Conspectus materiae tomi XXII, fasciculi 4

La revue est consacrée à la Théorie des Nombres
The journal publishes papers on the Theory of Numbers
Die Zeitschrift veröffentlicht Arbeiten aus der Zahlentheorie
Журнал посвящен теории чисел

L'adresse de la Rédaction et de l'échange
Address of the Editorial Board and of the exchange
Die Adresse der Schriftleitung und des Austausches
Адрес редакции и книгообмена

ACTA ARITHMETICA
ul. Śniadeckich 8, 00-950 Warszawa (Poland)

Les auteurs sont priés d'envoyer leurs manuscrits en deux exemplaires
The authors are requested to submit papers in two copies
Die Autoren sind gebeten um Zusendug von 2 Exemplare jeder Arbeit
Рукописи статьей редакция просит предлагать в двух экземплярах

PRINTED IN POLAND

W R O C Ł A W S K A D R U K A R N I A N A U K O W A

ACTA ARITHMETICA
XXII (1973)

A method in diophantine approximation V

by

Charles F. Osgood (Washington, D.C.)*

In this paper we shall prove several theorems which allow us to make statements concerning the arithmetical properties of the Taylor series coefficients of the functions in *any* fundamental system of solutions of a linear homogeneous differential equation of the type treated in part II of this series of papers (see [2]), at $m \geqslant 1$ distinct Gaussian rational points. As was seen in [2] and [3] these solutions need *not* be entire.

Let z denote a complex variable; let D denote $\frac{d}{dz}$; let l denote a fixed integer larger than or equal to one; and let each $g_j(z)$, for $1 \leqslant j \leqslant l$, denote a polynomial of degree exactly $j-1$ with coefficients which are parameters $\beta_1 = \gamma_1 + i\delta_1, \ldots, \beta_{\frac{l(l+1)}{2}} = \gamma_{\frac{l(l+1)}{2}} + i\delta_{\frac{l(l+1)}{2}}$ that takes values in $Q(i)$ (the Gaussian field). Suppose that $y_1, \ldots, y_l$ denote any l linearly independent solutions of

$$y = \sum_{j=1}^{l} g_j(z) D^j y. \tag{1}$$

Suppose further that $z_1, \ldots, z_k, \ldots, z_m$ denote any $m \geqslant 1$ distinct point in $Q(i)$. Set each $z_k = x_k + iu_k$, where x_k and u_k each denote real numbers

THEOREM I. *There exists an effectively computable polynomial in $x_1, \ldots, x_m, u_1, \ldots, u_m, \gamma_1, \ldots, \gamma_{\frac{l(l+1)}{2}}, \delta_1, \ldots, \delta_{\frac{l(l+1)}{2}}$ with coefficients in Q, which does not vanish identically in $\gamma_1, \ldots, \gamma_{\frac{l(l+1)}{2}}, \delta_1, \ldots, \delta_{\frac{l(l+1)}{2}}$ for any choice of distinct $z_1, \ldots, z_m$ in $Q(i)$, such that except for those $(z_1, \ldots, z_m, \beta_1, \ldots, \beta_{\frac{l(l+1)}{2}})$ where this polynomial vanishes the $z_1, \ldots, z_m$ are distinct points of analyticity of the $y_1, \ldots, y_l$ and the field F generated over $Q(i)$ by the numbers $D^\varphi y_j(z_k)$, for $1 \leqslant j \leqslant l$, $1 \leqslant k \leqslant m$, and $0 \leqslant \varphi < \infty$, has dimension over $Q(i)$ at least m.*

* Mathematics Research Center, Naval Research Laboratory, Washington D. C. 20390.

In Theorem II below we obtain more insight into the exceptional case, i.e., when the polynomial in Theorem I vanishes. (Theorem I is actually a corollary of Theorem II.)

DEFINITIONS. Let $w_1 = w_1(z), \ldots, w_k = w_k(z), \ldots, w_m = w_m(z)$ denote the m distinct branches of the algebraic function defined by $\prod_{k=1}^{m}(w-z_k) = z$. Let $A(z) = A(z, z_1, \ldots, z_m, \beta_1, \ldots, \beta_{\frac{l(l+1)}{2}})$ denote the ml by ml matrix such that

$$\big(D^\theta y_j(w_k(z))\big) = A(z)\big(w_k^\delta(z)\, y_j^{(\varphi)}(w_k(z))\big),$$

where $0 \leqslant \theta \leqslant ml-1$, $1 \leqslant j \leqslant l$, $1 \leqslant k \leqslant m$, $0 \leqslant \delta \leqslant m-1$, $0 \leqslant \varphi \leqslant l-1$, and on each side of the equation immediately above the columns are indexed by the ordered pairs (j, k). Each entry of $A(z)$ is of the form $\big(\prod_{k=1}^{m} g_l(w_k(z))\big)^{-N}$ times an element of $Q[i, z, z_1, \ldots, z_m, \beta_1, \ldots, \beta_{\frac{l(l+1)}{2}}]$, for some positive integer N. For fixed $z_1, \ldots, z_m$, and fixed $\beta_1, \ldots, \beta_{\frac{l(l+1)}{2}}$ such that $g_l(z) \not\equiv 0$ let $ml-r$, for $0 \leqslant r \leqslant ml$, denote the rank of $A(z)$.

THEOREM II. *The matrix* $A(z, z_1, \ldots, z_m, \beta_1, \ldots, \beta_{\frac{l(l+1)}{2}})$ *is effectively computable and its determinant does not vanish identically in* $\beta_1, \ldots, \beta_{\frac{l(l+1)}{2}}$ *for any choice of complex numbers* $z_1, \ldots, z_m$. *If* $z_1, \ldots, z_m$ *are distinct elements of* $Q(i)$ *and the* $\beta_1, \ldots, \beta_{\frac{l(l+1)}{2}}$ *are elements of* $Q(i)$ *such that no* $g_l(z_k) = 0$, *then*: (i) *the integer* $ml-r$ *is effectively computable*; (ii) *the dimension of the vector space over* C *(the complex numbers) spanned by the* $y_j(w_k(z))$, *for* $1 \leqslant j \leqslant l$ *and* $1 \leqslant k \leqslant m$, *is exactly* $ml-r$; *and* (iii) *the dimension of the field* F, *from Theorem* I, *generated over* $Q(i)$ *by the* $D^\varphi y_j(z_k)$ *is at least* $m - rl^{-1}$.

One easily sees that the polynomial in Theorem I may be taken to be the sum of the squares of the absolute values of the coefficients of the powers of z in the numerator of the determinant of $A(z)$ times $\big|\prod_{k=1}^{m} g_l(z_k)\big|^2 \cdot \prod_{k<k_1} |z_k - z_{k_1}|^2$.

If this polynomial does not vanish we may apply part (iii) of Theorem II with $r = 0$ to prove Theorem I.

