(h)+r\big)$$

and

$$(3.19) \qquad t = S(h) + r = \sum_{i=1}^{n} h_i + r.$$

As before, accounting for anomalous blocks, we have

$$(3.20) \qquad |N(t, x, B_j)/t - I| \leqslant n(j-1)/t$$

where I is the right hand side of (3.18). Since

$$(3.21) \qquad \begin{aligned} n(j-1)/t &= n(j-1)/\big(S(h)+r\big) < n(j-1)/\Big(\big(\omega(m^n)\big)^{\frac{1}{2}+\delta_n} + r\Big) \\ &= n(j-1)/\Big(r + \big(m^{n-k}\omega(m^k)\big)^{\frac{1}{2}+\delta_n}\Big) \end{aligned}$$

where k, δ_n, and m are fixed [7, (2.29)], it follows that $\lim_{n\to\infty} n(j-1)/t = 0$. Thus, we have

$$(3.22) \qquad \lim_{t\to\infty} N(t, x, B_j)/t = \lim_{n\to\infty} I$$

where I can be written using Cauchy's limit theorem

$$(3.23) \qquad \lim_{n\to\infty} I = \big(\lim_{n\to\infty} N(B_j, E_n')/h_n + \lim_{n\to\infty} N(B_j, r)/h_n\big)/(1 + \lim_{n\to\infty} r/h_n).$$

Now as in the proof of Theorem 5, we distinguish 2 cases.

Case 1. $r < (\omega(m^{n+1}))^{\frac{1}{2}+c\delta}$. For the two limits in (3.23), we have for $h_n > (\omega(m^n))^{\frac{1}{2}+\delta_n}$ and n sufficiently large with $0 < c < 1$

$$(3.24) \qquad \begin{aligned} N(B_j, r)/h_n \leqslant r/h_n &< \big(\omega(m^{n+1})\big)^{\frac{1}{2}+c\delta}/\big(\omega(m^n)\big)^{\frac{1}{2}+\delta_n} \\ &= m^{\frac{1}{2}+c\delta}\big(\omega(m^n)\big)^{\frac{1}{2}+c\delta}/\big(\omega(m^n)\big)^{\frac{1}{2}+\delta_n}. \end{aligned}$$

Since $\delta_n \geqslant \delta \Rightarrow (\omega(m^n))^{\frac{1}{2}+\delta_n} \geqslant (\omega(m^n))^{\frac{1}{2}+\delta}$, (3.24) becomes

$$(3.25) \qquad N(B_j, r)/h_n \leqslant r/h_n < m^{\frac{1}{2}+c\delta}\big(\omega(m^n)\big)^{\frac{1}{2}+c\delta}/\big(\omega(m^n)\big)^{\frac{1}{2}+\delta}$$

which approaches zero for $0 < c < 1$ since $\omega(m)$ is unbounded for increasing n. Therefore, in (3.23), we have left

$$(3.26) \qquad \lim_{t\to\infty} N(t, x, B_j)/t = \lim_{n\to\infty} N(B_j, E_n')/h_n.$$

By Theorem 4, we have

$$(3.27) \qquad \lim_{t\to\infty} |N(t, x, B_j)/t - 1/g^j| = \lim_{n\to\infty} |N(B_j, E_n')/h_n - 1/g^j| < \lim_{n\to\infty} \varepsilon_n$$

where $j \leqslant [\log_g 1/\varepsilon_n]$ and

$$(3.28) \qquad \lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} O\Big(\log^{2/3}\omega(m^n)/\big(\omega(m^n)\big)^{\lambda}\Big) = 0$$

for any $\lambda = \frac{2}{3}\delta_n \geqslant \delta > 0$. [Note: The constant in the O is independent of n in the expression envolving $\omega(m^n)$ in (3.28).]

Therefore, we have obtained

$$(3.29) \qquad \lim_{t\to\infty} N(t, x, B_j)/t = 1/g^j$$

for Type A with g such that $(g, m) = 1$ and all $j \geqslant 1$, i.e. the construction $\theta(g, m) = \lim_{n\to\infty} \theta(g, m, n)$ is normal in all such $g \geqslant 2$.

