We have obtained the following theorem:

THEOREM 4. *If there exists 2 positive constants δ and β independent of n such that*

$$\delta < a_{n+1}\omega(M^{n+1})/S(n, M) < \beta \tag{3.5}$$

for $n = 1, 2, \ldots$ when $(g, m) > 1$ and $S(n, M) = \sum_{i=1}^{n} a_i\omega(M^i)$ such that g contains some but not all prime factors of m, then $x(g, m)$ in Theorem 3 is a transcendental of the non-Liouville type.

One can easily see that the same boundedness condition as in [2, Th. 2, p. 247] obtains as a requirement for the transcendental non-Liouville character of $x(g, m)$ since (3.5) becomes [2, (2.46)] when $(g, m) = 1$, i.e. $M = m$.

References

[1] R. G. Stoneham, *On (j, ε)-normality in the rational fractions*, Acta Arith. 16 (1970), pp. 221–237.

[2] — *A general arithmetic construction of transcendental non-Liouville normal numbers from rational fractions*, Acta Arith. 16 (1970), pp. 239–253.

[3] E. Borel, *Les probabilités dénombrables et leurs applications arithmétiques*, Rend. Circ. Mat. Palermo, 27 (1909), pp. 247–271.

THE CITY COLLEGE
OF THE CITY UNIVERSITY OF NEW YORK

Received on 29. 10. 1971 (150)

ACTA ARITHMETICA
XXII (1973)

Non-divisibility of some multiplicative functions

by

E. J. SCOURFIELD (London)

1. Introduction. Let $f(n)$ be an integer-valued multiplicative function with the property that there exists a polynomial $W(x)$ with integral coefficients such that $f(p) = W(p)$ for all primes p. Further let $N(n \leqslant x: P)$ denote the number of positive integers $n \leqslant x$ with the property P. Our aim in this paper is to find an estimate for

$$N\big(n \leqslant x:\ d \nmid f(n)\big)$$

for any integer $d > 1$. An estimate has been obtained by Narkiewicz in the case when d is squarefree, and we shall be able to derive an explicit formula for his constant A of Theorem II of [5] (see Corollary 1 of Theorem 1 in § 5 below). From Theorem I of [5], it is also easy to deduce an estimate for $N\big(n \leqslant x:\ p^a \nmid f(n)\big)$ for any prime p and any integer $a \geqslant 1$; for

$$N\big(n \leqslant x:\ p^a \nmid f(n)\big) = \sum_{\lambda=0}^{a-1} N\big(n \leqslant x:\ p^\lambda \| f(n)\big)$$

(where the notation $p^\lambda \| f(n)$ means that $p^\lambda | f(n)$ but $p^{\lambda+1} \nmid f(n)$), and an estimate for each term on the right follows from [5]. Thus the result of this paper will be new in the cases when d is neither squarefree nor a prime power.

Let $d = \prod_{i=1}^{r} p_i^{a_i}$, where the p_i are distinct primes and each $a_i \geqslant 1$, and let $S(p, \lambda)$ denote the set $\{n:\ p^\lambda \| f(n)\}$ of positive integers. Then we can state the main result of this paper:

THEOREM 1. *Suppose that $S_i = \bigcup_{\lambda_i=0}^{a_i-1} S(p_i, \lambda_i) \neq \emptyset$ (the empty set) for $i = 1, 2, \ldots, r$. Then there exist constants B, β, m (dependent on f and d) with $B > 0$, $0 \leqslant \beta \leqslant 1$, and $m \geqslant 0$, where β, m are defined explicitly by* (31) *and* (32), *such that as $x \to \infty$,*

(i) *if* $0 < \beta < 1$,

$$N\big(n \leqslant x:\ d \nmid f(n)\big) \sim Bx(\log\log x)^m(\log x)^{\beta-1};$$

(ii) *if* $\beta = 1$,

$$N\big(n \leqslant x:\ d \nmid f(n)\big) \sim Bx, \ \textit{where}\ B \leqslant 1;$$

(iii) *if* $\beta = 0$, $m > 0$,

$$N(n \leqslant x:\ d \nmid f(n)) \sim Bx (\log\log x)^{m-1} (\log x)^{-1};$$

(iv) *if* $\beta = 0 = m$,

$$N(n \leqslant x:\ d \nmid f(n)) = O(x^{1/2}).$$

In proving this result (in §§ 4 and 5), we shall use an indirect argument and apply Narkiewicz's results from [5] and [6], which we shall state in Theorem 2. From these results, we shall also be able to deduce fairly easily an asymptotic formula for

$$N(n \leqslant x:\ p_i^{a_i} \nmid f(n) \text{ for } i = 1, 2, \ldots, r)$$

(see Theorem 3), a quantity that clearly does not exceed $N(n \leqslant x: d \nmid f(n))$.

In his papers [5] and [6], Narkiewicz has studied extensively the problem of estimating $N(n \leqslant x:\ d \,\|\, f(n))$, where $d \,\|\, f(n)$ means that d is a unitary divisor of $f(n)$, so that $d \mid f(n)$ but $(d, f(n)/d) = 1$. Thus, in addition to raising the question considered in Theorem 1 above, it is a natural step to ask next whether one can obtain an asymptotic formula for

$$N(n \leqslant x:\ d \mid f(n)),$$

a problem considered further in Theorem 4. The quantity $N(n \leqslant x: d \mid f(n))$ is obviously related to $N(n \leqslant x:\ d \nmid f(n))$ by the formula

$$(1) \qquad N(n \leqslant x:\ d \mid f(n)) + N(n \leqslant x:\ d \nmid f(n)) = [x].$$

Narkiewicz's paper [5] sought to generalize a result [9] of the present author in which an estimate for

$$N(n \leqslant x:\ p^a \,\|\, f(n)) \qquad (a \geqslant 1,\ p \text{ prime})$$

was obtained in the special cases when $f(n)$ is Euler's function $\varphi(n)$ or one of the divisor functions defined by

$$\tau(n) = \sum_{d \mid n} 1, \qquad \sigma_\nu(n) = \sum_{d \mid n} d^\nu \qquad (\nu \text{ a positive integer}).$$

Previously, in [7], Rankin had considered the analogous problem of estimating $N(n \leqslant x:\ p \nmid \sigma_\nu(n))$. It was shown in [7] and [9] that in some cases an improvement could be obtained in a result of G. N. Watson [11] that gave the bound

$$(2) \qquad N(n \leqslant x:\ d \nmid \sigma_\nu(n)) = O\big(x (\log x)^{-1/\varphi(d)}\big)$$

when ν is odd. We shall see in § 7 that, by appealing to Theorem 1 of this paper, we can replace (2) by an asymptotic formula in all the remaining cases, and in addition we shall consider other applications.