There are results which sound analogous to Theorems I and II except that they involve the condition that the function or functions under discussion be entire (see [5], [6], [7], and [8]). Here, if (1) has even one non-entire solution, then it is possible to choose a fundamental system of solutions, by a simple vector space argument, which contains *no* entire solutions, nor does the vector space over the algebraic numbers spanned by these solutions contain any entire solutions except zero. We shall state two more theorems which shed further light on the behavior of the functions $y_1, \ldots, y_l$ and then give (in Theorem V) the key algebraic result which allows us to obtain these generalizations of the results of [2]. A rough version of Theorem V is that if the $y_j(z)$ each satisfy (1) then the $y_j(w_k(z))$ each satisfy a new linear differential equation which is also of type (1). Since [2] is used very extensively here, at the end of this paper are a list of corrections for [2].

THEOREM III. *If in Theorem* II *the functions* $y_1, \ldots, y_t$, *for some* $t < l$, *are each the difference of two branches of a solution of* (1) *then the dimension of* F *over* $Q(i)$ *is at least* $m + (mt-r)(l-t)^{-1}$.

Note that if $t = l-1$ and $r = 0$ the dimension is at least ml.

DEFINITIONS. Let $W_{\theta_1, \ldots, \theta_{ml}}(f_1, \ldots, f_{ml})(z)$ denote $\big|\big(D^{\theta_j} f_k(z)\big)\big|$ where $1 \leqslant j \leqslant ml$ and $1 \leqslant k \leqslant ml$. Let Z denote the integers.

THEOREM IV. *Under the conditions of Theorem* III *if* $t = l-1$ *and* $r = 0$ *then there exists a sequence of non-negative integers* $0 \leqslant \theta_1 < \theta_2 < \ldots < \theta_{ml}$ *such that for every* $\varepsilon > 0$ *there exists a* $c(\varepsilon) > 0$ *so that for all functions* $f(z)$ *with* $\big(f^{(\theta_1)}(0), \ldots, f^{(\theta_{ml})}(0)\big)$ *a nonzero element of* $(Z[i])^{ml}$,

$$\big|W_{\theta_1, \ldots, \theta_{ml}}\big(y_1(w_1), \ldots, y_j(w_k), \ldots, y_{l-1}(w_m);\; y_l(w_2) - y_l(w_1), \ldots, y_l(w_m) - y_l(w_1), f\big)(0)\big| \geqslant c(\varepsilon)\big(\max_{1 \leqslant t \leqslant ml} \{|f^{(\theta_t)}(0)|\}\big)^{-(ml+\varepsilon)}.$$

One would conjecture that if $g_l(z) = \beta_{\frac{l(l+1)}{2}} \cdot \prod_{k=1}^{l-1}(z - X_k)$ where the X_k denote parameters taking on distinct values in $Q(i)$ then for "almost all" $\beta_1, \ldots, \beta_{\frac{l(l-1)}{2}}, X_1, \ldots, X_l$ we do have $t = l-1$.

Our final theorem, Theorem V, will be followed by proofs of Theorems I–IV which are based on Theorem V. The remainder of the paper will be devoted to the proof of Theorem V.

Let l denote a positive integer and let each $g_j(z)$, for $1 \leqslant j \leqslant l$, denote an element of $Q[i, z]$ of degree less than j with $g_l(z) \not\equiv 0$. Consider the equation

$$y = \sum_{j=1}^{l} g_j(z) D^j y. \tag{2}$$

Let the $w_k(z) = w_k(z, z_1, \ldots, z_m)$, $1 \leqslant k \leqslant m$, denote the m different branches of the algebraic function $w = w(z)$ defined by

$$p(w) \stackrel{\text{def}}{=} \prod_{k=1}^{m}(w - z_k) = z$$

for m complex valued parameters $z_1, \ldots, z_m$.

THEOREM V. *For every* $\varepsilon > 0$ *there exist a positive integer* n *and* n *elements* $h_j = h_j(z, z_1, \ldots, z_m)$ *of* $Q[i, z, z_1, \ldots, z_m]$ *with* $h_n(z)$ *equal to a power of*

$$\prod_{k=1}^{m} p'\big(w_k(z, z_1, \ldots, z_m)\big) \text{ times } \prod_{k=1}^{m} g_l\big(w_k(z, z_1, \ldots, z_m)\big)$$

such that

(i) $\deg h_j \overset{\text{def}}{\equiv} \deg_z h_j(z, z_1, \ldots, z_m) < j$ *for each* $1 \leqslant j \leqslant l$,

(ii) $\left(\max\left\{\dfrac{\deg h_j}{j-\deg h_j}\right\}\right) - m\left(\max\left\{\dfrac{\deg g_j}{j-\deg g_j}\right\}\right) - (m-1) < \varepsilon$ *and*

(iii) $y = \sum_{j=1}^{n} h_j(z, z_1, \ldots, z_m) D^j y$ *is satisfied by every* $y\big(w_k(z)\big)$ *where* y *denotes any solution of* (2) *and* $w_k(z)$ *denotes any branch of* $w(z)$.

The possibility that $n > ml$ above appears to be actual. We note that ze^z satisfies the linear differential equation

$$(zD - z - 1)y = 0$$

of order one but not of form (1) and the linear differential equation $(D-1)^2 y = 0$ which is of form (1) but not of minimal order over $Q[i, z]$. Other examples may be constructed. Below let $\varrho = \exp(2\pi i k^{-1})$.

EXAMPLE OF THEOREM II. Around $z = \infty$ each $w_k(z) = \varrho^k z^{k^{-1}}$ plus terms of lower order in z, for any choice of complex numbers $z_1, \ldots, z_m$. It is then easy to show using growth arguments that the functions $\exp\big(w_k(z)\big)$, $1 \leqslant k \leqslant m$, are linearly independent over the complex numbers. Thus the Wronskian of the $w_k(z)$ does not vanish identically for any choice of $z_1, \ldots, z_m$. Letting $z_1, \ldots, z_m$ be any distinct elements of $Q(i)$ we see that the rank of $A(z)$ is exactly m. Thus we see that the field F over $Q(i)$ generated by $e^r, e^{2r}, \ldots, e^{mr}$ (for any nonzero r in $Q(i)$) has dimension over $Q(i)$ at least m, for each $m \geqslant 1$, hence, each e^r is transcendental. Also Theorem IV applies. The statement about diophantine approximation obtained is, of course, well known.

Comments and Examples (Added February 1972). In the general case in Theorem II we have that for any choice of l linearly independent solutions, $y_1(z), \ldots, y_l(z)$, of (1) either the dimension of F over $Q(i)$ is at least m or the $y_j\big(w_k(z)\big)$ are linearly dependent over C. In a future paper we shall be able to show, using a method which considers the asymptotic expansions of the $y_j(z)$ about $z = \infty$ into series involving exponentials, that *even* in the event that the $y_j\big(w_k(z)\big)$ are *dependent* the dimension of F over $Q(i)$ is still at least m. One is led to consider such expansions originally because they offer a different way of testing for the linear independence of the $y_j\big(w_k(z)\big)$, i.e. by using arguments of the type applied above to show that the functions $e^{w_k(z)}$ are independent. (Similarly we shall show in this future paper that Theorem III holds with $r = 0$ even if the $y_j\big(w_k(z)\big)$ are linearly dependent.)

