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On the diophantine equations $\prod_0^k x_i - \sum_0^k x_i = n$ and $\sum_0^k \frac{1}{x_i} = \frac{a}{n}$ *

by

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1. Introduction. We are concerned with the equations

(1) $$\prod_0^k x_i - \sum_0^k x_i = n$$

and

(2) $$\sum_0^k \frac{1}{x_i} = \frac{a}{n},$$

where a, k and n are given integers and the unknowns x_i take positive integral values. Equation (1) was first considered by Schinzel [9] in the case $n = 0$; he observed that for every k there exists a trivial solution, namely $(1, \ldots, 1, 2, k+1)$. Misiurewicz (quoted in [9], Bemerkung) proved that in the case $n = 0$, apart from any permutation of the x_i's, equation (1) has no solutions different from the trivial one when $k+1 = 2, 3, 4, 6, 24, 114, 174, 444$, while for any other $k < 1000$ there is at least one other solution(1). Later Schinzel conjectured (see [2], p. 238) that there is a $k > 1$ such that, for every sufficiently large n, (1) is soluble in integers $x_i > 1$. Note that equation (1) has for any n, k the trivial solution $(1, \ldots, 1, 2, n+k+1)$.

Leonardo Pisano [5] proved in 1202 that for any $a > 0$, $n > 0$, there is a k such that (2) is soluble with $x_i \neq x_j$ for $i \neq j$; obviously k depends on a and n. Many authors (see e. g. [3], [4], [8], [10], [11]) have been concerned with related problems; a classical topic is the investigation of conditions for a positive rational number to be the sum of distinct

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(1) Note added in proof. With the aid of a computer, the result of Misiurewicz has been extended to all $k < 10^4$ (Amer. Math. Monthly 78(1971), pp. 1021-1022).

reciprocals of finitely many integers, belonging to a given sequence of natural numbers. Here we drop the restriction $x_i \neq x_j$; on the other hand we assume that k has a fixed value, independent of a and n. First of all we note that if $\frac{a}{n}$ is the sum of $k+1$ unit fractions:

$$\frac{a}{n} = \frac{1}{x_0} + \frac{1}{x_1} + \ldots + \frac{1}{x_k},$$

then it is also the sum of $k+2$, $k+3, \ldots$ unit fractions, since one may replace $\frac{1}{x_k}$ with $\frac{1}{2x_k} + \frac{1}{2x_k}$, etc. Hence (2) is trivially soluble when $a \leqslant k+1$.

In case $k=1$, one can easily prove that the equation $\frac{a}{n} = \frac{1}{x} + \frac{1}{y}$, $(a, n) = 1$, is soluble in positive integers x and y if and only if there exist d_1, d_2 such that $d_1 | n$, $d_2 | n$ and $d_1 + d_2 \equiv 0 \pmod a$. A proof is given in [11], Lemma 2. It follows that when $a > 2$ there are infinitely many n for which $\frac{a}{n} = \frac{1}{x} + \frac{1}{y}$ is insoluble. For instance, one may take n to be any prime $p \equiv 1 \pmod a$.

In case $k=2$, Schinzel conjectured ([10], p. 25) that for every $a > 0$, if $n > n_0(a)$, $\frac{a}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is soluble in positive integers x, y, z. Vaughan [12] has recently proved that the number of natural numbers $n \leqslant N$ for which $\frac{a}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is insoluble is $\ll N \exp\{-C(a)(\log N)^{2/3}\}$, $C(a) > 0$.

The aim of the present paper is to prove results of Vaughan's type for equations (1) and (2); it will then follow that for any a, k such that $a > k+1$, $k > 1$, the asymptotic density of the natural numbers n for which either (1) or (2) is insoluble is zero.

Since for $k > 1$, $x_i > 1$ the inequality $\prod_0^k x_i > \sum_0^k x_i$ holds, we may assume throughout that $n > 0$, $k > 1$ and, in order to avoid the trivial solution of equation (1), we shall impose the condition $x_i > 1$.

Our result is:

THEOREM 1. *Let $E_k(N)$ denote the number of natural numbers $n \leqslant N$ for which*

$$\prod_0^k x_i - \sum_0^k x_i = n \tag{1}$$

is insoluble in integers $x_i > 1$, and let $E_{a,k}(N)$ denote the number of natural numbers $n \leqslant N$ for which

$$\sum_0^k \frac{1}{x_i} = \frac{a}{n} \tag{2}$$

is insoluble in positive integers x_i. Then for $N \to \infty$

$$E_k(N) \ll N \exp\{-C(k)(\log N)^{1-\frac{1}{k}}\}, \tag{3}$$

$$E_{a,k}(N) \ll N \exp\{-C(a, k)(\log N)^{1-\frac{1}{k}}\}, \tag{4}$$

with $C(k) > 0$, $C(a, k) > 0$.