Case 2. $r > (\omega(m^{n+1}))^{\frac{1}{2}+c\delta}$. If we again apply Lemma 7 to (3.18), we obtain, using Cauchy's limit theorem

$$(3.30) \qquad \begin{aligned} \lim_{t\to\infty} |N(t, x, B_j)/t - 1/g^j| &< \lim_{n\to\infty} (h_n\varepsilon_n + r\varepsilon_r)/(h_n + r) \\ &= \lim_{n\to\infty} \big(\varepsilon_n + (\varepsilon_r - \varepsilon_n)\big)/(1 + h_n/r) = 0 \end{aligned}$$

where $\varepsilon_n \to 0$ and $\varepsilon_r \to 0$ as in (3.28) above for $\lambda > 0$. Finally, we have shown that $\lim_{n\to\infty} x(g, m, n) = x(g, m)$ is normal in the base g, i.e.

$$(3.31) \qquad \lim_{t\to\infty} N(t, x, B_j)/t = 1/g^j$$

for all j since by Lemma 7, $j \leqslant [\log_g (h_n + r)/(h_n\varepsilon_n + r\varepsilon_r)]$ which increases without bound as n increases. Q.E.D.

In proving the rest of Theorem 5 for some fixed integral $b > 1$ repetitions of the periods Z_i/m^i, it will deepen our understanding of the particular class of normal numbers we have been constructing if we analyze the behavior of the particular limits which may or may not have required the assumption of a divergent increasing sequence a_n in [7]. At the same time, we obtain a proof of the validity of (3.0) in the present paper which is basically the same result as [7, p. 242, (2.0)] assuming $a_i = b$ for all i. By a somewhat simpler argument for [7, p. 245, (2.15)] related to the anomalous blocks and similar to (3.4) of this paper, we have

$$(3.32)\qquad R_n = n(j-1)/t < n(j-1)/a_{n-1}\omega(m^{n-1}) = n(j-1)/a_{n-1}m^{n-1-d}\omega(m^d)$$

where d is some fixed positive integer. Clearly, even for a fixed $a_{n-1} = b$, the $R_n \to 0$ as $n \to \infty$, both in [7, (2.15)] and in the present case in (3.4), i.e. for this limit in [7], $a_n \to \infty$ was not necessary due to the rapid increase of the $\omega(m^n)$ for $n \to \infty$.

On the other hand, for the limits $N(B_j, r)/P_n \leqslant r/P_n$ in [7] with P_n as defined in [7, (2.10)], we have by a simpler argument here than that given in [7, p. 245, (2.15)],

$$(3.33)\qquad N(B_j, r)/P_n \leqslant r/P_n < m\omega(m^{n-1})/\big(a_{n-1}\omega(m^{n-1}) + km\,\omega(m^{n-1})\big) = m/(a_{n-1} + km)$$

which clearly shows that even though here in (3.33), we tried to utilize the rapid increase of the $\omega(m^{n-1})$ to the best advantage, it was not the crucial quantity.

We had to require that $a_{n-1} \to \infty$ so that $\lim\limits_{n\to\infty}\big(N(B_j, r)/P_n, r/P_n\big) = 0$. In the present paper, similar to (3.33), we simply introduce the fixed integer $b > 1$ in the denominator of the upper bound in (3.8). Therefore, the $\lim\limits_{n\to\infty}\Big(\big(N(B_j, r) \leqslant r\big)/bS(n, m)\Big) = 0$ for case 1 depends upon the diminishing ratio as n increases of the magnitude $\omega(m^{n+1})^{\frac{1}{2}+\delta}$ within the period E_{n+1} of Z_{n+1}/m^{n+1} and the growth of the previous period $\omega(m^n)$ even if it is repeated a fixed $b > 1$ number of times.

In general, we have from (3.2)

$$(3.34)\qquad N(t, x, B_j)/t = \Big(b\sum_{i=1}^{n} N(B_j, E_i) + N(B_j, r)\Big)/\big(bS(n, m) + r\big)$$

which shows that in case 1 for Theorem 5 that (3.10) and (3.11) would be independent of any $b > 1$. The same comments apply to case 2, i.e. $b > 1$ entering (3.13) and (3.14) in the same way as (3.34) but with no affect on the result. The proof of Theorem 5 is now complete.

