On the diophantine equations \[ \prod_{i=0}^{k} x_i - \sum_{i=0}^{k} x_i = n \] and \[ \sum_{i=0}^{n} \frac{1}{x_i} = \frac{a}{n} \]

by

C. Viola (Pisa)

1. Introduction. We are concerned with the equations

\[ \prod_{i=0}^{k} x_i - \sum_{i=0}^{k} x_i = n \] (1)

and

\[ \sum_{i=0}^{n} \frac{1}{x_i} = \frac{a}{n} \] (2)

where \(a, k\) and \(n\) are given integers and the unknowns \(x_i\) take positive integral values. Equation (1) was first considered by Schinzel [9] in the case \(n = 0\); he observed that for every \(k\) there exists a trivial solution, namely \((1, \ldots, 1, 2, k+1)\). Misurowicz (quoted in [9], Bemerkung) proved that in the case \(n = 0\), apart from any permutation of the \(x_i\)'s, equation (1) has no solutions different from the trivial one when \(k+1 = 2, 3, 4, 6, 24, 114, 174, 444\), while for any other \(k < 1000\) there is at least one other solution\(^{(*)}\). Later Schinzel conjectured (see [2], p. 238) that there is a \(k > 1\) such that, for every sufficiently large \(n\), (1) is solvable in integers \(x_i > 1\). Note that equation (1) has for any \(n\) \(k\) the trivial solution \((1, \ldots, 1, 2, n + k + 1)\).

Leonardo Pisano [5] proved in 1202 that for any \(a > 0, n > 0\), there is a \(k\) such that (2) is solvable with \(x_i \neq x_j\) for \(i \neq j\); obviously \(k\) depends on \(a\) and \(n\). Many authors (see e.g. [3], [4], [8], [10], [11]) have been concerned with related problems; a classical topic is the investigation of conditions for a positive rational number to be the sum of distinct

\(^{*}\) Research supported by Consiglio Nazionale delle Ricerche.

\(^(*)\) Note added in proof. With the aid of a computer, the result of Misurowicz has been extended to all \(k < 10^4\) (Amor. Math. Monthly 78(1971), pp. 1021-1022).
is insoluble in integers \( x_i > 1 \), and let \( E_{n,k}(N) \) denote the number of natural numbers \( n \leq N \) for which

\[
\sum_{i=0}^{n-1} \frac{1}{x_i} = \frac{a}{n}
\]

is insoluble in positive integers \( x_i \). Then for \( N \to \infty \)

\[
E_{n}(N) \ll N \exp\{-O(k)(\log N)^{-1/k}\},
\]

\[
E_{n,k}(N) \ll N \exp\{-O(a,k)(\log N)^{-1/k}\},
\]

with \( O(k) > 0, \ O(a, k) > 0 \).

We follow the sieving process already used by Vaughan [12]. However, the method he uses to adapt equation (2) in case \( k = 2 \) to the sieve (which gives the exponent 2/3 rather than our 1/2) does not extend to \( k > 2 \) in an obvious way. We outline here our method; consider for instance equation (1). Write (1) in the form

\[
n + a_1 + \ldots + a_k = n_0(a_1 \ldots a_k - 1) \quad (x_i > 1),
\]

or

\[
n = -\sum_{i=1}^{k} a_i \mod \prod_{i=1}^{k} a_i - 1 \quad (n \geq \prod_{i=1}^{k} a_i - 1);
\]

if \( m > 0 \) is any integer and \( (x_1, \ldots, x_k) \) is such that \( \prod_{i=1}^{k} a_i - 1 = m \), \( x_i > 1 \), \( x_i \) integers, then (1) is soluble for every \( n = -\sum_{i=1}^{k} a_i \mod m \), \( n \geq m \).

A number of such \( n \) up to \( N \) can be sifted out, provided \( m \) runs through the sequence of all prime numbers up to \( \sqrt{N} \). Any upper bound sieve estimate will therefore give us an upper bound for \( E_{k}(N) \).

Some difficulties arise from the fact that for a given prime \( p \) two different \( k \)-factorizations \( (x_1, \ldots, x_k) \), \( (x'_1, \ldots, x'_k) \) of \( p+1 \) may be such that

\[
\sum_{i=1}^{k} a_i \equiv \sum_{i=1}^{k} a'_i \pmod{p}.
\]

A lower bound \( o_{k}(p) \) for the number of \( k \)-factorizations \( (x_1, \ldots, x_k) \) \( \pmod{p} \), is then obtained by means of Lemma 1, which allows us to apply the sieve for any \( k \).