We follow the sifting process already used by Vaughan [12]. However, the method he uses to adapt equation (2) in case $k=2$ to the sieve (which gives the exponent $2/3$ rather than our $1/2$) does not extend to $k > 2$ in an obvious way. We outline here our method; consider for instance equation (1). Write (1) in the form

$$n + x_1 + \ldots + x_k = x_0(x_1 \ldots x_k - 1) \quad (x_i > 1),$$

or

$$n \equiv -\sum_1^k x_i \bmod \prod_1^k x_i - 1 \quad (n \geqslant \prod_1^k x_i - 1);$$

if $m > 0$ is any integer and $(x_1, \ldots, x_k)$ is such that $\prod_1^k x_i - 1 = m$, $x_i > 1$, x_i integers, then (1) is soluble for every $n \equiv -\sum_1^k x_i \pmod m$, $n \geqslant m$. A number of such n up to N can be sifted out, provided m runs through the sequence of all prime numbers up to $\sqrt{N}$. Any upper bound sieve estimate will therefore give us an upper bound for $E_k(N)$.

Some difficulties arise from the fact that for a given prime p two different k-factorizations $(x_1, \ldots, x_k)$, $(x_1', \ldots, x_k')$ of $p+1$ may be such that $\sum_1^k x_i \equiv \sum_1^k x_i' \pmod p$. A lower bound $\omega_k(p)$ for the number of k-factorizations (x_i), (x_i'), ... of $p+1$, such that $\sum_1^k x_i \not\equiv \sum_1^k x_i', \ldots \pmod p$, is then obtained by means of Lemma 1, which allows us to apply the sieve for any k.

Theorem 1 is easily deduced from Theorem 2, which gives the average order of the above function $\omega_k(p)$.

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2. First some preliminary definitions. The ordered k-tuple $(x_1, \ldots, x_k)$ is called an *admissible k-factorization of* ν whenever $\prod_1^k x_i = \nu$, $x_i > 1$, x_i integers. Let (x_i) and (x_i') be two admissible k-factorizations of ν. We say that (x_i) is *r-equivalent to* (x_i') and write $(x_i) \approx_r (x_i')$ $(1 \leqslant r \leqslant k)$, when

$$\sum_1^r x_i = \sum_1^r x_i'.$$

LEMMA 1. *Let $(x_1, \ldots, x_k)$, $(x_1', \ldots, x_k')$ be two admissible k-factorizations of ν; let $1 \leqslant s \leqslant r \leqslant k$ and*

$$(x_1, \ldots, x_k) \approx_r (x_1', \ldots, x_k').$$

If

$$x_i \geqslant \nu^{2/3^i}, \quad x_i' \geqslant \nu^{2/3^i} \quad (i = 1, \ldots, s),$$

then

$$x_i = x_i' \quad (i = 1, \ldots, s).$$

Proof. Clearly we may assume $r \geqslant 2$. Suppose first $s = 1$. We have

$$\xi = x_2 \ldots x_k = \frac{\nu}{x_1} \leqslant \nu^{1/3},$$

$$\xi' = x_2' \ldots x_k' = \frac{\nu}{x_1'} \leqslant \nu^{1/3};$$

hence

$$|x_1 - x_1'| = \Big|\sum_2^r x_i - \sum_2^r x_i'\Big| < \max\Big\{\sum_2^r x_i, \sum_2^r x_i'\Big\} \leqslant \max\Big\{\prod_2^k x_i, \prod_2^k x_i'\Big\} = \max\{\xi, \xi'\} \leqslant \nu^{1/3}.$$

If $x_1 \neq x_1'$, then $|\xi - \xi'| \geqslant 1$, whence

$$|x_1 - x_1'| = \nu\frac{|\xi - \xi'|}{\xi\xi'} \geqslant \nu^{1/3},$$

a contradiction.

In the general case the proof is by induction on s. We assume that $x_i \geqslant \nu^{2/3^i}$, $x_i' \geqslant \nu^{2/3^i}$ $(i = 1, \ldots, s-1)$ implies $x_i = x_i'$ $(i = 1, \ldots, s-1)$.