One may construct many additional examples: For any equation of type (1) if we define l solutions by $y_j^{(k)}(z_1) = \delta_j^{k+1}$, for $0 \leqslant k \leqslant l-1$, then the field F determined by the $y_1, \ldots, y_l$ at $z_1, \ldots, z_m$ has dimension over $Q(i)$ of at least m. Also consider the equation $y = D(zD - \alpha)y$ for any α in $Q(i)$ which is not a rational integer. A fundamental system of solutions of the above equation is $y_1 = z^{\frac{1}{2}\alpha} J_\alpha(2i\sqrt{z})$ and $y_2 = z^{\frac{1}{2}\alpha} J_{-\alpha}(2i\sqrt{z})$, where J_n denotes the Bessel function of order n. Since y_2 is not an E-function (in the sense of Siegel) if n is not in Q we can say nothing about the transcendentality of its values at algebraic z. Further, the methods of [5]–[8] *would* apply to determining the dimension over $Q(i)$ of the field F_1 generated by the power series coefficients of y_1 at $z = z_1, \ldots, z_m$; however, if α is not an integer, we would know nothing about the field F_2 generated by the power series coefficients of $y_1 + \pi y_2$ and $y_1 + \pi^2 y_2$, since neither one of these functions is entire nor is any nonzero linear combination of them with algebraic coefficients.

Section I

Proof of Theorem II. We shall assume Theorem V in this section and prove it later in Section II.

In a moment we shall carefully evaluate the Wronskian of $y_1\big(w_1(z)\big), \ldots, y_l\big(w_m(z)\big)$. First wish to show that this quantity is not identically zero as a function of $z, \beta_1, \ldots, \beta_{\frac{l(l+1)}{2}}$ for any choice of $z_1, \ldots, z_m$. The Example of Theorem II gives us a clue as to why this is true. Consider $\prod_{j=1}^{l} (D - r_j)y = 0$ where each $r_j \in Q(i)$, no r_j is zero, and no two $|r_j|$'s are equal. Our functions $y_j\big(w_k(z)\big)$ may be taken to be the $e^{r_j w_k(z)}$ which near $z = \infty$ look very much like $e^{r_j \varrho^k z^{k^{-1}}}$. By growth arguments, then, the $e^{r_j w_k(z)}$ are linearly independent over C, and we are through.

Since

$$\left(\prod_{k=1}^{m} p'\big(w_k(z)\big)\right)\big(p'\big(w_{k_1}(z)\big)\big)^{-1}$$

equals a symmetric polynomial in the $w_k(z)$, $k \neq k_1$, with coefficients in the field $Q(i)$ it may be written effectively over $Q(i)$ as a polynomial in the coefficients of $\big(p(w) - z\big)\big(w - w_{k_1}(z)\big)^{-1}$; hence, $D\big(w_{k_1}(z)\big)$ may be written effectively as a linear combination of $1, w_{k_1}(z), \ldots, w_{k_1}^{m-1}(z)$ with coefficients each of the form an element of

$$Q[i, z, z_1, \ldots, z_m] \text{ times } \left(\prod_{k=1}^{m} p'\big(w_k(z)\big)\right)^{-1}.$$

Using this result and equation (1) repeatedly one may effectively write the Wronskian of $y_1(w_1(z)), \ldots, y_l(w_m(z))$ as the determinant of an ml by ml matrix $A(z)$, with entries each of the form an element of $Q[i, z, z_1, \ldots, z_m, \beta_1, \ldots, \beta_{\frac{l(l+1)}{2}}]$ times negative powers of $\left(\prod_{k=1}^{m} p'(w_k(z))\right)$ and $\left(\prod_{k=1}^{m} g_l(w_k(z))\right)$, times the determinant of a matrix which when written in matrix block notation looks like $(\Omega_{t,k})$, where $0 \leqslant t \leqslant m-1$ and $1 \leqslant k \leqslant m$ are the "row" and "column" parameters respectively, and each $\Omega_{t,k} = \left(w_k^t(z) D^\theta y_j(w_k(z))\right)$ where $0 \leqslant \theta \leqslant l-1$ and $1 \leqslant j \leqslant l$ are the row and column parameters, respectively.

Using elementary "row" operations one may obtain a matrix to replace the second matrix above by one in which no block appear below the "main diagonal" of blocks. Thus the second matrix, Δ, has determinant equal to a rational function of $w_1(z), \ldots, w_m(z)$ times

$$\prod_{k=1}^{m} \left(W(y_1, \ldots, y_l)(w_k(z))\right),$$

where W denotes the ordinary Wronskian. For any $z_0 \in C$ if we substitute for $y_1, \ldots, y_l$ any l polynomials such that each $\left(y_j^{(\theta)}(w_k(z_0))\right)$ equals the identity matrix (here $0 \leqslant \theta \leqslant l-1$ and $1 \leqslant j \leqslant l$) for every $1 \leqslant k \leqslant m$, then $\Delta(z_0)$ looks like $(w_k^t(z_0))$ tensor product the l identity matrix where $1 \leqslant k \leqslant m$ and $0 \leqslant t \leqslant m-1$. Thus here

$$\det(\Delta(z_0)) = \left(\det(w_k^t(z_0))\right)^l.$$

It follows that generally

$$\det(\Delta(z)) = \left(\det(w_k^t(z))\right)^l \prod_{k=1}^{m} \left(W(y_1, \ldots, y_l)(w_k(z))\right).$$

We may obtain two important facts from the above formula: If $z_1, \ldots, z_m, \beta_1, \ldots, \beta_{\frac{l(l+1)}{2}}$ are such that no two z_k's are equal and no $g_l(z_k)$ vanishes then the rank of $A(z)$ is the dimension of the vector space over C spanned by the $y_j(w_k(z))$ and, under these same conditions, $\Delta(0) \neq 0$.

Let us join the zeros of $g_l(z)$, $\prod_{k=1}^{m} g_l(w_k(z))$, and $\prod_{k=1}^{m} p'(w_k(z))$ to $z = \infty$ by a simple curve $\gamma = \gamma(t)$ composed of line segments and such that for sufficiently large $|\gamma(t)|$ the imaginary part of $\gamma(t)$ equals some fixed negative real number. Let X denote the extended plane minus the "cut" γ. Since γ is simple the region X is simply connected and we may define $w_1(z), \ldots, w_m(z)$ as analytic functions on X (here each $w_k(z)$ is asymptotic to $\varrho^{k-1}|x^{m-1}|$ on the positive real axis). If $y(z)$ is any function analytic on X then each of $y(w_1(z)), \ldots, y(w_m(z))$ is defined in an open disk about some sufficiently large positive integer. If $y(z)$ is a solution of (1) then

it is analytic on X and we may continue each $y(w_k(z))$ to be an analytic function on all of X. Notice that analytically continuing $y(w_k(z))$, $0 \leqslant k \leqslant m-1$ times in the positive direction around the circumference of a large circle with center at zero, we obtain $y(w_{k+1}(z))$. Thus the $y(w_k(z))$, $1 \leqslant k \leqslant m$, are analytic continuations of each other along paths which avoid the zeros of

$$\left(\prod_{k=1}^{m} g_l(w_j(z))\right)\left(\prod_{k=1}^{m} p'(w_k(z))\right).$$