Theorem 1 is easily deduced from Theorem 2, which gives the average order of the above function \( o_{k}(p) \).

The author wishes to thank E. Bombieri and R. C. Vaughan for comments and helpful suggestions.
2. First some preliminary definitions. The ordered $k$-tuple $(x_1, \ldots, x_k)$ is called an admissible $k$-factorization of $\nu$ whenever $\prod_{i=1}^{k} x_i = \nu$, $x_i > 1$, integers. Let $(x_i)$ and $(x'_i)$ be two admissible $k$-factorizations of $\nu$. We say that $(x_i)$ is $r$-equivalent to $(x'_i)$ and write $(x_i) \approx_r (x'_i)$ ($1 \leq r \leq k$), when

$$\sum_{i=1}^{r} x_i = \sum_{i=1}^{r} x'_i.$$  

**Lemma 1.** Let $(x_1, \ldots, x_k)$, $(x'_1, \ldots, x'_k)$ be two admissible $k$-factorizations of $\nu$; let $1 \leq s \leq r \leq k$ and

$$(x_1, \ldots, x_k) \approx_s (x'_1, \ldots, x'_s).$$

If $x_i \geq \nu^{i/3}$, $x'_i \geq \nu^{i/3}$ ($i = 1, \ldots, s$), then

$$x_i = x'_i \quad (i = 1, \ldots, s).$$

**Proof.** Clearly we may assume $r \geq 2$. Suppose first $s = 1$. We have

$$\xi = x_2 \cdots x_k = x'_2 \cdots x'_k \leq \nu^{1/3};$$

hence

$$|x_1 - x'_1| = \frac{\sum_{i=2}^{r} x_i - \sum_{i=2}^{r} x'_i}{\sum_{i=2}^{r} 1} \leq \max \left\{ \sum_{i=2}^{r} x_i, \sum_{i=2}^{r} x'_i \right\} \leq \max \left\{ \prod_{i=2}^{r} x_i, \prod_{i=2}^{r} x'_i \right\} = \max \{\xi, \xi'\} \leq \nu^{1/3};$$

This $\xi = \xi'$, then $|x_1 - x'_1| \geq 1$, whence

$$|x_1 - x'_1| = \nu^{1/3} \rightarrow \nu^{1/3},$$

a contradiction.

In the general case the proof is by induction on $s$. We assume that $x_i \geq \nu^{i/3}$, $x'_i \geq \nu^{i/3}$ ($i = 1, \ldots, s - 1$) implies $x_i = x'_i$ ($i = 1, \ldots, s - 1$). Let $x_i \geq \nu^{i/3}$, $x'_i \geq \nu^{i/3}$ ($i = 1, \ldots, s$). Then

$$(x_1 \cdots x_{s-1})^{3/2} x_{s} \geq \nu^{\frac{3}{2} \sum_{i=1}^{s-1} \frac{1}{3} + \frac{1}{2}} = \nu^{s/3};$$

(5)

$$x_1 \geq \left(\frac{\nu}{x_1 \cdots x_{s-1}}\right)^{2/3}.$$  

and

(6)

$$x'_1 \geq \left(\frac{\nu}{x'_1 \cdots x'_{s-1}}\right)^{2/3}.$$  

It follows from $x_i = x'_i$ ($i = 1, \ldots, s - 1$) that $(x_1, \ldots, x_k)$, $(x'_1, \ldots, x'_k)$ are admissible $(k - s + 1)$-factorizations of

$$\frac{\nu}{x_1 \cdots x_{s-1}} \sim \frac{\nu}{x'_1 \cdots x'_{s-1}},$$

and

$$(x_1, \ldots, x_k) \approx_{s-1} (x'_1, \ldots, x'_k);$$

from (5), (6) and the previous argument for the case $s = 1$ we deduce that $x_s = x'_s$, which completes the proof of the lemma.