Let $x_i \geqslant \nu^{2/3^i}$, $x_i' \geqslant \nu^{2/3^i}$ $(i = 1, \ldots, s)$. Then

$$(x_1 \ldots x_{s-1})^{2/3} x_s \geqslant \nu^{\frac{2}{3}\sum_1^{s-1}\frac{2}{3^i} + \frac{2}{3^s}} = \nu^{2/3},$$

(5) $$x_s \geqslant \left(\frac{\nu}{x_1 \ldots x_{s-1}}\right)^{2/3},$$

and

(6) $$x_s' \geqslant \left(\frac{\nu}{x_1' \ldots x_{s-1}'}\right)^{2/3}.$$

It follows from $x_i = x_i'$ $(i = 1, \ldots, s-1)$ that $(x_s, \ldots, x_r, \ldots, x_k)$, $(x_s', \ldots, x_r', \ldots, x_k')$ are admissible $(k-s+1)$-factorizations of

$$\frac{\nu}{x_1 \ldots x_{s-1}} = \frac{\nu}{x_1' \ldots x_{s-1}'},$$

and

$$(x_s, \ldots, x_r, \ldots, x_k) \approx_{r-s+1} (x_s', \ldots, x_r', \ldots, x_k');$$

from (5), (6) and the previous argument for the case $s = 1$ we deduce that $x_s = x_s'$, which completes the proof of the lemma.

Note that if (x_i) and (x_i') are admissible k-factorizations of $p+1$, then it follows from

$$1 < \sum_1^k x_i \leqslant \prod_1^k x_i = p+1,$$

$$1 < \sum_1^k x_i' \leqslant \prod_1^k x_i' = p+1$$

that

$$(x_1, \ldots, x_k) \approx_k (x_1', \ldots, x_k')$$

if and only if

(7) $$\sum_1^k x_i \equiv \sum_1^k x_i' \pmod p.$$

Also if $p \equiv -1 \pmod a$ and (y_i), (y_i') are admissible k-factorizations of $\dfrac{p+1}{a}$, we have

$$(y_1, \ldots, y_k) \approx_{k-1} (y_1', \ldots, y_k')$$

if and only if

(8) $$\sum_1^{k-1} y_i \equiv \sum_1^{k-1} y_i' \pmod p.$$

We also require some well-known results which we state here as lemmas.

LEMMA 2 (Brun–Titchmarsh). *If $q \leqslant x^a$, $0 < a < 1$, $(q, l) = 1$, then*

$$\pi(x; q, l) \ll \frac{x}{\varphi(q)\log x}.$$

LEMMA 3 (Bombieri [1]). *For any $A > 0$ there is $B > 0$ such that*

$$\sum_{q \leqslant x^{1/2}(\log x)^{-B}} \max_{y \leqslant x} \max_{(q,l)=1} \left| \pi(y; q, l) - \frac{\operatorname{li} y}{\varphi(q)} \right| \ll x(\log x)^{-A}.$$

LEMMA 4. *Let* $d_k(n) = \sum_{\substack{(x_1, \ldots, x_k) \\ x_1 \ldots x_k = n}} 1$. *Then*

$$\sum_{n \leqslant x} \frac{d_k^l(n)}{\varphi(n)} \ll (\log x)^{k^l}.$$

Proof.

$$\sum_{n \leqslant x} \frac{d_k^l(n)}{\varphi(n)} = \sum_{n \leqslant x} \frac{d_k^l(n)}{n} \sum_{s|n} \frac{\mu^2(s)}{\varphi(s)} = \sum_{s \leqslant x} \frac{\mu^2(s)}{\varphi(s)} \sum_{\substack{n \leqslant x \\ s|n}} \frac{d_k^l(n)}{n}$$

$$= \sum_{s \leqslant x} \frac{\mu^2(s)}{s\,\varphi(s)} \sum_{r \leqslant x/s} \frac{d_k^l(rs)}{r} \leqslant \sum_{s \leqslant x} \frac{\mu^2(s)\, d_k^l(s)}{s\,\varphi(s)} \sum_{r \leqslant x/s} \frac{d_k^l(r)}{r}$$

$$\leqslant \left(\sum_{s=1}^{\infty} \frac{d_k^l(s)}{s\,\varphi(s)} \right) \left(\sum_{r \leqslant x} \frac{d_k^l(r)}{r} \right).$$

The lemma follows from [6], Lemma 1.1.2 by partial summation.