In § 8, the last section of this paper, we shall obtain for most χ the result analogous to Theorem 1 for the generalized divisor function $\sigma_\nu(n, \chi)$ defined by

$$\sigma_\nu(n, \chi) = \sum_{d \mid n} \chi(d) d^\nu,$$

where ν is a positive integer, χ is a real non-principal character (mod Q') for some fixed $Q' > 1$. We considered the same function in [10], when we extended Narkiewicz's result [5] to cover a slightly more general class of functions that included $\sigma_\nu(n, \chi)$, and deduced an estimate for $N(n \leqslant x:\ p^a \,\|\, \sigma_\nu(n, \chi))$.

2. Some notation and preliminary results. As above, write $d = \prod_{i=1}^{r} p_i^{a_i}$, where $a_i \geqslant 1$ for each i and the p_i are distinct primes, and let

$$D = p_1 p_2 \ldots p_r, \qquad A = (a_1, a_2, \ldots, a_r),$$

and more generally let

$$\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are non-negative integers; in the following pages, λ_i will satisfy $0 \leqslant \lambda_i < a_i$ for $i = 1, 2, \ldots, r$. In particular, we shall denote the ordered r-tuples $(1, 1, \ldots, 1)$, $(0, 0, \ldots, 0)$ by I, O respectively.

If d' is a unitary divisor of d ($d' \,\|\, d$ in the above notation), so that

$$d' = \prod_{j=1}^{l} p_{i_j}^{a_{i_j}},$$

where the primes $p_{i_1}, p_{i_2}, \ldots, p_{i_l}$ (in this order) form a subsequence of the finite sequence $p_1, p_2, \ldots, p_r$, then let

$$D' = p_{i_1} p_{i_2} \ldots p_{i_l}, \qquad A' = (a_{i_1}, a_{i_2}, \ldots, a_{i_l}), \qquad \Lambda' = (\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_l}).$$

We shall use the notation $\Lambda' < A'$ to mean that $0 \leqslant \lambda_{i_j} < a_{i_j}$ for $j = 1, 2, \ldots, l$, so that $\sum_{\Lambda' < A'}$ stands for the sum over all such ordered sets Λ', with a similar meaning for $\sum_{\Lambda < A}$. If $d' = p_i^{a_i}$ for some i $(1 \leqslant i \leqslant r)$, then $A' = (a_i)$, $\Lambda' = (\lambda_i)$ and for convenience we shall denote these throughout by a_i, λ_i respectively.

Let

$$(3) \qquad S(D', \Lambda') = \{n:\ p_i^{\lambda_i} \| f(n) \text{ for each } p_i \mid D'\}.$$

Then in particular

$$S(D, A) = \{n:\ d \,\|\, f(n)\}, \qquad S(D, O) = \{n:\ (d, f(n)) = 1\},$$

and, as in the statement of Theorem 1,

$$S(p_i, \lambda_i) = \{n:\ p_i^{\lambda_i} \| f(n)\}.$$

Define

$$a(D, \Lambda;\, n) = \begin{cases} 1 & \text{if } n \in S(D, \Lambda), \\ 0 & \text{otherwise} \end{cases}$$

(so that $a(D, \Lambda; n) = 1$ if and only if $d \,\|\, f(n)$);

$$b(D, \Lambda; n) = \begin{cases} 1 & \text{if } p_i^{\lambda_i} \nmid f(n) \text{ for } i = 1, 2, \ldots, r, \\ 0 & \text{otherwise} \end{cases}$$

(defined provided $\Lambda \geqslant I$, where this notation means that $\lambda_i \geqslant 1$ for each i);

$$c(D, \Lambda; n) = \begin{cases} 1 & \text{if } \prod_{i=1}^{r} p_i^{\lambda_i} \nmid f(n), \\ 0 & \text{otherwise} \end{cases}$$

(defined provided that $\Lambda \neq O$). Since f is multiplicative, $f(1) = 1$ and hence clearly

$$a(D, O; 1) = 1, \quad a(D, \Lambda; 1) = 0 \quad \text{if } \Lambda \neq O,$$
$$b(D, \Lambda; 1) = 1, \quad c(D, \Lambda; 1) = 1 \quad \text{for each } \Lambda.$$

We observe also that $a(D, O; n) = 1$ if and only if $\big(D, f(n)\big) = 1$.

LEMMA 1. *For each* $n \geqslant 1$,

(i) $$b(D, A; n) = \sum_{\Lambda < A} a(D, \Lambda; n);$$

(ii) $$c(D, A; n) = -\sum_{\substack{D' \mid D \\ D' \neq 1}} \mu(D') b(D', A'; n),$$

where μ is the Möbius function.

Proof. Since $p^a \nmid f(n)$ if and only if one of

$$p \nmid f(n), \quad p \,\|\, f(n), \quad \ldots, \quad p^{a-1} \,\|\, f(n)$$

holds, (i) follows immediately.