Note that if $(x_i)$ and $(x'_i)$ are admissible $k$-factorizations of $p + 1$, then it follows from

$$1 < \sum_{i=1}^{k} x_i \leq \prod_{i=1}^{k} x_i = p + 1,$$

$$1 < \sum_{i=1}^{k} x'_i \leq \prod_{i=1}^{k} x'_i = p + 1$$

that

$$(x_1, \ldots, x_k) \approx_h (x'_1, \ldots, x'_k)$$

if and only if

$$(7) \quad \sum_{i=1}^{k} x_i = \sum_{i=1}^{k} x'_i \mod p.$$  

Also if $p = -1 \mod 4$ and $(y_i)$, $(y'_i)$ are admissible $k$-factorizations of $\frac{p + 1}{a}$, we have

$$(8) \quad \sum_{i=1}^{k} y_i = \sum_{i=1}^{k} y'_i \mod p.$$  

We also require some well-known results which we state here as lemmas.

**Lemma 2 (Brun–Titchmarsh).** If $g \leq \nu^2$, $0 < a < 1$, $(g, l) = 1$, then

$$\pi(x; y, l) \ll \frac{x}{\nu \log \nu}.$$
Lemma 3 (Bombieri [1]). For any $A > 0$ there is $B > 0$ such that
\[
\sum_{\nu} \max_{l < B} \left| \sum_{n < \nu} \frac{\nu y}{\varphi(q)} \right| \ll (\log y)^{-A}.
\]

Lemma 4. Let $d_k(n) = \sum_{\nu \mid \sigma, \nu \equiv 0 \pmod{h}} 1$. Then
\[
\sum_{\nu < \sigma} \frac{d_k(n)}{\varphi(n)} \ll (\log x)^k.
\]

Proof.
\[
\sum_{\nu < \sigma} \frac{d_k(n)}{\varphi(n)} = \sum_{\nu < \sigma} \frac{d_k(n)}{\varphi(n)} \sum_{\nu < \sigma} \frac{\varphi(s)}{\varphi(n)} \sum_{\nu < \sigma} \frac{d_k(n)}{\nu} = \sum_{\nu \mid \sigma} \frac{\varphi(s)}{\varphi(n)} \sum_{\nu \mid \sigma} \frac{d_k(n)}{\nu} \sum_{\nu \mid \sigma} \frac{d_k(n)}{\nu} \ll \left( \sum_{\nu \mid \sigma} \frac{d_k(n)}{\nu} \right) \left( \sum_{\nu \mid \sigma} \frac{d_k(n)}{\nu} \right).
\]

The lemma follows from [6], Lemma 1.1.2 by partial summation.

Lemma 5 (Montgomery [7]). If $\omega(p) (0 \leq \omega(p) < p)$ residue classes modulo $p$ are removed from the first $N$ natural numbers for each prime $p \leq \sqrt{N}$, then the number $Z$ of natural numbers which remain satisfies
\[
Z \leq \frac{4N}{\sum_{\nu < \sqrt{N}} \mu^2(m) \prod_{p \mid \nu} \frac{\omega(p)}{p - \omega(p)}}.
\]

3. First consider equation (1). Let $\sqrt{N} < n < \sqrt{N}$; suppose there are a prime $p \leq \sqrt{N}$ and an admissible $k$-factorization $(x_i)$ of $p + 1$ such that $n = -\sum_{i=1}^{k} x_i (\mod p)$. Then
\[
x_0 = \frac{n + \sum_{i=1}^{k} x_i}{p} > 1,
\]
whence
\[
\prod_{i=1}^{k} x_i - \sum_{i=1}^{k} x_i = n, \quad x_i > 1.
\]

Suppose now that, for every prime $p \leq \sqrt{N}$, there are at least \( \omega_k(p) \) admissible $k$-factorizations of $p + 1$, no two of which are $k$-equivalent to one another. By (7), there are at least $\omega_k(p)$ residue classes modulo $p$ such that, for any $n$ belonging to one of them, $\sqrt{N} \ll n < N$, equation (1) is soluble. Hence, by Lemma 5,
\[
E_k(N) \leq \frac{4N}{\sum_{\nu < \sqrt{N}} \mu^2(m) \prod_{p \mid \nu} \frac{\omega_k(p)}{p - \omega_k(p)}},
\]
If $(x_1, \ldots, x_k) \neq (x'_1, \ldots, x'_k)$ are two $k$-factorizations of $p + 1$ such that $x_i \equiv (p + 1)^{\mu_i} \pmod{\nu}$, $x'_i \equiv (p + 1)^{\mu'_i} \pmod{\nu}$ ($i = 1, \ldots, k$), then $(x_1, \ldots, x_k), (x'_1, \ldots, x'_k)$ are admissible and, by Lemma 1 with $s = r = \sqrt{N}$,
\[
(x_1, \ldots, x_k) \equiv (x'_1, \ldots, x'_k).
\]