Suppose that $y_1(z)$ and $y_2(z)$ are two solutions of (1) such that continuing $y_1(z)$ along a path $\gamma_1 = \gamma_1(t)$ which avoids the zeros of $g_l(z)$ we obtain $y_2(z)$. Without loss of generality we may assume that (i) $p(\gamma_1(0))$ and $p(\gamma_2(1))$ are in X and (ii) $\gamma_1(t)$ also avoids the points in $\{w_j$ (the zeros of $\prod_{k=1}^{m} p'(w_k(z))$), for $1 \leqslant j \leqslant m\}$. Set $\gamma_2(t) = p(\gamma_1(t))$. Near $z_0 = p(\gamma_1(0))$, $\gamma_1(t) = w_{k_1}(\gamma_2(t))$ for some $1 \leqslant k_1 \leqslant m$. It follows that for all $0 \leqslant t \leqslant 1$, $\gamma_1(t) = w_{k_1}(\gamma_2(t))$, where here we mean by $w_{k_1}(z)$ the analytic continuation of $w_{k_1}(z)$ along γ_2. Thus our analytic continuation of $y_1(w_{k_1}(z))$ along $\gamma_2(t)$ is $y_2(w_{k_2}(z))$ for *some* $1 \leqslant k_2 \leqslant m$. By what we have already seen then it follows that each $y_1(w_{k_1}(z))$ may be analytically continued into *each* $y_2(w_{k_2}(z))$, for *all* $1 \leqslant k_1, k_2 \leqslant m$, along a path which avoids the zeros of

$$\left(\prod_{k=1}^{m} g_l(w_j(z))\right)\left(\prod_{k=1}^{m} p'(w_k(z))\right).$$

We wish now to apply the Proposition from [2] and Theorem V of the present paper in order to obtain a statement of diophantine approximation involving the $y_j(w_k(z)) - y_j(w_1(z))$ for all $1 \leqslant j \leqslant l$ and $2 \leqslant k \leqslant m$ (for use in the proof of Theorem II). Setting $X = C$ and $a(z) = 0$ in the Proposition we could obtain such a result immediately except that there is no *one* curve $\mathscr{C}$ along which we can continue each $y_j(w_k(z))$ to obtain $y_j(w_1(z))$. Thus a stronger version of the Proposition is needed. Such a stronger version would follow immediately from a version of Theorem III of [2] in which instead of one linear operator σ (corresponding to analytic continuation about one curve $\mathscr{C}$) we allow operators $\sigma_1, \ldots, \sigma_j, \ldots, \sigma_n$ which each satisfy the hypotheses of σ for some subspace V of U_l. For each $1 \leqslant j \leqslant n$ we may define $\overline{T}_j$, $\overline{U}^{(j)}$, and each $\overline{U}_i^{(j)}$, for every $1 \leqslant j \leqslant n$, in the same manner that $\overline{T}$, $\overline{U}$, and each $\overline{U}_i$ were defined using σ. Then we set

$$\overline{U} = \overline{U}^{(1)} \oplus \ldots \oplus \overline{U}^{(n)} \quad \left(\text{with every } \overline{u}(x) = \sum_{j=1}^{n} \overline{u}^{(j)}(x)\right),$$

each

$$\overline{U}_i = \overline{U}_i^{(1)} \oplus \ldots \oplus \overline{U}_i^{(n)} \quad \left(\text{with every } \overline{u}_i(x) = \sum_{j=1}^{n} \overline{u}_i^{(j)}(x)\right),$$

and

$$\overline{T}\big((\overline{U}^{(1)}, \ldots, \overline{U}^{(n)})\big) = \big(\overline{T}_1(\overline{U}^{(1)}), \ldots, \overline{T}_n(\overline{U}^{(n)})\big).$$

The proof in the present case goes through because the "old" proof holds in each component.

Thus by our strengthened version of the Proposition of [2] and by Theorem V of the present paper we see that there exists $n = n(\varepsilon)$ such that

$$(3)\quad \max_{0\leqslant\theta\leqslant n-1}\Big\{\Big\|\Big|\sum_{j=1}^{l}\sum_{k=2}^{m} A_{j,k}D^{\theta}\big[y_j(w_k(0)) - y_j(w_1(0))\big]\Big|\Big\|\Big\} \geqslant c(\varepsilon)\,\big(\max_{j,k}\{|A_{j,k}|\}\big)^{-(md+\varepsilon)}$$

for all nonzero $(m-1)l$-tuples of Gaussian integers $A_{j,k}$, where $d = \max\left\{\dfrac{\deg g_j}{j-\deg g_j}\right\}$ and $\|z\|$ denotes the distance from z to the nearest Gaussian integer.

If the $z_1, \ldots, z_m$ are each regular points of the $y_j(z)$'s then every

$$\sum_{j=1}^{l}\sum_{k=2}^{m} A_{j,k}D^{\theta}\big[y_j(w_k(0)) - y_j(w_1(0))\big]$$

in (3) may be expressed as a linear combination over $Q(i)$ of the

$$(4)\qquad \sum_{j=1}^{l}\sum_{k=2}^{m} A_{j,k}\big[w_k^t(0)\,y_j^{(\theta)}(w_k(0)) - w_1^t(0)\,y_j^{(\theta)}(w_1(0))\big]$$

for $0 \leqslant \theta \leqslant l-1$ and $0 \leqslant t \leqslant m-1$. Thus we need only consider in (3) a maximum over the ml numbers above (for any choice of the $A_{j,k}$ such that

$$\sum_{j=1}^{l}\sum_{k=2}^{m} A_{j,k}\big(y_j(w_k(z)) - y_j(w_1(z))\big) \not\equiv 0)$$

if we are willing to, possibly, change the $c(\varepsilon) > 0$.

Now recall that $\Delta(0) \neq 0$. Subtracting the columns of $\Delta(0)$ involving y_j and $w_1(0)$ from the columns involving y_j and $w_k(0)$, for $k = 2, \ldots, m$ and for each $1 \leqslant j \leqslant l$, we see that the $(m-1)l$ by ml matrix of coefficients of the $A_{j,k}$ in (4) has rank $(m-1)l$. Thus, extracting an $(m-1)l$ by $(m-1)l$ nonsingular submatrix from this latter matrix by deleting l rows, corresponding to l ordered pairs (t, θ), and calling the new row parameter v one may construct $(m-1)l$ functions

$$Y_s(z) \stackrel{\text{def}}{=} \sum_{j=1}^{l}\sum_{k=2}^{m} B_{j,k,s}\big(y_j(w_k(z))\big)$$

such that except for the l deleted ordered pairs (t, θ)

(a) each $\sum_{j=1}^{l}\sum_{k=2}^{m} B_{j,k,s}\big[w_k^t(0)\,y_j^{(\theta)}(w_k(0)) - w_1^t(0)\,y_j^{(\theta)}(w_k(0))\big]$ belongs to $Q(i)$,

(b) each $B_{j,k,s}$ belongs to the field F, and

(c) $|B_{j,k,s}| \neq 0$.