Now consider (ii). From the definition, we observe that

$$1 - c(D, A;\, n) = \begin{cases} 1 & \text{if } d \mid f(n), \\ 0 & \text{if } d \nmid f(n). \end{cases}$$

Clearly $d \mid f(n)$ if and only if $p_i^{a_i} \mid f(n)$ for $i = 1, 2, \ldots, r$, that is if and only if $1 - c(p_i, a_i; n) = 1$ for $i = 1, 2, \ldots, r$. Hence

$$(4) \quad c(D, A; n) = 1 - \prod_{i=1}^{r} \big(1 - c(p_i, a_i; n)\big) = -\sum_{\substack{D' \mid D \\ D' \neq 1}} \mu(D') \prod_{p_i \mid D'} c(p_i, a_i; n).$$

Moreover $\prod_{p_i \mid D'} c(p_i, a_i; n) = 1$ if and only if $p_i^{a_i} \nmid f(n)$ for each $p_i \mid D'$, and hence, by definition, if and only if $b(D', A'; n) = 1$. Thus

$$\prod_{p_i \mid D'} c(p_i, a_i; n) = b(D', A'; n),$$

and hence the result of (ii) follows from (4).

COROLLARY 1.

$$b(D, I; n) = a(D, O; n).$$

COROLLARY 2.

$$c(D, A; n) = -\sum_{\substack{D' \mid D \\ D' \neq 1}} \mu(D') \sum_{\Lambda' < A'} a(D', \Lambda'; n).$$

Let

$$\mathcal{N}(D', \Lambda';\, x) = \sum_{n \leqslant x} a(D', \Lambda'; n) = N\big(n \leqslant x\colon\, p_i^{\lambda_i} \| f(n) \text{ for each } p_i \mid D'\big)$$

(so that, clearly, $\mathcal{N}(D', \Lambda'; x) = 0$ for all x if and only if $S(D', \Lambda') = \emptyset$); then we have

COROLLARY 3.

$$\sum_{n \leqslant x} b(D, A;\, n) = N\big(n \leqslant x\colon\, p_i^{a_i} \nmid f(n) \text{ for } i = 1, 2, \ldots, r\big)$$
$$= \sum_{\Lambda < A} \mathcal{N}(D, \Lambda; x).$$

COROLLARY 4.

$$\sum_{n \leqslant x} c(D, A; n) = N\big(n \leqslant x\colon\, d \nmid f(n)\big)$$
$$= -\sum_{\substack{D' \mid D \\ D' \neq 1}} \mu(D') \sum_{\Lambda' < A'} \mathcal{N}(D', \Lambda'; x).$$

These Corollaries follow immediately from the Lemma. Since an estimate for $\mathcal{N}(D, A; x)$ is given in all cases by Narkiewicz in [5] and [6], it follows that we can formally deduce an expression for the sums of

Corollaries 3 and 4. However further discussion (given in §§ 4 and 5) is needed before we can obtain the precise order of magnitude of the dominant terms in this expression. Since in general the sign of $\mu(D')$ varies with D', it is clear that in considering the sum of Corollary 4, we have the added difficulty of determining whether or not the terms $\mathcal{N}(D', \Lambda'; x)$ with the greatest order of magnitude cancel each other out.

The following Lemmas will be needed later on in the proof.

LEMMA 2. *If k is an integer > 1, then as $x \to \infty$,*

(i) $$\sum_{\substack{n \leqslant x \\ (n,k)=1}} 1 \sim \frac{\varphi(k)}{k} x;$$

(ii) $$\sum_{\substack{n \leqslant x \\ (n,k)=1}} |\mu(n)| \sim 6\pi^{-2} \prod_{p|k} \frac{p}{p+1} x.$$

These results follow, for example, from Lemmas 3.4 and 5.2 of [1], or can be deduced directly from well known results.

Let h, k, $\varkappa$ denote positive integers with $k > 1$ and $(h, k) = 1$, and let $\pi_\varkappa(h, k; x)$ denote the number of squarefree positive integers $n \leqslant x$ of the form

(5) $$n = q_1 q_2 \ldots q_\varkappa, \quad \text{where} \quad q_i \equiv h \pmod k \quad (i = 1, 2, \ldots, \varkappa),$$

and where $q_1, \ldots, q_\varkappa$ are primes; thus n has exactly $\varkappa$ prime factors that are all different and all lie in the same congruence class as $h \pmod k$.

LEMMA 3. *If $\varkappa \geqslant 1$, then as $x \to \infty$,*

$$\pi_\varkappa(h, k; x) \sim \big((\varkappa-1)!\big)^{-1} \big(\varphi(k)\big)^{-\varkappa} x (\log\log x)^{\varkappa-1} (\log x)^{-1}.$$

This is a special case of a result of Delange (Théorème 28 of [2]). The particular case $h = k = 1$ is also established, for example, in Theorem 437 of [4].

3. A discussion of Narkiewicz's results. First of all we set up the notation needed in order to state Narkiewicz's results, and then we investigate further some of the constants that appear. Where possible, we shall use the same notation as Narkiewicz. We recall that r, D, Λ were defined in § 2, and that $S(D, \Lambda)$ is given by (3).

We define the constant $M(D, \Lambda)$ whenever $S(D, \Lambda) \neq \emptyset$. If $S(D, O) \neq \emptyset$, let $M(D, O) = 0$. However when $\Lambda \neq O$, we need some rather cumbersome notation in order to define $M(D, \Lambda)$; (as Narkiewicz explains, this complication arises because $a(D, \Lambda; n)$ is not in general a multiplicative function of n). Let $R(D, \Lambda)$ $(\Lambda \neq O)$ denote the set of matrices $T = (t_{ij})$ with r rows and an arbitrary number m of columns $T_1, \ldots, T_m$

whose entries are non-negative integers that satisfy the following conditions:

(6)
(i) none of the columns $T_1, \ldots, T_m$ is the column O (consisting of zeros);
(ii) $\sum_{j=1}^{m} t_{ij} = \lambda_i$ for $i = 1, 2, \ldots, r$;
(iii) the columns of T always occur in a "lexicographical" order, obtained by assuming that identical columns are adjacent to each other, and that otherwise for each pair i, j with $i \neq j$, T_i comes before T_j when one of the following holds:
$$t_{1i} < t_{1j}; \quad t_{1i} = t_{1j} \text{ and } t_{2i} < t_{2j}; \quad \ldots;$$
$$t_{vi} = t_{vj} \text{ for } v = 1, 2, \ldots, r-1 \text{ and } t_{ri} < t_{rj}.$$