We may therefore define
\[
\omega_k(p) = f_k(p + 1, 1),
\]
with
\[
f_k(p, \xi) = \sum_{x_1 \neq x'_1, \ldots, x_k \neq x'_k} 1.
\]

Consider now equation (2). Suppose there are a prime $p \equiv -1 (\mod a)$ and an admissible $k$-factorization $(y_1, \ldots, y_k)$ of $p + 1$ such that $n = \sum_{i=1}^{k} y_i (\mod p)$; then
\[
y_1 + \cdots + y_{k-1} + y_{k-1} = y_0 p = y_5 (y_1 \cdots y_k - 1),
\]
\[
y_0 + y_1 + \cdots + y_{k-1} + y_{k-1} = a y_0 y_1 \cdots y_k,
\]
whence
\[
a \frac{n}{n y_0 y_1 \cdots y_k} = \frac{1}{x_0} + \frac{1}{x_1} + \cdots + \frac{1}{x_k},
\]
with
\[
x_0 = n y_1 \cdots y_k, \quad x_1 = n y_0 y_2 \cdots y_k, \quad \ldots, \quad x_k = y_0 y_1 \cdots y_k.
\]

Suppose that for each $p \equiv -1 (\mod a)$ there are at least $\omega_k(p)$ admissible $k$-factorizations of $p + 1 (\mod a)$, no two of which are equivalent to one another. By (8), there are at least $\omega_k(p)$ residue classes modulo $p$ such that, for any $n$ belonging to one of them, equation (2) is soluble. It follows from Lemma 5
\[
E_{a, k}(N) \leq \frac{4N}{\sum_{\nu < \sqrt{N}} \mu^2(m) \prod_{p \mid \nu} \frac{\omega_k(p)}{p - \omega_k(p)}},
\]
with

\[\omega_{n,k}(p) = \begin{cases} f_k \left( \frac{p+1}{a}, \frac{p+1}{a} \right), & \text{if } p \equiv -1 \pmod{a}, \ p \geq n, \\ 0, & \text{otherwise,} \end{cases}\]

where \(f_k\) is defined by (11). For if \((y_1, \ldots, y_k) \neq (y'_1, \ldots, y'_k)\) are \(k\)-factorizations of \(\frac{p+1}{a} > 1\) such that

\[y_i > \left( \frac{p+1}{a} \right)^{1/\delta^i}, \quad y'_i > \left( \frac{p+1}{a} \right)^{1/\delta^i} \quad (i = 1, \ldots, k),\]

then

\((y_1, \ldots, y_k) \neq_k (y'_1, \ldots, y'_k),\)

by Lemma 1 with \(s = r = k - 1\).

By (10) and (13),

\[\omega_k(p) = \omega_{n,k}(p).\]

4. Theorem 2.

\[(\log \xi)^{k-1} \ll \omega_{n,k}(p) \ll (\log \xi)^{k-1}.\]

Proof. By partial summation, it suffices to prove that

\[(\log \xi)^{k-1} \ll \sum_{p < \xi} \omega_{n,k}(p) \ll (\log \xi)^{k-1}.\]

The upper bound. If \(p = -1 \pmod{a}\), \(p \geq n\), then

\[\omega_{n,k}(p) = \sum_{\frac{p+1}{a} = x_1, \ldots, x_k} \sum_{r = \frac{p+1}{a}} 1 = \sum_{\frac{p+1}{a}} \sum_{\frac{p+1}{a}} 1\]

\[= \sum_{r \leq \frac{p+1}{a}} \sum_{x_1 \ldots x_k = r ^k} 1 = \sum_{r \leq \frac{p+1}{a}} d_{k-1}(r).\]

Hence

\[\sum_{p < \xi} \omega_{n,k}(p) \ll \sum_{p = -1(n)} \sum_{r \leq \frac{p+1}{a}} d_{k-1}(r) \ll \sum_{r \leq \frac{p+1}{a}} d_{k-1}(r) \sum_{p = -1(n)} 1\]

\[\ll \sum_{r \leq \frac{p+1}{a}} d_{k-1}(r) \pi(\xi; r, -1).\]