All of the above comments apply for Theorem 6 on portions of periods since $b > 1$ would enter into (3.18) and (3.30) as it does in (3.34).

The summation result similar to (3.16) for the portions of the periods construction for any $b > 1$ is

$$(3.35)\qquad \theta(g, m) = \sum_{n=1}^{\infty}(g^{bh_n} - 1)[Zg^{h_n}/m^n]/(g^{h_n} - 1)g^{bS(h)}$$

which for $b = 1$ reduces to (3.16).

In (3.0), if we set $Z_n = (p^n - 1)/(p - 1)$, i.e. we have

$$(3.36)\qquad Z_n/p^n = (1 + p + p^2 + \ldots + p^{n-1})/p^n = (p^n - 1)/(p - 1)p^n,$$

then $Z_{n+1}/p^{n+1} = (p^{n+1} - 1)/(p - 1)p^{n+1}$. At the same time, let $b = p$,

$$S(n, p) = p\sum_{i=1}^{n}\omega(p^i) = p(p^n - 1),$$

and therefore, we obtain, when g is a primitive root,

$$(3.37)\qquad x(g, p) = \sum_{n=0}^{\infty} 1/p^{n+1}g^{p^{n+1}-p} = g^p\sum_{n=0}^{\infty} 1/p^{n+1}g^{p^{n+1}}.$$

Thus, we have (1.0) since we can drop the g^p as a g-adic shift in the representation which would not affect the normality, and then replace n by $n - 1$ in the sum.

4. Transcendence, Brouwer conjecture. In [7, p. 247, Th. 2], we proved that these constructions lead to transcendentals of the non-Liouville type if there exist 2 positive constants δ and β independent of n such that

$$(4.0)\qquad \delta < a_{n+1}\omega(m^{n+1})/S(n, m) < \beta$$

where $S(n, m) = \sum_{i=1}^{n} a_i\omega(m^i)$ and $a_n \to \infty$ as $n \to \infty$.

We prove the following theorem which gives a sharper view of the significance of this requirement for transcendence in (4.0).

THEOREM 7. *If δ and β exist such that $\delta < a_{n+1}\omega(m^{n+1})/S(n, m) < \beta$ where δ and β are positive constants, $S(n, m) = \sum_{i=1}^{n} a_i\omega(m^i)$, $\omega(m^i) = \mathrm{ord}_{m^i}g$, and $a_n \to \infty$ as $n \to \infty$, or a_n is constant, then*

(i) *$x(g, m)$ in [7, (2.0)] and (3.0) is a transcendental non-Liouville normal number;*

(ii) *the $a_n = O(B^n)$ where $B \geqslant 1$ is some positive constant, or more specifically; increasing a_n for a fixed a_1, β, m, and k, must be such that*

$$(4.1)\qquad a_{n+1} < a_1\beta m^{k-2}\big((\beta + 1)/m\big)^{n-1}$$

for $\beta > m - 1$; and,

(iii) $\lim\limits_{n\to\infty} a_{n+1}/a_n = c \geqslant 1$ *where c is some constant.*

Proof. From (4.0), we have

$$a_{n+1}\omega(m^{n+1}) < \beta\big(a_1\omega(m)+a_2\omega(m^2)+\ldots+a_n\omega(m^n)\big) \tag{4.2}$$

and proceeding recursively for $n = 1, 2, \ldots$, we find

$$a_2\omega(m^2) < \beta a_1\omega(m), \quad a_3\omega(m^3) < \beta(\beta+1)a_1\omega(m), \quad \ldots.$$

Hence, we obtain

$$a_{n+1} < a_1\beta m^{k-2}\big((\beta+1)/m\big)^{n-1} \tag{4.3}$$

where $k \geqslant 1$ is fixed for a given m depending on $\omega(m) = \omega(m^2) = \ldots = \omega(m^k)$ but $\omega(m^{k+1}) = m\omega(m^k)$, etc. Clearly, (4.3) shows that $a_n = O(B^n)$ which, therefore, has a linear exponential growth where we must set $B = (\beta+1)/m > 1$ so that $a_n \to \infty$ as $n \to \infty$.