LEMMA 5 (Montgomery [7]). *If $\omega(p)$ $(0 \leqslant \omega(p) < p)$ residue classes modulo p are removed from the first N natural numbers for each prime $p \leqslant \sqrt{N}$, then the number Z of natural numbers which remain satisfies*

$$Z \leqslant \frac{4N}{\sum_{m \leqslant \sqrt{N}} \mu^2(m) \prod_{p|m} \frac{\omega(p)}{p - \omega(p)}}.$$

3. First consider equation (1). Let $\sqrt{N} \leqslant n \leqslant N$; suppose there are a prime $p \leqslant \sqrt{N}$ and an admissible k-factorization (x_i) of $p+1$ such that $n \equiv -\sum_1^k x_i \pmod{p}$. Then

$$x_0 = \frac{n + \sum_1^k x_i}{p} > 1,$$

whence

$$\prod_0^k x_i - \sum_0^k x_i = n, \quad x_i > 1.$$

Suppose now that, for every prime p, $p \leqslant \sqrt{N}$, there are at least $\omega_k(p)$ admissible k-factorizations of $p+1$, no two of which are k-equivalent to one another. By (7), there are at least $\omega_k(p)$ residue classes modulo p such that, for any n belonging to one of them, $\sqrt{N} \leqslant n \leqslant N$, equation (1) is soluble. Hence, by Lemma 5,

$$E_k(N) \leqslant \frac{4N}{\sum_{m \leqslant \sqrt{N}} \mu^2(m) \prod_{p|m} \frac{\omega_k(p)}{p - \omega_k(p)}} + \sqrt{N}. \tag{9}$$

If $(x_1, \ldots, x_k) \neq (x_1', \ldots, x_k')$ are two k-factorizations of $p+1$ such that $x_i \geqslant (p+1)^{2/3^i}$, $x_i' \geqslant (p+1)^{2/3^i}$ $(i = 1, \ldots, k)$, then $(x_1, \ldots, x_k)$, $(x_1', \ldots, x_k')$ are admissible and, by Lemma 1 with $s = r = k$,

$$(x_1, \ldots, x_k) \not\sim_k (x_1', \ldots, x_k').$$

We may therefore define

$$\omega_k(p) = f_k(p+1, p+1), \tag{10}$$

with

$$f_k(\nu, \xi) = \sum_{\substack{(x_1, \ldots, x_k) \\ x_1 \ldots x_k = \nu \\ x_i \geqslant \xi^{2/3^i}}} 1. \tag{11}$$

Consider now equation (2). Suppose there are a prime $p \equiv -1 \pmod{a}$ and an admissible k-factorization $(y_1, \ldots, y_k)$ of $\frac{p+1}{a}$ such that $n \equiv -\sum_1^{k-1} y_i \pmod{p}$; then

$$y_1 + \ldots + y_{k-1} + n = y_0 p = y_0(a y_1 \ldots y_k - 1),$$

$$y_0 + y_1 + \ldots + y_{k-1} + n = a y_0 y_1 \ldots y_k,$$

$$\frac{a}{n} = \frac{y_0 + y_1 + \ldots + y_{k-1} + n}{n y_0 y_1 \ldots y_k} = \frac{1}{x_0} + \frac{1}{x_1} + \ldots + \frac{1}{x_k},$$

with

$$x_0 = n y_1 \ldots y_k, \quad x_1 = n y_0 y_2 \ldots y_k, \quad \ldots, \quad x_k = y_0 y_1 \ldots y_k.$$

Suppose that for each $p \equiv -1 \pmod{a}$ there are at least $\omega_{a,k}(p)$ admissible k-factorizations of $\frac{p+1}{a}$, no two of which are $(k-1)$-equivalent to one another. By (8), there are at least $\omega_{a,k}(p)$ residue classes modulo p such that, for any n belonging to one of them, equation (2) is soluble. It follows from Lemma 5

$$E_{a,k}(N) \leqslant \frac{4N}{\sum_{m \leqslant \sqrt{N}} \mu^2(m) \prod_{p|m} \frac{\omega_{a,k}(p)}{p - \omega_{a,k}(p)}}, \tag{12}$$

with

$$(13)\qquad \omega_{a,k}(p) = \begin{cases} f_k\left(\dfrac{p+1}{a}, \dfrac{p+1}{a}\right), & \text{if} \quad p \equiv -1 \ (\mathrm{mod}\ a),\ p \geqslant a, \\ 0, \ \text{otherwise}, \end{cases}$$

where f_k is defined by (11). For if $(y_1, \ldots, y_k) \neq (y_1', \ldots, y_k')$ are k-factorizations of $\frac{p+1}{a} > 1$ such that

$$y_i \geqslant \left(\frac{p+1}{a}\right)^{2/3^i}, \quad y_i' \geqslant \left(\frac{p+1}{a}\right)^{2/3^i} \quad (i = 1, \ldots, k),$$

then

$$(y_1, \ldots, y_k) \not\approx_{k-1} (y_1', \ldots, y_k'),$$

by Lemma 1 with $s = r = k-1$.