Therefore, by Lemmas 2 and 4,

\[\sum_{p < \xi} \omega_{n,k}(p) \ll \frac{\xi}{\log \xi} \sum_{r < \xi^{1/\delta}} \frac{d_{k-1}(r)}{\varphi(r)} \ll \frac{\xi}{(\log \xi)^{k-1}}.\]

The lower bound.

\[\sum_{p < \xi} \omega_{n,k}(p) \gg \sum_{\frac{p+1}{a}} f_k \left( \frac{p+1}{a}, \xi \right) = \sum_{\frac{p+1}{a}} \sum_{r \leq \frac{p+1}{a}} \sum_{\frac{p+1}{a}} 1\]

\[= \sum_{\frac{p+1}{a}} \sum_{\frac{p+1}{a}} 1 \gg \sum_{\frac{p+1}{a}} \sum_{\frac{p+1}{a}} 1\]

\[= \sum_{r \leq \frac{p+1}{a}} \sum_{\frac{p+1}{a}} d_{k-1}(r) \ll \pi(\xi; ar, -1) - \pi(\frac{\xi}{2}; ar, -1).\]

whence

\[\sum_{p < \xi} \omega_{n,k}(p) \gg \left( \left\lfloor \frac{\xi}{2} \right\rfloor - \frac{\xi}{2} \right) \sum_{r \leq \frac{p+1}{a}} f_k \left( r, \xi^{1/\delta} \right) + \]

\[+ \sum_{r \leq \frac{p+1}{a}} f_k \left( r, \xi^{1/\delta} \right) \left\{ \pi(\xi; ar, -1) - \frac{\xi}{\varphi(ar)} \right\} - \pi(\frac{\xi}{2}; ar, -1) - \frac{\xi}{\varphi(ar)}.\]

First consider the term

\[R_{n,k}(\xi) = \sum_{r \leq \frac{p+1}{a}} f_k \left( r, \xi^{1/\delta} \right) \times\]

\[\times \left\{ \pi(\xi; ar, -1) - \frac{\xi}{\varphi(ar)} \right\} - \pi(\frac{\xi}{2}; ar, -1) - \frac{\xi}{\varphi(ar)}.\]
The Cauchy–Schwarz inequality yields
\begin{align*}
|R_{a,b}(\xi)| &\ll \left( \sum_{r \leq \frac{x}{2a}} \frac{f_{b-1}(r, \xi^{2b})}{\varphi(ar)} \right)^{1/2} \left( \sum_{r \leq \frac{x}{2a}} \varphi(ar) \left[ \frac{\varphi(\xi; ar, -1) - \frac{1}{2} \frac{\xi}{\varphi(ar)}}{\varphi(ar)} \right] \right)^{1/2} \\
&\ll \left( \sum_{r \leq \frac{x}{2a}} \frac{d_{b-1}(r)}{\varphi(r)} \right)^{1/2} \left( \sum_{s \leq \frac{x}{2a}} \varphi(s) \left[ \frac{\varphi(\xi; s, -1) - \frac{1}{2} \frac{\xi}{\varphi(s)}}{\varphi(s)} \right] \right)^{1/2}.
\end{align*}

By Lemma 4
\begin{equation}
\left( \sum_{r \leq \frac{x}{2a}} \frac{d_{b-1}(r)}{\varphi(r)} \right)^{1/2} \ll (\log \xi)^{2/3},
\end{equation}
and by Lemma 2 with \( \alpha = 1/3 \)
\begin{equation}
\varphi(s) \left[ \frac{\varphi(\xi; s, -1) - \frac{1}{2} \frac{\xi}{\varphi(s)}}{\varphi(s)} \right] \ll \frac{\xi}{\log \xi}.
\end{equation}

Lemma 3 gives, for any \( A > 0 \),
\begin{equation}
\left( \sum_{s \leq \frac{x}{2a}} \varphi(s) \left[ \frac{\varphi(\xi; s, -1) - \frac{1}{2} \frac{\xi}{\varphi(s)}}{\varphi(s)} \right] \right) \ll \xi (\log \xi)^{-A}.
\end{equation}