Also from (4.1), we may write

$$\lim_{n\to\infty} a_{n+1}\omega(m^{n+1})/a_n\omega(m^n) = m\lim_{n\to\infty} a_{n+1}/a_n, \tag{4.4}$$

hence; the transcendence condition implies that $\lim_{n\to\infty} a_{n+1}/a_n = c$ where $c \geqslant 1$ is some constant. We can set $\beta = cm$, then such a choice would always satisfy our transcendence condition independent of n. Also, note that now $B = (\beta+1)/m = (cm+1)/m > 1$ for $m \geqslant 2$, thus the a_n will increase appropriately. For the case of $B = 1$, from (4.3), we have $a_n = b$ (a fixed positive integer) which we have now shown gives a normal number in Theorem 5, hence; $b < b\beta m^{k-2}\big((\beta+1)/m\big)^{n-1}$ or

$$1/m^{k-2} < \big((\beta+1)/m\big)^{n-1} \tag{4.5}$$

which clearly holds for $k \geqslant 1$ and any fixed choice of $\beta > 1$ independent of n such that $(\beta+1)/m > 1$ or $\beta > m-1$ for $m \geqslant 2$. Q. E. D.

According to Theorem 7 above, the transcendence condition implies a broad class of repetition sequences a_n, i.e. of linear exponential order. As a few examples, these can be integer valued polynomials, $a_n = P(n) = b_r n^r + b_{r-1}n^{r-1} + \ldots$ where the b_i are positive integers for which sequences; $\lim_{n\to\infty} a_{n+1}/a_n = 1$ $(c = 1)$, $a_n = 2^n$, for which $\lim_{n\to\infty} a_{n+1}/a_n = 2$ $(c = 2)$, etc. On the other hand, if we choose $a_n = 2^{n^2}$ for which $\lim_{n\to\infty} a_{n+1}/a_n$ is unbounded, we still obtain a normal number $x(g, m)$, but the transcendence condition does not hold since such a_n are not of a linear exponential order. Therefore, it is an open question as to the irrational character of such a normal number construction using $a_n = 2^{n^2}$ for the repetition sequence.

In 1925, L. E. J. Brouwer [2, p. 3] and others in the intuitionist school of mathematical logic often stated the following as a possible "undecidable" proposition. Can we *prove* that some prescribed block, like say, 0123456789, appears in the infinite expansion of an irrational like π in the base 10? In [6, p. 235], we showed how a small advance on this question could be attained using the results we have obtained. We rephrased the question for rationals and showed that the same question now can be answered completely for (j, ε)-normal rational fractions of Type A, i.e. we can prove that an arbitrary block like 0123456789 *will* make a first appearance somewhere in set of digits of one period, and, of course, repeat itself a number of times in this set. For example, as a further emphasis of these surprising results, it is interesting that we can prove, starting somewhere in the approximately 2.652×10^{1230} digits of $1/17^{1000}$, we will find π exactly to at least 1230 places in

$$1/17^{1000} = .b_1b_2\ldots b_k 314159265389793\ldots314159\ldots, \tag{4.6}$$

and it will repeat this same set about 12 times. The same can be said of "e", $\sqrt{2}$, γ (Euler's constant), etc., *any combination* of up to about 1230 digits will be there with the required frequency. This is the meaning of the (j, ε)-normal phenomenon for rationals. Of course, this kind of result could be stated probabilistically as Borel did, i.e. "any real number has a probability of one of being absolutely normal" [1, p. 198], i.e. normal in any positive integer base $\geqslant 2$. Today most of the results concerning normal numbers are stated in measure-theoretic terminology but are non-constructive, "almost all real numbers are absolutely normal". But it is quite another question to be able to *prove* these kind of statements about some specific real number like $1/17^{1000}$ without computing out the obviously impossibly large set of 10^{1230} digits to see if it is true! In [6, p. 235], we said that we could not say where a prescribed block would make its first appearance in the period but now, with the results of this paper, we can make a definite advance on this question.