Applying the argument leading to (3) to the $Y_s(z)$ and rewriting the different

$$\sum_{s=1}^{(m-1)l} A_s Y_s^{(\theta)}(0),$$

with each $A_s \in Z[i]$, as linear combinations over $Q(i)$ of the ml different

$$\sum_{s=1}^{(m-1)l}\Big(A_s \sum_{j=1}^{l}\sum_{k=2}^{m} B_{j,k,s}\big(w_k^t(0)\,y_j^{(\theta)}(w_k(0)) - w_1^t(0)\,y_j^{(\theta)}(w_k(0))\big)\Big)$$

we see that we need only consider the above forms for those l ordered pairs (t, θ) such that the rows indexed by them were deleted from the matrix of coefficients of the $A_{j,k}$'s above.

Thus, we have l linear forms in $(m-1)l$ variables over A which cannot all assume values in $Z[i]$ unless $\sum_{s=1}^{(m-1)l} A_s Y_s(z) \equiv 0$. We have already seen that the $y_j(w_k(z))$ span a vector space of dimension exactly $ml-r$ over C. Therefore the $y_j(w_k(z)) - y_j(w_1(z))$, $1 \leqslant j \leqslant l$ and $2 \leqslant k \leqslant m$, span a vector space over C of dimension at least $(m-1)l-r$. It follows since $|B_{j,k,s}| \neq 0$ that the $Y_s(z)$ span a vector space over C (and hence over $Q(i)$) of dimension at least $(m-1)l-r$. Thus $\sum_{s=1}^{(m-1)l} A_s Y_s(z) \equiv 0$ has at most an r dimensional solution space over $Q(i)$.

In each of the l forms above choose a basis, beginning with 1, for the vector space over $Q(i)$ spanned by 1 and the coefficients of the form. If none of these bases ever has at least $\big((m-1)l-r\big)l^{-1}+1 = m-rl^{-1}$ elements in it we can choose the A_s's in $Z[i]$ such that every form equals an element of $Z[i]$ *and* $\sum_{s=1}^{(m-1)l} A_s Y_s(z) \not\equiv 0$, since at most $(m-1)l-(r+l)$ equations in $(m-1)l$ unknowns have an $(r+l)$-dimensional solution space. This contradiction proves that some basis has at least $\big((m-1)l-r\big)l^{-1}+1$ elements in it and we are through. This proves Theorem II.

Proof of Theorem III. Except that we must work with $(m-1)l+t$ functions, not $(m-1)l$ functions, the same argument may be used as in the proof of Theorem II. We then obtain $l-t$ forms in $(m-1)l+t$ variables with the coefficients in F. Theorem III follows immediately.

Proof of Theorem IV. If $r = 0$ and $t = l-1$ we may first follow the previous argument as far as inequality (3). If the $y_j(w_k(z))$ and the

$f_{ml+1}, \ldots, f_n$ are a fundamental system of solutions of the equation obtained from Theorem V then, since $z = 0$ is a regular point of the equation, it follows that there exist nonnegative integers $\theta_1 < \theta_2 < \ldots < \theta_{ml}$ such that the sub-determinant of the Wronskian of the $y_j(w_k(z))$ and the $f_t(z)$ at $z = 0$ formed by removing the last $n-ml$ columns and all rows except the θ_1st, $\ldots$, θ_{ml}th, is nonzero. If

$$y(z) = \sum_{j=1}^{l}\sum_{k=1}^{m} c_{j,k} y_j(w_k(z)),$$

where the $c_{j,k}$ denote arbitrary constants, then each derivative of y at $z = 0$ may be written as a linear combination over $Q(i)$ of the different

$$\sum_{j=1}^{l}\sum_{k=1}^{m} c_{j,k}\big(w_k(0)\big)^{\varphi} y_j^{(\theta)}\big(w_k(0)\big)$$

for $0 \leqslant \varphi \leqslant m-1$ and $0 \leqslant \theta \leqslant l-1$.

It follows by what was shown above that $y^{(\theta_1)}(0), \ldots, y^{(\theta_{ml})}(0)$ are linearly independent over $Q(i)$. Thus each $D^{\theta}y(0)$ $(\theta = 0, 1, \ldots, n-1)$ may be written as a linear combination over $Q(i)$ of the $D^{\theta_t}y(0)$ $(1 \leqslant t \leqslant ml)$ with coefficients in $Q(i)$ (hence independent of the $c_{j,k}$). So in (3) we need only consider the cases where $\theta = \theta_1, \ldots, \theta = \theta_{ml}$.

Where $A_1, \ldots, A_{ml-1}$ denote parameters which are to take on values in $Z[i]$, either one may find $U_1(z)$, a linear combination over $C[A_1, \ldots, A_{ml-1}]$ of the elements of

$$\{h_1(z), \ldots, h_{ml-1}(z)\}$$
$$\stackrel{\text{def}}{=} \{y_l(w_k(z)), 1 \leqslant k \leqslant m, 2 \leqslant j \leqslant k\} \cup \{y_1(w_k(z)) - y_1(w_1(z)), 2 \leqslant k \leqslant m\}$$

such that $D^{\theta_t}U_1(0) = A_t$ for $1 \leqslant t \leqslant ml-1$ or one may find $U_2(z)$, a nonzero linear combination of the $h_j(z)$, such that $D^{\theta_t}U_2(0) = 0$ for each $1 \leqslant t \leqslant ml-1$. This last possibility leads to a violation of (3) since either $D^{\theta_{ml}}U_2(0) = 0$ or, without loss of generality, we may take $D^{\theta_{ml}}U_2(0)$ to be one. Thus the first case holds.

We may use Cramer's rule to write

$$U_1(z) = \sum_{j=1}^{ml-1} A_j(\Delta)^{-1} Y_j(z)$$

where the $Y_j(z)$ are linear combinations of the $h_j(z)$ and Δ is the Wronskian of $h_1(z), \ldots, h_{ml-1}(z)$ at $z = 0$. Thus on the left hand side of (3) we have

$$\Big\|\sum_{j=1}^{ml-1} A_j(\Delta)^{-1} Y_j^{(\theta_{ml})}(0)\Big\|.$$

This latter quantity may be replaced in (3) by

$$\Big|\sum_{j=1}^{ml-1} A_j(\Delta)^{-1} Y_j^{(\theta_{ml})}(0) + A_{ml}\Big|$$

for any A_{ml} in $Z[i]$. In fact for some $c(\varepsilon) > 0$ the inequality in (3) holds with

$$\Big|\sum_{j=1}^{ml-1} A_j Y_j^{(\theta_{ml})}(0) + A_{ml}\Delta\Big|$$

on the left hand side, for any nonzero $(A_1, \ldots, A_{ml})$ in $(Z[i])^{ml}$. We may write each $Y_j^{(\theta_{ml})}(0)$ as a linear combination of the $h_j^{(\theta_{ml})}(0)$ for $1 \leqslant j \leqslant m-l$. Set each $A_t = f^{(\theta_t)}(0)$ for $1 \leqslant t \leqslant ml$. One may write $\Big|\sum_{j=1}^{ml-1} A_j Y_j^{(\theta_{ml})}(0) + A_{ml}\Big|$, then, as a linear form in the $h_j^{(\theta_{ml})}(0)$, $1 \leqslant j \leqslant ml-1$, and $f^{(\theta_{ml})}(0)$. Expanding the determinant in Theorem V along the bottom row we obtain up to a $\pm$ sign the linear form in the $h_j^{(\theta_{ml})}(0)$ and $f^{(\theta_{ml})}(0)$ just obtained above. This proves Theorem IV.