For any column $T_k = (t_{ik})$ of r elements, let $u_k = \prod_{i=1}^{r} p_i^{t_{ik}+1}$, and let $N(T_k)$ denote the number of integers x that satisfy

$$1 \leqslant x \leqslant u_k, \quad (x, u_k) = 1 \quad \text{and} \quad \prod_{i=1}^{r} p_i^{t_{ik}} \| W(x).$$

Thus, in particular, if D is a prime p, so that $r = 1$, T_k has one element t (say), $u_k = p^{t+1}$ and $N(T_k)$ $(= N(t)$, say) is the number of integers x satisfying

$$1 \leqslant x \leqslant p^{t+1}, \quad p \nmid x \quad \text{and} \quad p^t \| W(x).$$

In this case we define

(7) $$\tau = \inf_{t \geqslant 1} \{t: N(t) > 0\}$$

if the set is non-empty, so that $p \mid W(x)$ for at least one x coprime to p; then clearly if $p \nmid x$, either $p \nmid W(x)$ or $p^\tau | W(x)$, and moreover $p^\tau \| W(x)$ for at least one x coprime to p.

Returning to the general case when D has r factors, let $\Theta(D)$ denote the set of all possible columns $T_k \neq O$ for which $N(T_k) > 0$. For any column $T_k = (t_{ik})$ of r elements (regarded here as an r-tuple) and any prime q, let

$$H(D, T_k; q; s) = \Big\{\sum_{j=1}^{\infty} a(D, T_k; q^j) q^{-js}\Big\} \Big\{\sum_{j=0}^{\infty} a(D, O; q^j) q^{-js}\Big\}^{-1}$$

and for any $T \in R(D, \Lambda)$ with columns T_k, let

(8) $$A(T; s) = \sum\nolimits^{*} \prod_{j=1}^{\varrho} H(D, T_{i_j}; q_j; s) \quad (\operatorname{Re} s \geqslant 1),$$

where $T_{i_1}, T_{i_2}, \ldots, T_{i_e}$ denote all those columns of T that do not lie in $\Theta(D)$ and where $\sum^*$ denotes the sum over all ordered sets of distinct primes $(q_1, q_2, \ldots, q_e)$, and let $A(T, s) = 1$ if the sum is empty (so that $T_k \in \Theta(D)$ for every column T_k of T). Denote by $R_1(D, A)$ the set of those matrices $T \in R(D, A)$ for which $A(T, 1) \neq 0$. If $R_1(D, A) \neq \emptyset$, let $M(D, A)$ denote the greatest possible number of columns $T_k \in \Theta(D)$ of any $T \in R_1(D, A)$, so that $M(D, A) \geqslant 0$, and $M(D, A) = 0$ if $\Theta(D) = \emptyset$. Finally let $R_0(D, A)$ denote the set of those matrices $T \in R_1(D, A)$ that have exactly $M(D, A)$ columns $T_k \in \Theta(D)$; clearly $R_0(D, A) \subseteq R_1(D, A) \subseteq R(D, A)$, and $R_0(D, A) = \emptyset$ implies $R_1(D, A) = \emptyset$, so that $R_0(D, A) = \emptyset$ if and only if $A(T, 1) = 0$ for all $T \in R(D, A)$. This completes the definition of $M(D, A)$ when $A \neq O$.

Let $X(d)$ denote the number of integers x satisfying $1 \leqslant x \leqslant d$ and $(xW(x), d) = 1$, and let

$$\alpha(d) = X(d)/\varphi(d). \tag{9}$$

We are now in a position to state the main results of Narkiewicz's papers [5] and [6]; these give an estimate for

$$\mathcal{N}(D, A; x) = \sum_{n \leqslant x} a(D, A; n)$$

in the cases D prime, D composite, respectively, for any $A \geqslant O$.

THEOREM 2.

(i) *If* $S(D, A) = \emptyset$, $\mathcal{N}(D, A; x) = 0$ *for all* x, *and if* $A \neq O$, $S(D, A) = \emptyset$ *if and only if* $R_0(D, A) = \emptyset$.

Suppose that $S(D, A) \neq \emptyset$; *then as* $x \to \infty$,

(ii) *if* $\alpha(d) \neq 0$,

$$\mathcal{N}(D, A; x) \sim C_1(D, A)x(\log\log x)^{M(D,A)}(\log x)^{\alpha(d)-1},$$

where $C_1(D, A) > 0$;

(iii) *if* $\alpha(d) = 0$, $M(D, A) \neq 0$,

$$\mathcal{N}(D, A; x) \sim C_2(D, A)x(\log\log x)^{M(D,A)-1}(\log x)^{-1},$$

where $C_2(D, A) > 0$;

(iv) *if* $\alpha(d) = 0 = M(D, A)$, *there exists* $\varepsilon(d) > 0$ *such that for all sufficiently large* x,

$$0 < \mathcal{N}(D, A; x) = O(x^{1-\varepsilon(d)}).$$

COROLLARY. *If* $S(D, A) \neq \emptyset$ *and* $\alpha(d) = 1$, *then* $M(D, A) = 0$ *and*

$$\mathcal{N}(D, A; x) \sim C_1(D, A)x, \quad \textit{where} \quad 0 < C_1(D, A) \leqslant 1.$$

This Corollary is an immediate consequence of case (ii) of the Theorem; for clearly $\mathcal{N}(D, A; x) \leqslant x$, so $C_1(D, A) > 1$ is impossible. Moreover if $\alpha(d) = 1$, $(W(x), d) = 1$ whenever $(x, d) = 1$, and so $N(T_k) = 0$ for any $T_k \neq O$, whence $\Theta(D) = \emptyset$ and $M(D, A) = 0$.

Our next objective is to obtain for later use further information about some of the quantities in Theorem 2. First we observe that if $S(D, A) \neq \emptyset$, then there exists n_0 such that $d \| f(n_0)$ and it follows that $S(p_i, a_i) \neq \emptyset$ for $i = 1, 2, \ldots, r$. Thus we have

LEMMA 4. *If* $S(D, A) \neq \emptyset$, *then* $S(p_i, a_i) \neq \emptyset$ *for* $i = 1, 2, \ldots, r$. *Hence if* $S(p_s, a_s) = \emptyset$ *for one prime* $p_s | D$, *then* $S(D', A') = \emptyset$ *for each* $D' | D$ *for which* $p_s | D'$.