By (10) and (13),

$$(14)\qquad \omega_k(p) = \omega_{1,k}(p).$$

4. **Theorem 2.**

$$(\log \xi)^{k-1} \ll_{a,k} \sum_{p \leqslant \xi} \frac{\omega_{a,k}(p)}{p} \ll (\log \xi)^{k-1}.$$

Proof. By partial summation, it suffices to prove that

$$(15)\qquad \xi (\log \xi)^{k-2} \ll_{a,k} \sum_{p \leqslant \xi} \omega_{a,k}(p) \ll \xi (\log \xi)^{k-2}.$$

The upper bound. If $p \equiv -1 \ (\mathrm{mod}\ a)$, $p \geqslant a$, then

$$\omega_{a,k}(p) = \sum_{\substack{x_1 \ldots x_k = \frac{p+1}{a} \\ x_i \geqslant \left(\frac{p+1}{a}\right)^{2/3^i}}} 1 \leqslant \sum_{\substack{x_1 \ldots x_k = \frac{p+1}{a} \\ x_1 \geqslant \left(\frac{p+1}{a}\right)^{2/3}}} 1 = \sum_{\substack{x_1 \mid \frac{p+1}{a} \\ x_1 \geqslant \left(\frac{p+1}{a}\right)^{2/3}}} \sum_{x_2 \ldots x_k = \frac{p+1}{a x_1}} 1$$

$$= \sum_{\substack{r \mid \frac{p+1}{a} \\ r \leqslant \left(\frac{p+1}{a}\right)^{1/3}}} \sum_{x_2 \ldots x_k = r} 1 = \sum_{\substack{r \mid \frac{p+1}{a} \\ r \leqslant \left(\frac{p+1}{a}\right)^{1/3}}} d_{k-1}(r).$$

Hence

$$\sum_{p \leqslant \xi} \omega_{a,k}(p) \leqslant \sum_{\substack{p \leqslant \xi \\ p \equiv -1 \, (a)}} \sum_{\substack{r \mid \frac{p+1}{a} \\ r \leqslant \left(\frac{p+1}{a}\right)^{1/3}}} d_{k-1}(r) \leqslant \sum_{r \leqslant \left(\frac{\xi+1}{a}\right)^{1/3}} d_{k-1}(r) \sum_{\substack{p \leqslant \xi \\ p \equiv -1 \, (ar)}} 1$$

$$\leqslant \sum_{r \leqslant (\xi+1)^{1/3}} d_{k-1}(r) \pi(\xi; r, -1).$$

Therefore, by Lemmas 2 and 4,

$$\sum_{p \leqslant \xi} \omega_{a,k}(p) \ll \frac{\xi}{\log \xi} \sum_{r \leqslant (\xi+1)^{1/3}} \frac{d_{k-1}(r)}{\varphi(r)} \ll \xi (\log \xi)^{k-2}.$$

The lower bound.

$$\sum_{p \leqslant \xi} \omega_{a,k}(p) \geqslant \sum_{\substack{\frac{\xi}{2} < p \leqslant \xi \\ p \equiv -1 \, (a)}} f_k\left(\frac{p+1}{a}, \xi\right) = \sum_{\substack{\frac{\xi}{2} < p \leqslant \xi \\ p \equiv -1 \, (a)}} \sum_{\substack{x_1 \ldots x_k = \frac{p+1}{a} \\ x_i \geqslant \xi^{2/3^i}}} 1$$

$$= \sum_{\substack{\frac{\xi}{2} < p \leqslant \xi \\ p \equiv -1 \, (a)}} \sum_{\substack{x_1 \mid \frac{p+1}{a} \\ x_1 \geqslant \xi^{2/3}}} \sum_{\substack{x_2 \ldots x_k = \frac{p+1}{a x_1} \\ x_i \geqslant \xi^{2/3^i}}} 1 \geqslant \sum_{\substack{\frac{\xi}{2} < p \leqslant \xi \\ p \equiv -1 \, (a)}} \sum_{\substack{r \mid \frac{p+1}{a} \\ r \leqslant \frac{\xi^{1/3}}{2a}}} \sum_{\substack{x_2 \ldots x_k = r \\ x_i \geqslant \xi^{2/3^i}}} 1$$

$$= \sum_{r \leqslant \frac{\xi^{1/3}}{2a}} \sum_{\substack{x_1 \ldots x_{k-1} = r \\ x_i \geqslant \xi^{2/3^{i+1}}}} \sum_{\substack{\frac{\xi}{2} < p \leqslant \xi \\ p \equiv -1 \, (ar)}} 1$$

$$= \sum_{r \leqslant \frac{\xi^{1/3}}{2a}} f_{k-1}(r, \xi^{1/3}) \left\{ \pi(\xi; ar, -1) - \pi\left(\frac{\xi}{2}; ar, -1\right) \right\},$$