Section II

In this section we shall prove Theorem V. We begin with a ring-theoretic lemma. Let $S \stackrel{\text{def}}{=} S(r, s)$, for any pair of positive integers r and s, denote the subring of the noncommutative ring $Q[i, N, z_1, \ldots, z_m, D, zD]$ generated by all monomials in which the degree in zD divided by the degree in D is less than or equal to rs^{-1}. (Here $N, z_1, \ldots, z_m$ denote $m+1$ complex valued parameters, z denotes a complex variable, and D denotes $\dfrac{d}{dz}$.)

LEMMA I. *The ring S satisfies the ascending chain condition on left ideals.*

Proof. We wish to see first that S is generated over $Q(i)$ by $N, z_1, \ldots, z_m$ and $(zD)^0 D^{\theta_0}, (zD)^1 D^{\theta_1}, \ldots, (zD)^r D^{\theta_r}$ where $\theta_0 = 1$, $\theta_r = s$, and—generally—each θ_j is chosen to be the unique integer such that $j(\theta_j)^{-1} \leqslant rs^{-1} < j(\theta_j - 1)^{-1} \leqslant +\infty$ (here $j(0)^{-1} = +\infty$). Note that $1 = \theta_0 \leqslant \theta_1 \leqslant \ldots \leqslant \theta_r = s$.

All that we need to do to show the above set of elements generate S is to see that we may write each $(zD)^a D^b$, where $ab^{-1} \leqslant rs^{-1}$, as a polynomial in the $(zD)^j D^{\theta_j}$, since any element of S may be written as a linear combination over $Q[i, N, z_1, \ldots, z_m]$ of such terms $(zD)^a D^b$. (Use $(zD)D = D(Dz-1)$ and $D(zD) = (zD+1)D$, repeatedly.) We proceed by induction on a. If $a \leqslant r$ then we may write $(zD)^a D^b = \big((zD)^a D^{\theta_a}\big) D^{b-\theta_a}$ where $b-\theta_a \geqslant 0$ by the definition of θ_a. If $a > r$ then since $ab^{-1} \leqslant rs^{-1}$ we must have $b > s$. Note that

$$(a-r)(b-s)^{-1} \leqslant r/s.$$

Thus we may write $(zD)^a D^b$ as $(zD)^r D^s\big((zD)^{a-r} D^{b-s}\big)$ plus other monomials in S of the form an element of $Q[i, N, z_1, \ldots, z_m]$ times $(zD)^\alpha D^\beta$ where $\alpha\beta^{-1} \leqslant rs^{-1}$ and $\alpha < a$. By induction on a it follows that $N, z_1, \ldots, z_m$, $(zD)^0 D^{\theta_0}, \ldots, (zD)^r D^{\theta_r}$ generate S.

Let S_j, $0 \leqslant j \leqslant r$, denote the ring generated over $Q(i)$ by $N, z_1, \ldots$ $\ldots, z_m$, $(zD)^0 D^{\theta_0}, \ldots, (zD)^j D^{\theta_j}$. Then $S_r = S(r, s)$. Also, choosing $0 \leqslant k \leqslant j$ so that $k(\theta_k)^{-1} = \max\limits_{0\leqslant t\leqslant j} \{t\theta_t^{-1}\}$, $S_j = S(k, \theta_k)$ since, for every $0 \leqslant t \leqslant k$ we have $t\theta_t^{-1} \leqslant k\theta_k^{-1} \leqslant rs^{-1} < t(\theta_t - 1)^{-1} \leqslant +\infty$, while for $k+1 \leqslant t \leqslant j$ each $t\theta_t^{-1} \leqslant k\theta_k^{-1}$.

We shall next show by induction on j that S_j has a. c. c. (the ascending chain condition on left ideals). If $j = 0$ the ring is Noetherian and we are through. Suppose that $0 \leqslant j \leqslant r-1$ and that S_j has a. c. c. . Every element of S_{j+1} may be written as a linear combination over $Q[i, N, z_1, \ldots$ $\ldots, z_m]$ of products of the different $(zD)^t D^{\theta_t}$, for $0 \leqslant t \leqslant j+1$, in some order. Where $0 \leqslant t \leqslant j$,

$$(zD)^{j+1} D^{\theta_{j+1}} (zD)^t D^{\theta_t} = \big((zD + \theta_{j+1})^t D^{\theta_t}\big)(zD - \theta_t)^{j+1} D^{\theta_{j+1}}. \tag{5}$$

Since $\theta_{j+1} \geqslant \theta_j$ we see that

$$\big((j+1)-1\big)(\theta_{j+1})^{-1} = j(\theta_{j+1})^{-1} \leqslant j(\theta_j)^{-1} \leqslant k(\theta_k)^{-1}.$$

Therefore in (5) we have $(zD+\theta_{j+1})^t D^{\theta_t}\big((zD)^{j+1} D^{\theta_{j+1}}\big)$ plus an element of S_j. Since we have $t\theta_t^{-1} \leqslant k\theta_k^{-1}$, $(zD+\theta_{j+1})^t D^{\theta_t}$ belongs to S_j also. By induction on the maximal number of factors of $(zD)^{j+1} D^{\theta_{j+1}}$ which appear in any monomial it follows that every element of S_{j+1} may be written as a polynomial in $(zD)^{j+1} D^{\theta_{j+1}}$ with coefficients on the left from S_j, in at least one way.

Let L denote an arbitrary left ideal of S_{j+1}. Let J_n, for $n = 0, 1, \ldots$ denote the left ideal in S_j consisting of the set of all coefficients of $\big((zD)^{j+1} D^{\theta_{j+1}}\big)^n$ in *all* polynomials of degree at most n in $(zD)^{j+1} D^{\theta_{j+1}}$ with coefficients on the left from S_j which represent elements of L.