(However it does not follow that the converse of this result holds, for it might happen that $S(D, A) = \emptyset$ whilst $S(p_i, a_i) \neq \emptyset$ for $i = 1, 2, \ldots, r$.)

LEMMA 5. $\alpha(d)$ *is multiplicative, and*

$$\alpha(d) = \alpha(D) = \prod_{i=1}^{r} \left(1 - \frac{\theta(p_i)}{p_i - 1}\right),$$

where for any prime p, $\theta(p)$ *is the number of integers* x *satisfying* $1 \leqslant x \leqslant p-1$ *and* $p \mid W(x)$.

Proof. $\alpha(d)$ is defined by (9). It is an exercise in elementary number theory to prove that, since $xW(x)$ is a polynomial in x with integer coefficients, $X(d)$ is multiplicative and

$$X(p^a) = p^{a-1}X(p), \quad X(p) = p - (\theta(p) + 1).$$

The result now follows from the well known properties of $\varphi(d)$. We observe that $\alpha(d)$ depends on D but not on the r-tuple A.

COROLLARY. *For all* d, $0 \leqslant \alpha(d) \leqslant 1$ *and*

(i) $\alpha(d) = 1$ *if and only if* $\theta(p) = 0$ *for all primes* $p \mid d$, *so that* $(W(x), d) = 1$ *whenever* $(x, d) = 1$;

(ii) $\alpha(d) = 0$ *if and only if* $\theta(p) = p - 1$ *for some prime* $p \mid d$;

(iii) $\max\limits_{\substack{D' \mid D \\ D' \neq 1}} \alpha(D') = \max\limits_{p \mid D} \alpha(p) = \max\limits_{p \mid D} \left(1 - \dfrac{\theta(p)}{p-1}\right)$.

This is an immediate consequence of the lemma; the last part follows from the fact that if $p \nmid D'$,

$$0 \leqslant \alpha(pD') = \alpha(p)\alpha(D') \leqslant \max(\alpha(p), \alpha(D')) \leqslant 1.$$

We recall that $M(D, A)$ was defined before the statement of Theorem 2, and that $M(D, A)$ is not defined unless $S(D, A) \neq \emptyset$.

LEMMA 6. *For each $p_i|D$ with $\alpha(p_i)\neq 1$, define τ_i by (7), and let $\varkappa_i = [(a_i-1)/\tau_i]$, and put $\varkappa_i = 0$ when $\alpha(p_i) = 1$. Suppose that*

$$S = \bigcup_{\Lambda<A} S(D, \Lambda) \neq \emptyset$$

and let

$$l(d) = \max_{\substack{\Lambda<A\\ S(D,\Lambda)\neq\emptyset}} M(D, \Lambda).$$

Then

(i) *if* $\alpha(d) = 1$, $l(d) = 0$;

(ii) *if* $0<\alpha(d)<1$, $l(d) = \sum_{i=1}^{r} \varkappa_i$;

(iii) *if* $\alpha(d) = 0$, $l(d) = \min_{\substack{1\leqslant i\leqslant r\\ \alpha(p_i)=0}} \varkappa_i$.

Proof. (i) It follows from the Corollary to Theorem 2 that if $\alpha(d) = 1$ then $M(D,\Lambda) = 0$ for all $\Lambda < A$ such that $S(D,\Lambda)\neq\emptyset$, and hence $l(d) = 0$.

(ii) Since $0<\alpha(d)<1$, it follows from the Corollary to Lemma 5 that $\alpha(p)>0$ for all $p|d$ and that $\alpha(p)<1$ for at least one prime $p|d$, so suppose without loss of generality that $\alpha(p_i)<1$ for $1\leqslant i\leqslant j$ and $\alpha(p_i) = 1$ for $j<i\leqslant r$; then τ_i is defined for $1\leqslant i\leqslant j$. If $T_k = (t_{ik}) \neq O$ is a column such that

$$t_{ik} = \tau_i \text{ or } 0 \quad \text{for } 1\leqslant i\leqslant j, \qquad t_{ik} = 0 \quad \text{for } j<i\leqslant r,$$

then $N(T_k)>0$; for we can choose x_i for $i = 1, 2, \ldots, r$ so that $p_i \nmid x_i$ and $p_i^{t_{ik}} \| W(x_i)$, and then if x_0 is a solution of the simultaneous congruences $x\equiv x_i \pmod{p_i^{t_{ik}+1}}$ $(i = 1, 2, \ldots, r)$, it is clear that x_0 contributes 1 to $N(T_k)$. If, however, $0<t_{ik}<\tau_i$ for some $i\leqslant j$ or $t_{ik}>0$ for some $i>j$, then for that value of i, $p_i^{t_{ik}}\| W(x)$ for no x coprime to p_i and so $N(T_k) = 0$.

Assume first that $\varkappa = \sum_{i=1}^{j}\varkappa_i > 0$, so that at least one $\varkappa_i$ is non-zero. Let T^* be the $r\times\varkappa$ matrix defined by $T^* = (t_{ik}^*)$, where, if $1\leqslant i\leqslant j$, the ith row of T^* has $\varkappa_i$ elements equal to τ_i and the rest equal to zero, and, if $j<i\leqslant r$, the ith row of T^* consists entirely of zeros, where each column of T^* has exactly one non-zero element, and where the columns of T^* are in a lexicographical order as described in (6 (iii)); thus

$$t_{ik}^* = \begin{cases} \tau_i & \text{if } \varkappa_i\neq 0,\ \varkappa-\varkappa_1-\ldots-\varkappa_{i-1}-\varkappa_i < k \leqslant \varkappa-\varkappa_1-\ldots-\varkappa_{i-1} \\ & \qquad\qquad \text{and} \quad 1\leqslant i\leqslant j, \\ 0 & \text{otherwise.}\end{cases}$$