From (17), (18), (19) and (20) we obtain
\begin{equation}
|R_{a,b}(\xi)| \ll (\log \xi)^{2/3} \left( \frac{\xi}{\log \xi} \right) (\log \xi)^{-A} \ll \xi (\log \xi)^{-A}.
\end{equation}

For the main term in (16) we have
\begin{align*}
\sum_{r \leq \frac{x}{2a}} \frac{f_{b-1}(r, \xi^{2b})}{\varphi(ar)} &\ll \frac{1}{a} \sum_{r \leq \frac{x}{2a}} \frac{f_{b-1}(r, \xi^{2b})}{r} = \frac{1}{a} \sum_{r \leq \frac{x}{2a}} \sum_{s \leq \frac{x}{2a}} \sum_{a s = r} \frac{1}{r} \\
&= \frac{1}{a} \sum_{a, a_{2a} \cdots a_{(b-1)2a}} \frac{1}{a_{2a} \cdots a_{b-1}}.
\end{align*}

Now, if
\begin{equation}
a_i \ll \frac{2^{i+1}}{3} \frac{1}{\xi^{2/3}} \quad (i = 1, \ldots, b-1),
\end{equation}
we have
\begin{equation}
a_1 \cdots a_{b-1} \ll \frac{1}{\xi^{2/3}} \left( \frac{2^{b-1}}{3} \frac{1}{\xi^{2/3}} \right) = \frac{2^{b-1}}{3} \frac{1}{\xi^{2/3}} \leq \frac{2^{b-1}}{2a},
\end{equation}
provided \( \xi \gg C_{a,b} = (2a)^{2b-1} \), hence if \( \xi \) is large
\begin{equation}
\sum_{a_1 \cdots a_{b-1} \leq \frac{2^{b-1}}{2a}} \frac{1}{a_1 \cdots a_{b-1}} \gg \sum_{a_1 \cdots a_{b-1} \leq \frac{2^{b-1}}{2a}} \frac{1}{a_1 \cdots a_{b-1}}.
\end{equation}

Since
\begin{equation}
\left( \frac{\xi}{\log \xi} \right) \frac{\xi}{\log \xi} \gg \frac{\xi}{\log \xi},
\end{equation}
we obtain from (22) and (23)
\begin{equation}
\left( \frac{\xi}{\log \xi} \right) \frac{\xi}{\log \xi} \gg \sum_{r \leq \frac{x}{2a}} \frac{f_{b-1}(r, \xi^{2b})}{\varphi(ar)} \gg \xi (\log \xi)^{-b}.\tag{24}
\end{equation}

(16), (21) and (24) together give at once
\begin{equation}
\sum_{\nu < \xi} \omega_{a,b}(\nu) \gg \xi (\log \xi)^{-b},\tag{16}
\end{equation}
which proves the theorem.