Define a mapping σ_j from S_j to S_j by $D^{\theta_{j+1}} s_j = \sigma_j(s_j) D^{\theta_{j+1}}$ for all elements s_j of S_j. Obviously σ_j is well defined since if $t_1 D^{\theta_{j+1}} = t_2 D^{\theta_{j+1}}$, where t_1 and t_2 belong to S_j, we would have $(t_1 - t_2) D^{\theta_{j+1}} = 0$, which would say that $t_1 = t_2$. If $\sigma_j(s_j) = 0$ this implies $D^{\theta_{j+1}} s_j = 0$ which certainly means that $s_j = 0$. Further σ_j is a homomorphism since $D^{\theta_{j+1}}(s_{j,1} s_{j,2})$ $= \sigma_j(s_{j,1})\sigma_j(s_{j,2}) D^{\theta_{j+1}}$ for every pair of elements $s_{j,1}$ and $s_{j,2}$ belonging to S_j. Therefore σ_j is an isomorphism. The mapping σ_j is onto S_j since $\sigma_j(D) = D$ and $\sigma_j(zD - \theta_{j+1}) = zD$. Thus σ_j is an automorphism of S_j.

From (5), if $0 \leqslant t \leqslant j$, $\big((zD)^{j+1} D^{\theta_{j+1}}\big)\big((zD)^t D^{\theta_t}\big)$ equals $\sigma_j\big((zD)^t D^t\big)$ times $(zD)^{j+1} D^{\theta_{j+1}}$ plus an element of S_j. It follows that then for $n = 0, 1, \ldots$, $\sigma_j(J_n) \subseteq J_{n+1}$. Therefore, we have $J_0 \subseteq \sigma_j^{-1}(J_1) \subseteq \sigma_j^{-2}(J_2) \subseteq \ldots$ Each $\sigma_j^{-n}(J_n)$ is a left ideal in S_j since σ_j^{-1} is an automorphism of S_j. By a. c. c. in S_j there must exist some positive integer N such that $\sigma_j^{-N}(J_N)$ $= \sigma_j^{-(N+1)}(J_{N+1}) = \ldots$ Then for each $k \geqslant 0$, $J_{N+k} = \sigma_j^k(J_N)$. Since each J_n, $0 \leqslant n \leqslant N$, is finitely generated it is easy to see that L is finitely generated. This proves Lemma I.

Consider the functions

$$I_{mM+h} = \int\limits_a^{w_k(z)} \frac{(z-p(u))^M}{M!} u^h y(u)\,du$$

where y denotes any solution of (1), a denotes any point where $g_l(z)$ is nonzero, h denotes an integer between 0 and $m-1$, and M denotes a non-negative integer.

Lemma II. *There exists a sequence of functions $p_{mM+h} = p_{mM+h}(z, z_1, \ldots$ $\ldots, z_m)$, which in z are each polynomials of degree at most M, such that the functions $I_{mM+h} - p_{mM+h}$, for $M = 0, 1, \ldots$, generate a finitely generated left module over $Q[i, z_1, \ldots, z_m]$.*

Proof. Recall that we may rewrite (1) as

$$y = \sum_{1\leqslant k\leqslant l} K_k(zD) D^k y \tag{6}$$

where the $K_k(zD)$ belong to $Q[i, zD]$. Define the $p_t(z, z_1, \ldots, z_m)$ to be identically zero if $0 \leqslant t \leqslant lm$. Given any I_{mM+h} for $M \geqslant l$ and any $0 \leqslant h$ $\leqslant m-1$ integrate (6) by parts integrating $y(u)$ into

$$\sum_{1\leqslant k\leqslant l} K_k(uD-1) D^{k-1} y(u)$$

and differentiating the remaining factor. This gives a polynomial of degree at most M in z (with coefficients depending on $z_1, \ldots, z_m$) plus, since $M \geqslant 1$, a linear combination over $Q[i, z_1, \ldots, z_m]$ of terms of the form

$$\int\limits_a^{w_k(z)} \frac{d}{du}\left\{\frac{(z-p(u))^M}{M!} u^h\right\} u^t \big(D^{j-1} y(u)\big)\,du. \tag{7}$$

Integrate by parts repeatedly, integrating the $D^{j-1}y(u)$ until we have only terms of the form

$$\int\limits_a^{w_k(z)} \frac{(z-p(u))^{M-\theta}}{(M-\theta)!} u^{h_1} y(u)\,du \tag{8}$$

where $0 \leqslant \theta \leqslant j \leqslant l$ and each $0 \leqslant h_1 \leqslant m\theta + h + t - j$. We may write u^{h_1} as

$$\sum_{mk+\delta=0}^{ml+h+t-j} q_{k,\delta}(z, z_1, \ldots, z_m)\big(z-p(u)\big)^k u^\delta$$

with $0 \leqslant \delta < m$ where each $q_{k,\delta}(z, z_1, \ldots, z_m)$ belongs to $Q[i, z, z_1, \ldots, z_m]$ and has degree in z less than or equal to $\big(h_1 - (mk+\delta)\big)m^{-1}$ which is less than $\big(m\theta + h - (mk+\delta)\big)m^{-1}$, since $h_1 < m\theta + h$ (recall that in (7) we always have $j > t$).

Therefore we see that I_{mM+h} minus a function which in z is a polynomial of degree at most M equals a linear combination over $Q[i, z, z_1, \ldots$ $\ldots, z_m]$ of the I_{mM+h-v} (for $v = 1, 2, \ldots, mM+h$) with coefficient functions $r_v = r_v(z, z_1, \ldots, z_m)$ each having degree in z less than vm^{-1}. Note that, where $[x]$ is the greatest integer not exceeding x, $\left[\dfrac{mM+h-v}{m}\right] + \left[\dfrac{v}{m}\right] \leqslant M$ for $v = 1, 2, \ldots, mM+h$, thus we may define p_{mM+h}. This proves Lemma II.

We wish to examine the proof of Lemma II above more carefully. If j and t are as in (7) then, looking at all $r_v(I_{mM+h-v} - p_{mM+h-v})$ which are obtained from the term of (7) involving $u^t D^{j-1} y(u)$, we have

$$\min_v \{vm^{-1} - \deg_z r_v\} \geqslant (j-t)m^{-1}$$

and

$$\max_v \{\deg_z r_v\} \leqslant \big((m-1)j + t + h\big)m^{-1}.$$

Thus for all v, and all j and t in (7),

$$\max_v \{(\deg_z r_v)(vm^{-1} - \deg_z r_v)^{-1}\} \leqslant \max_{j,t} \{(m-1) + mt(j-t)^{-1} + h(j-t)^{-1}\}. \tag{9}$$

One may use, instead of (1) in (7),

$$\Big(1 - \sum_{1\leqslant k\leqslant l} K_k(zD)D^k\Big)^\varphi y(z) = 0, \tag{10}$$

for $\varphi = 1, 2, \ldots$ In (10) $\min\{j-t\} \geqslant \varphi$. Thus we would have that our upper bound in (9) may be replaced by

$$m - 1 + m \max_{j,t} \{t(j-t)^{-1}\} + \varepsilon \tag{11}$$

for any $\varepsilon > 0$, if φ is sufficiently large. (The maximum need only be taken over the pairs j, t occurring with $\varphi = 1$.)