whence

$$(16)\qquad \sum_{p \leqslant \xi} \omega_{a,k}(p) \geqslant \left(\operatorname{li} \xi - \operatorname{li} \frac{\xi}{2}\right) \sum_{r \leqslant \frac{\xi^{1/3}}{2a}} \frac{f_{k-1}(r, \xi^{1/3})}{\varphi(ar)} +$$

$$+ \sum_{r \leqslant \frac{\xi^{1/3}}{2a}} f_{k-1}(r, \xi^{1/3}) \left\{ \left[\pi(\xi; ar, -1) - \frac{\operatorname{li} \xi}{\varphi(ar)} \right] - \left[\pi\left(\frac{\xi}{2}; ar, -1\right) - \frac{\operatorname{li} \frac{\xi}{2}}{\varphi(ar)} \right] \right\}.$$

First consider the term

$$R_{a,k}(\xi) = \sum_{r \leqslant \frac{\xi^{1/3}}{2a}} f_{k-1}(r, \xi^{1/3}) \times$$

$$\times \left\{ \left[\pi(\xi; ar, -1) - \frac{\operatorname{li} \xi}{\varphi(ar)} \right] - \left[\pi\left(\frac{\xi}{2}; ar, -1\right) - \frac{\operatorname{li} \frac{\xi}{2}}{\varphi(ar)} \right] \right\}.$$

The Cauchy–Schwarz inequality yields

(17)
$$|R_{a,k}(\xi)| \leqslant \Big(\sum_{r\leqslant \xi^{1/3}} \frac{f_{k-1}^2(r,\,\xi^{1/3})}{\varphi(ar)}\Big)^{1/2}\Big(\sum_{r\leqslant \frac{\xi^{1/3}}{a}} \varphi(ar)\Big\{\Big[\pi(\xi;\,ar,\,-1)-\frac{\operatorname{li}\xi}{\varphi(ar)}\Big]-$$
$$-\Big[\pi\Big(\frac{\xi}{2};\,ar,\,-1\Big)-\frac{\operatorname{li}\frac{\xi}{2}}{\varphi(ar)}\Big]\Big\}^2\Big)^{1/2}$$
$$\leqslant \Big(\sum_{r\leqslant \xi} \frac{d_{k-1}^2(r)}{\varphi(r)}\Big)^{1/2}\Big(\sum_{s\leqslant \xi^{1/3}} \varphi(s)\Big\{\Big[\pi(\xi;\,s,\,-1)-\frac{\operatorname{li}\xi}{\varphi(s)}\Big]-$$
$$-\Big[\pi\Big(\frac{\xi}{2};\,s,\,-1\Big)-\frac{\operatorname{li}\frac{\xi}{2}}{\varphi(s)}\Big]\Big\}^2\Big)^{1/2}.$$

By Lemma 4

(18)
$$\Big(\sum_{r\leqslant \xi} \frac{d_{k-1}^2(r)}{\varphi(r)}\Big)^{1/2} \ll (\log \xi)^{k^2/2},$$

and by Lemma 2 with $\alpha = 1/3$

(19)
$$\varphi(s)\left|\Big[\pi(\xi;\,s,\,-1)-\frac{\operatorname{li}\xi}{\varphi(s)}\Big]-\Big[\pi\Big(\frac{\xi}{2};\,s,\,-1\Big)-\frac{\operatorname{li}\frac{\xi}{2}}{\varphi(s)}\Big]\right| \ll \frac{\xi}{\log \xi}.$$

Lemma 3 gives, for any $A > 0$,

(20)
$$\sum_{s\leqslant \xi^{1/3}}\left|\Big[\pi(\xi;\,s,\,-1)-\frac{\operatorname{li}\xi}{\varphi(s)}\Big]-\Big[\pi\Big(\frac{\xi}{2};\,s,\,-1\Big)-\frac{\operatorname{li}\frac{\xi}{2}}{\varphi(s)}\Big]\right| \ll \xi\,(\log \xi)^{-A}.$$

From (17), (18), (19) and (20) we obtain

(21)
$$|R_{a,k}(\xi)| \ll (\log \xi)^{k^2/2}\Big(\frac{\xi}{\log \xi}\,\xi(\log \xi)^{-A}\Big)^{1/2} \ll \xi(\log \xi)^{-A_1}.$$