5. Proof of Theorem 1. Let \( \omega_{a,b}(m) \) be, for any integer \( m > 0 \), the completely multiplicative function generated by \( \omega_{a,b}(p) \). Then
\begin{equation}
\sum_{m < c} \mu^2(m) \prod_{p \mid m} \omega_{a,b}(p) \gg \sum_{m < c} \mu^2(m) \prod_{p \mid m} \omega_{a,b}(p) \gg \sum_{m \leq c} \mu^2(m) \omega_{a,b}(m).
\end{equation}
Hence, in view of (9), (12) and (14), Theorem 1 is proved if we show that
\begin{equation}
\sum_{m \leq c} \mu^2(m) \omega_{a,b}(m) \gg \exp \left( C(a, b) (\log N)^{-1} \right).
\end{equation}
Since $\omega_{a,h}(m) \ll m^4$ for any $\eta > 0$, the Dirichlet series
\[ F_{a,k}(s) = \sum_{m=1}^{\infty} \frac{\mu^2(m)\omega_{a,k}(m)}{m^s} \]
converges for $\Re s > 1$. Also
\[ \sum_p \left| \frac{\omega_{a,h}(p)}{p} \right|^s < \infty; \]
hence, for any $\epsilon > 0$,
\[ F_{a,k}(1+\epsilon) = \prod_p \left( 1 - \frac{\omega_{a,h}(p)}{p^{1+\epsilon}} \right) \]
\[ = \exp \left\{ \sum_p \frac{\omega_{a,h}(p)}{p^{1+\epsilon}} + O(1) \right\} \approx \exp \sum_p \frac{\omega_{a,h}(p)}{p^{1+\epsilon}}. \]
It follows from Theorem 2 that
\[ \sum_p \omega_{a,h}(p) = \sum_{p \leq x} \omega_{a,h}(p) \geq x^{-\epsilon} \sum_{p \leq x} \omega_{a,h}(p) \gg_{a,h} X^{-\epsilon} (\log X)^{k-1}; \]
taking $X = \exp \frac{1}{\epsilon}$ we obtain
\[ \sum_p \frac{\omega_{a,h}(p)}{p^{1+\epsilon}} \gg_{a,h} \epsilon^{-(k-1)}. \]
Next
\[ \sum_p \frac{\omega_{a,h}(p)}{p^{1+\epsilon}} \geq \frac{\sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \omega_{a,h}(p)}{p^{1+\epsilon}} \leq \frac{\sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \omega_{a,h}(p)}{p^{1+\epsilon}} \]
\[ = \sum_{n=2}^{\infty} \frac{(2+2^{-\epsilon} - (n+1))}{p^{1+\epsilon}} \sum_{p \leq x} \omega_{a,h}(p) \]
\[ = (1-2^{-\epsilon}) \sum_{n=2}^{\infty} \frac{(n+1)^{1+\epsilon}}{p^{1+\epsilon}} \sum_{p \leq x} \omega_{a,h}(p); \]
hence by Theorem 2
\[ \sum_p \frac{\omega_{a,h}(p)}{p^{1+\epsilon}} \ll (1-2^{-\epsilon}) \sum_{n=2}^{\infty} (n+1)^{1+\epsilon} 2^{-n}. \]
Since
\[ \sum_{n=2}^{\infty} (n+1)^{1+\epsilon} 2^{-n} \ll_{\epsilon} \epsilon^{-k}, \]
we obtain
\[ \sum_p \frac{\omega_{a,h}(p)}{p^{1+\epsilon}} \ll_{\epsilon} \epsilon^{-k+\epsilon}. \]
Moreover
\[ \sum_{m \in \mathbb{N}} \frac{\mu^2(m)\omega_{a,k}(m)}{m} \geq \sum_{m \in \mathbb{N}} \frac{\mu^2(m)\omega_{a,k}(m)}{m^{1+\epsilon}} \]
\[ = F_{a,k}(1+\epsilon) - \sum_{m > \sqrt{N}} \frac{\mu^2(m)\omega_{a,k}(m)}{m^{1+\epsilon}}; \]
and (30) give, for any $\epsilon > 0$,
\[ \sum_{m \in \mathbb{N}} \frac{\mu^2(m)\omega_{a,k}(m)}{m} \gg \sum_{m \in \mathbb{N}} \frac{\mu^2(m)\omega_{a,k}(m)}{m^{1+\epsilon}} \]
\[ = F_{a,k}(1+\epsilon) - N^{-\epsilon} F_{a,k} \left( 1 + \frac{\epsilon}{2} \right). \]
We deduce from (26), (27), (28) and (31) that, for any sufficiently small $\epsilon$,\[ \sum_{m \in \mathbb{N}} \frac{\mu^2(m)\omega_{a,k}(m)}{m} \]
\[ \gg C_4 \exp \{ C_4(a, k) \epsilon^{-(b-1)} \} - C_3 N^{-\epsilon^2} \exp \{ C_4(k) \epsilon^{-(b-1)} \} \]
\[ = C_1 \exp \{ C_4(a, k) \epsilon^{-(b-1)} \} - C_3 \exp \left( - \frac{\epsilon}{4} \log N + C_4(k) \epsilon^{-(b-1)} \right). \]
Putting $\epsilon = (\frac{4}{C_4(k)})^{1/b} \cdot (\log N)^{-1/b}$ we obtain
\[ - \frac{\epsilon}{4} \log N + C_4(k) \epsilon^{-(b-1)} = 0. \]
Hence the above choice for $\epsilon$ gives, if $N$ is large,
\[ \sum_{m \in \mathbb{N}} \frac{\mu^2(m)\omega_{a,k}(m)}{m} \gg C_1 \exp \{ C(a, k) (\log N)^{1-\frac{1}{b}} \}, \]
and (25) is proved.
References


Received on 19. 1. 1972