By the remarks in the previous paragraph, $N(T_k^*)>0$ for each column T_k^* of T^*, and so $T_k^*\in\Theta(D)$ for $k = 1, 2, \ldots, \varkappa$. Clearly (see the sentence containing (8)) $A(T^*, 1) = 1$, and $T^*\in R_1(D, \Lambda^*)$ where $\Lambda^* = (\varkappa_1\tau_1, \varkappa_2\tau_2, \ldots, \varkappa_j\tau_j, 0, \ldots, 0)$, so that Λ^* is an r-tuple satisfying $\Lambda^*\neq O$ and $\Lambda^*<A$. It follows that $R_0(D,\Lambda^*)\neq\emptyset$ (whence $S(D,\Lambda^*)\neq\emptyset$ by Theorem 2(i)), and that $l(d)\geqslant M(D,\Lambda^*)\geqslant\varkappa$.

To complete the proof in the case $\varkappa\neq 0$, we have to show that $M(D,\Lambda)\leqslant\varkappa$ whenever $O\neq\Lambda<A$ and $R_1(D,\Lambda)\neq\emptyset$; (we recall that $M(D,O) = 0$ if $S(D,O)\neq\emptyset$). It follows from the last sentence in the first paragraph that if T has at least $\varkappa+1$ columns in $\Theta(D)$, then for at least one $i\leqslant j$, the sum of the elements in the ith row is $\geqslant(\varkappa_i+1)\tau_i\geqslant a_i$, so that $T\notin\bigcup_{O\neq\Lambda<A} R(D,\Lambda)$. Hence

$$l(d) = \max_{\substack{\Lambda<A\\ S(D,\Lambda)\neq\emptyset}} M(D,\Lambda)\leqslant\varkappa,$$

whence $l(d) = \varkappa$.

If $\varkappa = 0$, it follows similarly that if $T\in\bigcup_{O\neq\Lambda<A} R_1(D,\Lambda)$, then no column of T belongs to $\Theta(D)$, and hence $l(d) = 0$. This completes the proof of (ii).

(iii) Since $\alpha(d) = 0$, $\alpha(p) = 0$ for at least one prime $p|d$; suppose without loss of generality that $\alpha(p_i) = 0$ for $1\leqslant i\leqslant j$ and that $\alpha(p_i)>0$ for $j<i\leqslant r$. By Lemma 5, if $\alpha(p_i) = 0$, then $p_i | W(x)$ whenever $p_i\nmid x$. Hence by the remarks in the first paragraph of (ii), if T_k is the column (t_{ik}) and if $N(T_k)>0$, then

$$t_{ik}\begin{cases}\geqslant\tau_i & \text{for} \quad 1\leqslant i\leqslant j,\\ = 0 & \text{if} \quad j<i\leqslant r \text{ and } \alpha(p_i) = 1,\\ = 0 \text{ or } \geqslant\tau_i & \text{if} \quad j<i\leqslant r \text{ and } \alpha(p_i)<1.\end{cases}$$

The proof is now similar to that of (ii). Let $\varkappa_1 = \min_{1\leqslant i\leqslant j}\varkappa_i$. If $\varkappa_1>0$, let T^* be the $r\times\varkappa_1$ matrix defined by

$$T^* = \begin{pmatrix}\tau_1 & \ldots & \tau_1\\ \tau_2 & \ldots & \tau_2\\ \cdot & \cdot\cdot\cdot & \cdot\\ \tau_j & \ldots & \tau_j\\ 0 & \ldots & 0\\ \cdot & \cdot\cdot\cdot & \cdot\\ 0 & \ldots & 0\end{pmatrix}.$$

Then each column of T^* belongs to $\Theta(D)$, so $A(T^*, 1) = 1$, and $T^*\in R_1(D,\Lambda^*)$, where $\Lambda^* = (\varkappa_1\tau_1, \ldots, \varkappa_1\tau_j, 0, \ldots, 0)$, an r-tuple such that $O\neq\Lambda^*<A$, and $R_0(D,\Lambda^*)\neq\emptyset$. Hence $l(d)\geqslant M(D,\Lambda^*)\geqslant\varkappa_1$. If T has more than $\varkappa_1$ columns in $\Theta(D)$, then the sum of the elements in the first row of T is $\geqslant(\varkappa_1+1)\tau_1\geqslant a_1$ and so $T\notin\bigcup_{O\neq\Lambda<A} R(D,\Lambda)$. Thus $l(d)\leqslant\varkappa_1$,

whence $l(d) = \varkappa_1$. If $\varkappa_1 = 0$, no column of T belongs to $\Theta(D)$ when $T \in \bigcup_{0 \neq \Lambda < \Delta} R_1(D, \Lambda)$, and hence $l(d) = 0$. The result of (iii) now follows.

4. First stage of the proof of the main results. Our aim is to obtain estimates for the sums

$$\sum_{n \leqslant x} b(D, \Delta; n), \qquad \sum_{n \leqslant x} c(D, \Delta; n)$$

from Corollaries 3 and 4 of Lemma 1, Theorem 2 and our other Lemmas. First of all, we observe that we can assume that for each prime $p_i | D$,

$$S_i = \bigcup_{\lambda_i=0}^{a_i-1} S(p_i, \lambda_i) \neq \emptyset \quad (\text{where } p_i^{a_i} \| d). \tag{10}$$

For if not, we can write $D = D_1 D_2$ where $D_1 \geqslant 1$, $D_2 > 1$, and $S_i \neq \emptyset$ for each $p_i | D_1$, $S_i = \emptyset$ for each $p_i | D_2$. Then by Lemma 4 and Theorem 2 (i), $\mathcal{N}(D', \Lambda'; x) = 0$ whenever $D' | D$ but $D' \nmid D_1$, and $\Lambda' < \Delta'$. Hence from Corollaries 3 and 4 of Lemma 1,

$$\sum_{n \leqslant x} b(D, \Delta; n) = 0 \quad \text{whenever } D_2 > 1,$$

$$\sum_{n \leqslant x} c(D, \Delta; n) = -\sum_{\substack{D' | D_1 \\ D' \neq 1}} \mu(D') \sum_{\Lambda' < \Delta'} \mathcal{N}(D', \Lambda'; x) = \begin{cases} \sum_{n \leqslant x} c(D_1, \Delta_1; n) & \text{if } D_1 > 1, \\ 0 & \text{if } D_1 = 1. \end{cases} \tag{11}$$

Thus we can assume from now on that $S_i \neq \emptyset$ for $i = 1, 2, \ldots, r$.