For the main term in (16) we have

(22)
$$\sum_{r\leqslant \frac{\xi^{1/3}}{2a}} \frac{f_{k-1}(r,\,\xi^{1/3})}{\varphi(ar)} \geqslant \frac{1}{a}\sum_{r\leqslant \frac{\xi^{1/3}}{2a}} \frac{f_{k-1}(r,\,\xi^{1/3})}{r} = \frac{1}{a}\sum_{r\leqslant \frac{\xi^{1/3}}{2a}}\ \sum_{\substack{x_1\ldots x_{k-1}=r\\ x_i\geqslant \xi^{2/3^{i+1}}}} \frac{1}{r}$$
$$= \frac{1}{a}\sum_{\substack{x_1\ldots x_{k-1}\leqslant \frac{\xi^{1/3}}{2a}\\ x_i\geqslant \xi^{2/3^{i+1}}}} \frac{1}{x_1\ldots x_{k-1}}.$$

Now, if
$$x_i \leqslant \xi^{\frac{2}{3^{i+1}}+\frac{2}{3^{i+k}}} \quad (i = 1, \ldots, k-1),$$
we have
$$x_1\ldots x_{k-1} \leqslant \xi^{\sum_{i=1}^{k-1}\left(\frac{2}{3^{i+1}}+\frac{2}{3^{i+k}}\right)} = \xi^{\frac{1}{3}-\frac{1}{3^{2k-1}}} \leqslant \frac{\xi^{1/3}}{2a},$$
provided $\xi \geqslant C_{a,k} = (2a)^{3^{2k-1}}$; hence if ξ is large

(23)
$$\sum_{\substack{x_1\ldots x_{k-1}\leqslant \frac{\xi^{1/3}}{2a}\\ x_i\geqslant \xi^{2/3^{i+1}}}} \frac{1}{x_1\ldots x_{k-1}} \geqslant \sum_{\xi^{2/3^{i+1}}\leqslant x_i\leqslant \xi^{2/3^{i+1}+2/3^{i+k}}} \frac{1}{x_1\ldots x_{k-1}}$$
$$= \prod_{i=1}^{k-1}\ \sum_{\xi^{2/3^{i+1}}\leqslant x_i\leqslant \xi^{2/3^{i+1}+2/3^{i+k}}} \frac{1}{x_i}$$
$$\gg \prod_{i=1}^{k-1} \log\frac{\xi^{2/3^{i+1}+2/3^{i+k}}}{\xi^{2/3^{i+1}}} = 2^{k-1}3^{-\frac{3k(k-1)}{2}}(\log \xi)^{k-1}.$$

Since
$$\operatorname{li}\xi - \operatorname{li}\frac{\xi}{2} \gg \frac{\xi}{\log \xi},$$
we obtain from (22) and (23)

(24)
$$\Big(\operatorname{li}\xi-\operatorname{li}\frac{\xi}{2}\Big)\sum_{r\leqslant \frac{\xi^{1/3}}{2a}} \frac{f_{k-1}(r,\,\xi^{1/3})}{\varphi(ar)} \gg_{a,k} \xi(\log \xi)^{k-2}.$$

(16), (21) and (24) together give at once
$$\sum_{p\leqslant \xi} \omega_{a,k}(p) \gg_{a,k} \xi(\log \xi)^{k-2},$$
which proves the theorem.

5. Proof of Theorem 1. Let $\omega_{a,k}(m)$ be, for any integer $m > 0$, the completely multiplicative function generated by $\omega_{a,k}(p)$. Then
$$\sum_{m\leqslant \sqrt{N}} \mu^2(m)\prod_{p|m}\frac{\omega_{a,k}(p)}{p-\omega_{a,k}(p)} \geqslant \sum_{m\leqslant \sqrt{N}} \mu^2(m)\prod_{p|m}\frac{\omega_{a,k}(p)}{p} = \sum_{m\leqslant \sqrt{N}} \frac{\mu^2(m)\,\omega_{a,k}(m)}{m}.$$
Hence, in view of (9), (12) and (14), Theorem 1 is proved if we show that

(25)
$$\sum_{m\leqslant \sqrt{N}} \frac{\mu^2(m)\,\omega_{a,k}(m)}{m} \gg \exp\{C(a,k)(\log N)^{1-\frac{1}{k}}\}.$$

Since $\omega_{a,k}(m) \ll_\eta m^\eta$ for any $\eta > 0$, the Dirichlet series

$$F_{a,k}(s) = \sum_{m=1}^{\infty} \frac{\mu^2(m)\,\omega_{a,k}(m)}{m^s}$$

converges for $\operatorname{Re} s > 1$. Also

$$\sum_p \left(\frac{\omega_{a,k}(p)}{p}\right)^2 < \infty;$$

hence, for any $\varepsilon > 0$,

$$(26)\qquad F_{a,k}(1+\varepsilon) = \prod_p \left(1 + \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}}\right)$$

$$= \exp\left\{\sum_p \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} + O(1)\right\} \asymp \exp \sum_p \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}}.$$