For convenience in stating results, we have used "O" estimates, but if desired, we can work more quantitatively than the use of "O" often implies. For example, let us *calculate* quite precisely the j bounds on blocks B_j that can be found with certainty within, approximately, the square roots of period lengths. Placing (2.29) into (2.7), and letting $\omega(m) = \omega(p^a) = \omega(17^{1001}) = 16\cdot17^{1000}$, we obtain rather precisely an ε which bounds the discrepancy in (2.1). Setting $h > \big(\omega(m)\big)^{\frac{1}{2}+\delta}$, $N(\beta)-N(\alpha) = N(B_j, g)$, $\alpha = B_j/g^j$, and $\beta = (B_j+1)/g^j$ in (2.1), we have for g a primitive root mod p^2

$$|N(B_j, g)/h - 1/g^j| < \varepsilon \tag{4.7}$$

where we shall set $D = 17$ in (2.29) then place (2.29) in (2.7) and choose

$$\varepsilon = 2\left(\frac{17\,\omega(17^{1001})\log^2\omega(17^{1001})\,K}{h^2} + \frac{1}{17^{2002}}\right)^{1/3} \tag{4.8}$$

with

$$K = \Big(6/\pi^2 \sum_{r=1}^{17^{1001}-1} 1/r^2 + 6/\pi^2 17^{1001} \sum_{r=1}^{17^{1001}-1} 1/r\Big) \cong 1. \tag{4.9}$$

Since the first sum in (4.9) is very close to one (to an order of $1/17^{1001}$) and the second to zero of about the same order, we set $K \cong 1$, hence

$$\varepsilon = 2\cdot 17^{1/3}\big(\log 16\cdot 17^{1000}/(16\cdot 17^{1000})^{\delta}\big)^{2/3} \tag{4.10}$$

for $h > (16\cdot 17^{1000})^{\frac{1}{2}+\delta}$. Hence, for j bounds to keep $N(B_j, g)/h > 0$, we have

$$\log_{10} 1/\varepsilon = (2\delta/3)(1231.65) - (2/3)(3.0899) \cong 821\,\delta - 2. \tag{4.11}$$

Thus, if $\delta = 1/10$, then we have $j \leqslant 80$ for any $h > 17^{511}$. So that in any set of $h > 17^{511}$ digits from *within* the full period of $16\cdot 17^{1000}$ digits in $1/17^{1001}$ (i. e. a total of about 17^{489} such sets), we know that *any* combination of up to about 80 digits will appear with certainty in each such set. One might look at this result and say, it seems highly "probable" that such a comparatively small set of 80 arbitrarily chosen digits "should" appear within such a large set as 17^{511} digits, but on the other hand we are not concerned with "probabilities" in our approach. We are concerned with *provable* certainties. In this sense, continuing with the Wallis nth product representation of $\pi/4$ given by

$$P_n(\pi/4) = \prod_{i=1}^{n} \big(1 - 1/(2i+1)^2\big) = p_n/q_n$$

which is of Type A as we asserted in [6, p. 235], we can now refine the statement we made in [6], i.e. that the block 0123456789 *will occur* in the whole period of this representation. We can now say that the block 0123456789 will occur with certainty within the block of digits that commences the period slightly greater than the square root of the period, i.e. $\sqrt{\omega(q_n)}$ (for n sufficiently large). However, even though we have sharpened our result in [6], we cannot yet say whether the block in question is in that portion which is $\pi/4$ exactly or in the part which will change as n increases.

Finally, the results here also have important implications for the analytical properties of multiplicative congruential [10] pseudo-random number generators which have been in wide use for a number of years. To date, there has not been available any mathematical basis on which to prove the assumed "randomness" or uniformity of sequences of residues taken from within the periods of particular generators. Theorems 1–4 that we offer here on the uniform ε-distribution of sequences of normalized residues within the periods will furnish such a basis. We will present such results in the near future.

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