Given any $\varepsilon_1 > 0$ we may choose two positive integers r and s such that

$$(m-1) + m\big(\max\{t(j-t)^{-1}\}\big) + \varepsilon_1 > rs^{-1} > (m-1) + m\big(\max\{t(j-t)^{-1}\}\big).$$

If φ is taken to be sufficiently large then we will have

$$rs^{-1} > (m-1) + m\big(\max\{t(j-t)^{-1}\}\big) + \varepsilon > \max_v \{(\deg_z r_v)(vm^{-1} - \deg_z r_v)^{-1}\}.$$

For a possibly even larger φ

$$rs^{-1} > \max_v \{(\deg_z r_v)(vm^{-1} - \deg_z r_v - 1)^{-1}\}$$

since

$$\min\{vm^{-1} - \deg_z r_v\} \geqslant \min\{(j-t)m^{-1}\} \geqslant \varphi m^{-1},$$

which goes to $+\infty$ with φ.

Notice from the definition of the I_{mM+h}, for every $M \geqslant 1$, $DI_{mM+h} = I_{m(M-1)+h}$. Thus if we have used (10) instead of (6) in the proof of Lemma II and now apply D^{N+l} to the resultant equations, where N is a nonzero integral valued parameter (and we assume that $N \geqslant M+1 \geqslant 1$), we may write for each $0 \leqslant h \leqslant m-1$

$$D^{N-M} I_h = \sum_{j=0}^{m-1} U_{j,M,h}(N, z_1, \ldots, z_m, zD, D) D^{N-M+1} I_j \tag{12}$$

where the $U_{j,M,h} = U_{j,M,h}(N, z_1, \ldots, z_m, zD, D)$ belong to $Q[i, N, z_1, \ldots$ $\ldots, z_n, zD, D]$ and each monomial of each $U_{j,M,h}$ has its degree in D less than or equal to

$$\max_v \{(\deg_z r_v)(vm^{-1} - \deg_z r_v - 1)^{-1}\} < r/s.$$

Thus the $U_{j,M,h}$ are in $S = S(r, s)$.

We may use (12) to write for $M = 1, 2, \ldots, D^{N-M}\bar{I} = \theta_M D^{N-M+1}\bar{I}$ where $\bar{I}$ is a column vector containing $I_0, \ldots, I_{m-1}$ and θ_M is an m by m matrix with elements in $S(r, s)$. In each case we have

$$D^{N-M}\bar{I} = \Big(\prod_{j=0}^{M-1} \theta_{M-j}\Big) D^N \bar{I} \quad \text{if} \quad N \geqslant M+1. \tag{13}$$

The components of the column vectors $\Big(\prod_{j=0}^{M-1} \theta_{M-j}\Big) D^N \bar{I}$, for $M = 1, 2, \ldots$, generate a finitely generated left module L over $S(r, s)$, since they are all contained in a finitely generated module over $S(r, s)$ and $S(r, s)$ has a.c.c. Similarly where $p'(w) = a_0 + a_1 w + \ldots + a_{m-1} w^{m-1}$ and $\bar{P} = (a_0, \ldots, a_{m-1})$ the module generated by the

$$\bar{P}\Big(\Big(\prod_{j=0}^{M-1} \theta_{M-j}\Big) D^N \bar{I}\Big), \quad \text{for} \quad M = 1, 2, \ldots$$

is finitely generated and there exists a positive integer M_1 such that

$$\bar{P}\Big(\Big(\prod_{j=0}^{M_1-1}\theta_{M_1-j}\Big)D^N\bar{I}\Big)$$

may be written as a linear combination over $S(r,s)$ of the

$$\bar{P}\Big(\Big(\prod_{j=0}^{M_1-1-k}\theta_{M_1-j-k}\Big)D^N\bar{I}\Big),\quad \text{for}\quad 1\leqslant k\leqslant M_1-1.$$

Setting $N = M_1+1$ we obtain, using (13) an equation of type (1) in $\bar{P}D\bar{I} = y\big(w_k(z)\big)$ in which

$$\max\{(\deg_z g_j)(j-\deg_z g_j)^{-1}\} < rs^{-1}.$$

Since the $\big(\prod_{k=1}^{m} g_l\big(w_k(z)\big)\big)^{\varphi} D^{\varphi} y\big(w_k(z)\big)$ generate a finitely generated module over the Noetherian ring $Q[i,z]$ we see that there exists a linear homogeneous differential equation in $y\big(w_k(z)\big)$ with coefficients in $Q[i,z]$ which is satisfied by every $y_j\big(w_k(z)\big)$ and which has a power of $\prod_{k=0}^{m} g_l\big(w_k(z)\big)$ for the coefficient of the highest order derivative of $y\big(w_k(z)\big)$. If we add a suitably high derivative of this second linear differential equation to the equation must obtained above for $y\big(w_k(z)\big)$ we shall have satisfied part (iii) of Theorem V as well as parts (i) and (ii). This proves Theorem V.

The following corrections should be made in [2]. On page 385 in line 11 it should be A_{j+1} instead of B_j and, in line 13, $D^{l-1}y_j(z_1)$ instead of $y_j(z_1)$. On page 390, line 14, the statement of uniformity is not quite correct since in the proof referred to there was at one point a choice of a basis over Q from among the $T^i(y(x_1))$ and then each $T^i y(x_1)$ was expressed as a linear combination, over Q, of these basis elements. One may repair this by choosing a basis over Q from among the different $T^i(\sum_j c_j y_j(x_1))$ where the c_j are arbitrary constants and proceeding as before. On page 390 in the seventh line from the bottom instead of a Gaussian integer A_0 there should be an l-tuple of Gaussian integers $A_{0,i}$ and in the sixth line from the bottom we should have $A_{0,i}$ not A_0. Finally, on page 391, line 13, V should be a subspace of $U_l \subseteq U$, not U_1.

References

[1] Charles F. Osgood, *A method in diophantine approximation*, Acta Arith. 12 (1966), pp. 111–128.

[2] — *A method in diophantine approximation II*, Acta Arith. 13 (1967), pp. 383–393.

[3] — *A method in diophantine approximation III*, Acta Arith. 16 (1969), pp. 5–22.

[4] — *A method in diophantine approximation IV*, Acta Arith. 16 (1969), pp. 23–40.

[5] Charles F. Osgood, *Some theorems on diophantine approximation*, Trans. Amer. Math. Soc. 123 (1966), pp. 64–87.

[6] — *Theorems about the derivatives of certain entire functions at algebraic points, (Part I)*, Koninkl. Nederl. Akad. Van Wetensch. Proc. Ser. A 70 (1967), pp. 431–443.

[7] — *Theorems about the derivatives of certain entire functions at algebraic points (Part II)*, Koninkl. Nederl. Akad. Van Wetensch. Proc. Ser. A 70 (1967), pp. 547–555.

[8] E. G. Straus, *On entire functions with algebraic derivatives at certain algebraic points*, Annals of Math. 52 (1) (1950), pp. 180–198.

Received on 30. 7. 1971 (183)