From Theorem 2 we can deduce immediately an estimate for the sum

$$\mathcal{L}(D, \Delta; x) = \sum_{n \leqslant x} b(D, \Delta; n) = \sum_{\Lambda < \Delta} \mathcal{N}(D, \Lambda; x)$$

by Corollary 3 of Lemma 1. In the Theorem below, $\alpha(d)$ and $l(d)$ are the numbers given by Lemmas 5 and 6.

THEOREM 3. (i) *If $S(D, \Lambda) = \emptyset$ for every $\Lambda < \Delta$ (including $\Lambda = 0$), then*

$$\mathcal{L}(D, \Delta; x) = 0 \quad \textit{for all } x.$$

Suppose that $\bigcup_{\Lambda < \Delta} S(D, \Lambda) \neq \emptyset$; then as $x \to \infty$,

(ii) *if $\alpha(d) \neq 0$,*

$$\mathcal{L}(D, \Delta; x) \sim E_1(D, \Delta)\, x (\log\log x)^{l(d)} (\log x)^{\alpha(d)-1},$$

where

$$E_1(D, \Delta) = \sum_{\substack{\Lambda < \Delta \\ M(D,\Lambda) = l(d)}} C_1(D, \Lambda) > 0;$$

(iii) *if $\alpha(d) = 0$, $l(d) \neq 0$,*

$$\mathcal{L}(D, \Delta; x) \sim E_2(D, \Delta)\, x (\log\log x)^{l(d)-1} (\log x)^{-1},$$

where

$$E_2(D, \Delta) = \sum_{\substack{\Lambda < \Delta \\ M(D,\Lambda) = l(d)}} C_2(D, \Lambda) > 0;$$

(iv) *if $\alpha(d) = 0 = l(d)$, there exists $\varepsilon(d) > 0$ such that for all sufficiently large x,*

$$0 < \mathcal{L}(D, \Delta; x) = O(x^{1-\varepsilon(d)}).$$

The sums for $E_j(D, \Delta)$ $(j = 1, 2)$ are non-empty by our assumptions and by Theorem 2 each term is positive, and hence it follows that $E_j(D, \Delta) > 0$ for $j = 1, 2$.

We turn now to the rather more difficult problem of estimating

$$\mathcal{M}(D, \Delta; x) = \sum_{n \leqslant x} c(D, \Delta; n) = -\sum_{\substack{D' | D \\ D' \neq 1}} \mu(D') \sum_{\substack{\Lambda' < \Delta' \\ S(D', \Lambda') \neq \emptyset}} \mathcal{N}(D', \Lambda'; x) \tag{12}$$

by Corollary 4 of Lemma 1 and Theorem 2 (i); since we are assuming that $S_i \neq \emptyset$ (see (10)) for each $p_i | D$, it follows from Theorem 2 that the inner sum of (12) is certainly non-zero when $D' = p_i$ $(i = 1, 2, \ldots, r)$. Our next objective is to determine the terms $\mathcal{N}(D', \Lambda'; x)$ on the right side of (12) with the greatest order of magnitude. We then have to show that when we combine these terms according to (12), we obtain an expression of the same (and not of a smaller) order of magnitude as these individual terms; this we shall accomplish in § 5.

Define

$$\beta(D) = \max_{\substack{D' | D \\ D' \neq 1}} \alpha(D'), \qquad m(d) = \max_{\substack{D' | D, \Lambda' < \Delta' \\ \alpha(D') = \beta(D)}} M(D', \Lambda'); \tag{13}$$

(we recall that $M(D', \Lambda')$ is not defined unless $S(D', \Lambda') \neq \emptyset$). By the Corollary to Lemma 5,

$$0 \leqslant \beta(D) \leqslant 1 \quad \text{and} \quad \beta(D) = \max_{1 \leqslant i \leqslant r} \alpha(p_i), \tag{14}$$

and by Lemma 6,

$$m(d) = \max_{d^*} l(d^*) \tag{15}$$

where d^* denotes a unitary divisor of d satisfying $\alpha(d^*) = \beta(D)$ and

$\bigcup_{A^*<A^*} S(D^*, A^*) \neq \emptyset$ (with an obvious notation). From Theorem 2 and equations (12) to (15), it follows that

(i) if $\beta(D) \neq 0$,

$$(16)\qquad \mathscr{M}(D, A;\ x) = \{B_1(D, A)+o(1)\}x(\log\log x)^{m(d)}(\log x)^{\beta(D)-1}$$

where

$$B_1(D, A) = -\sum_{\substack{D'|D,\, D'\neq 1\\ \alpha(D')=\beta(D)}} \mu(D') \sum_{\substack{A'<A'\\ M(D',A')=m(d)}} C_1(D', A');$$

(ii) if $\beta(D) = 0$, $m(d) \neq 0$,

$$(17)\qquad \mathscr{M}(D, A;\ x) = \{B_2(D, A)+o(1)\}x(\log\log x)^{m(d)-1}(\log x)^{-1}$$

where

$$B_2(D, A) = -\sum_{\substack{D'|D\\ D'\neq 1}} \mu(D') \sum_{\substack{A'<A'\\ M(D',A')=m(d)}} C_2(D', A');$$