It follows from Theorem 2 that

$$\sum_p \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} \geqslant \sum_{p \leqslant X} \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} \geqslant X^{-\varepsilon} \sum_{p \leqslant X} \frac{\omega_{a,k}(p)}{p} \gg_{a,k} X^{-\varepsilon} (\log X)^{k-1};$$

taking $X = \exp\frac{1}{\varepsilon}$ we obtain

$$(27)\qquad \sum_p \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} \gg_{a,k} \varepsilon^{-(k-1)}.$$

Next

$$\sum_p \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} = \sum_{n=0}^{\infty} \sum_{2^n \leqslant p < 2^{n+1}} \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} \leqslant \sum_{n=0}^{\infty} 2^{-\varepsilon n} \sum_{2^n \leqslant p < 2^{n+1}} \frac{\omega_{a,k}(p)}{p}$$

$$= \sum_{n=0}^{\infty} (2^{-\varepsilon n} - 2^{-\varepsilon(n+1)}) \sum_{p < 2^{n+1}} \frac{\omega_{a,k}(p)}{p}$$

$$= (1 - 2^{-\varepsilon}) \sum_{n=0}^{\infty} 2^{-\varepsilon n} \sum_{p < 2^{n+1}} \frac{\omega_{a,k}(p)}{p};$$

hence by Theorem 2

$$\sum_p \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} \ll (1 - 2^{-\varepsilon}) \sum_{n=0}^{\infty} (n+1)^{k-1} 2^{-\varepsilon n}.$$

Since

$$\sum_{n=0}^{\infty} (n+1)^{k-1} 2^{-\varepsilon n} \ll_k \varepsilon^{-k},$$

we obtain

$$(28)\qquad \sum_p \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} \ll_k \varepsilon^{-(k-1)}.$$

Moreover

$$(29)\qquad \sum_{m \leqslant \sqrt{N}} \frac{\mu^2(m)\,\omega_{a,k}(m)}{m} \geqslant \sum_{m \leqslant \sqrt{N}} \frac{\mu^2(m)\,\omega_{a,k}(m)}{m^{1+\varepsilon}}$$

$$= F_{a,k}(1+\varepsilon) - \sum_{m > \sqrt{N}} \frac{\mu^2(m)\,\omega_{a,k}(m)}{m^{1+\varepsilon}},$$

$$(30)\qquad \sum_{m > \sqrt{N}} \frac{\mu^2(m)\,\omega_{a,k}(m)}{m^{1+\varepsilon}} \leqslant \sum_{m=1}^{\infty} \left(\frac{m}{\sqrt{N}}\right)^{\varepsilon/2} \frac{\mu^2(m)\,\omega_{a,k}(m)}{m^{1+\varepsilon}} = N^{-\varepsilon/4} F_{a,k}\left(1 + \frac{\varepsilon}{2}\right);$$

(29) and (30) give, for any $\varepsilon > 0$,

$$(31)\qquad \sum_{m \leqslant \sqrt{N}} \frac{\mu^2(m)\,\omega_{a,k}(m)}{m} \geqslant F_{a,k}(1+\varepsilon) - N^{-\varepsilon/4} F_{a,k}\left(1 + \frac{\varepsilon}{2}\right).$$

We deduce from (26), (27), (28) and (31) that, for any sufficiently small ε,

$$\sum_{m \leqslant \sqrt{N}} \frac{\mu^2(m)\,\omega_{a,k}(m)}{m}$$

$$\geqslant C_1 \exp\left(C_2(a, k)\,\varepsilon^{-(k-1)}\right) - C_3 N^{-\varepsilon/4} \exp\left(C_4(k)\,\varepsilon^{-(k-1)}\right)$$

$$= C_1 \exp\left(C_2(a, k)\,\varepsilon^{-(k-1)}\right) - C_3 \exp\left(-\frac{\varepsilon}{4}\log N + C_4(k)\,\varepsilon^{-(k-1)}\right).$$

Putting $\varepsilon = \{4C_4(k)\}^{1/k} \cdot (\log N)^{-1/k}$ we obtain

$$-\frac{\varepsilon}{4}\log N + C_4(k)\,\varepsilon^{-(k-1)} = 0.$$

Hence the above choice for ε gives, if N is large,

$$\sum_{m \leqslant \sqrt{N}} \frac{\mu^2(m)\,\omega_{a,k}(m)}{m} \geqslant C_1 \exp\{C(a, k)(\log N)^{1-\frac{1}{k}}\} - C_3$$

$$\gg \exp\{C(a, k)(\log N)^{1-\frac{1}{k}}\},$$

and (25) is proved.

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