(iii) if $\beta(D) = 0 = m(d)$, then there exists $\varepsilon(d) > 0$ such that

$$(18)\qquad \mathscr{M}(D, A;\ x) = O(x^{1-\varepsilon(d)}).$$

Clearly, by definition, $\mathscr{M}(D, A;\ x) \geqslant 0$ in all cases, and hence it follows that $B_j(D, A) \geqslant 0$ for $j = 1, 2$, but it is not clear that these constants are strictly positive. In the next section we shall prove, partly by an indirect method, that

$$B_1(D, A) > 0 \quad\text{and}\quad B_2(D, A) > 0,$$

and an asymptotic formula for $\mathscr{M}(D, A;\ x)$ will follow in cases (i) and (ii) above. We shall consider separately the three cases

$$0 < \beta(D) < 1, \quad \beta(D) = 1, \quad \beta(D) = 0.$$

5. Completion of the proof of Theorem 1.

Case I: $0 < \beta(D) < 1$. It follows from (14) that $0 \leqslant \alpha(p_i) < 1$ for each i and that $\alpha(p_i) > 0$ for at least one i $(1 \leqslant i \leqslant r)$; suppose without loss of generality that

$$\alpha(p_i) = \beta(D) > 0 \text{ for } 1 \leqslant i \leqslant \nu, \quad\text{and}\quad \alpha(p_i) < \beta(D) \text{ for } \nu < i \leqslant r.$$

It follows from Lemma 5 that if $D'|D$ but D' is not one of $1, p_1, p_2, \ldots, p_\nu$, then $\alpha(D') < \beta(D)$, and hence $\alpha(D') = \beta(D)$ if and only if D' is one of $p_1, \ldots, p_\nu$, in which case $\mu(D') = \mu(p_i) = -1$. From (10), (15) and Lemma 6, we have

$$m(d) = \max_{1\leqslant i\leqslant \nu} l(p_i^{a_i}) = \max_{1\leqslant i\leqslant \nu} \varkappa_i;$$

suppose without loss of generality that $\varkappa_i = m(d)$ for $1 \leqslant i \leqslant j$ and $\varkappa_i < m(d)$ for $j < i \leqslant \nu$, so that $\beta(D) = \alpha(p_1)$, $m(d) = \varkappa_1$. Then we have by (16) that

$$(19)\qquad \mathscr{M}(D, A;\ x) \sim B_1(D, A)\, x(\log\log x)^{\varkappa_1}(\log x)^{\alpha(p_1)-1}$$

where

$$B_1(D, A) = \sum_{i=1}^{j} \sum_{\substack{0\leqslant\lambda_i<a_i\\ M(p_i,\lambda_i)=\varkappa_1}} C_1(p_i, \lambda_i) > 0$$

since the sum is a non-empty sum of positive terms by Theorem 2 and our assumption that $S_i \neq \emptyset$ for $i = 1, 2, \ldots, r$.

Case II: $\beta(D) = 1$. In this case, $\alpha(p_i) = 1$ for at least one i, and we may suppose without loss of generality that

$$\alpha(p_i) = 1 \text{ for } 1 \leqslant i \leqslant j \quad\text{and}\quad \alpha(p_i) < 1 \text{ for } j < i \leqslant r.$$

Let $D_1 = p_1p_2\ldots p_j$; then it follows from Lemma 5 that $\alpha(D') = 1$ for any $D'|D_1$ but that $\alpha(D') < 1$ if $D'|D$ but $D'\nmid D_1$. Let $d_1 = p_1^{a_1}\ldots p_j^{a_j}$ and in general let d' denote the unitary divisor of d that corresponds to a divisor D' of D. By Lemma 6(i), if $\bigcup_{A'<A'} S(D', A') \neq \emptyset$ and $D'|D_1$, then $l(d') = 0$, and hence by (15) $m(d) = 0$. It follows from (16) that for this case

$$(20)\qquad \mathscr{M}(D, A;\ x) = \{B_1(D, A)+o(1)\}x$$

where

$$B_1(D, A) = -\sum_{\substack{D'|D_1\\ D'\neq 1}} \mu(D') \sum_{\substack{A'<A'\\ S(D',A')\neq\emptyset}} C_1(D', A').$$

Since clearly $\mathscr{M}(D, A; x) \leqslant x$, we have $B_1(D, A) \leqslant 1$ in this case, but, except when D_1 is a prime, it is not yet clear that $B_1(D, A) > 0$; we deduce this from

LEMMA 7.

$$\mathscr{M}(D, A;\ x) \geqslant \sum_{\substack{n\leqslant x\\ (n, D_1)=1}} |\mu(n)|,$$

where D_1 is defined above.

Proof. Since

$$\mathscr{M}(D, A;\ x) = \sum_{n\leqslant x} e(D, A;\ n)$$

and $e(D, A;\ n)$ and $|\mu(n)|$ assume the values 1, 0 only, it is sufficient to prove that $e(D, A;\ n) = 1$ whenever $(n, D_1) = 1$ and $|\mu(n)| = 1$, that is whenever n is squarefree and coprime to D_1.

Now $\alpha(D_1) = 1$ and therefore by the Corollary to Lemma 5, $(W(x), D_1) = 1$ whenever $(x, D_1) = 1$. Since $W(p) = f(p)$ for every prime p and f

is multiplicative, it follows that $(f(n), D_1) = 1$ for every squarefree integer n with $(n, D_1) = 1$; but if $(f(n), D_1) = 1$, then $d_1 \nmid f(n)$ and so certainly $d \nmid f(n)$, giving $c(D, A; n) = 1$. This proves the Lemma.

An estimate for the sum on the right of this Lemma is given by Lemma 2 (ii), and hence we certainly have that for all sufficiently large x,

$$\mathscr{M}(D, A; x) \geqslant 3\pi^{-2}\left\{\prod_{i=1}^{j} \frac{p_i}{p_i+1}\right\} x. \tag{21}$$

It follows that $B_1(D, A) > 0$, for otherwise $\mathscr{M}(D, A; x) = o(x)$ by (20), which is false by (21). Thus we have shown that when $\beta(D) = 1$,

$$\mathscr{M}(D, A; x) \sim B_1(D, A)x \quad \text{where} \quad 0 < B_1(D, A) \leqslant 1. \tag{22}$$

Case III: $\beta(D) = 0$. Here we have $\alpha(D') = 0$ for every $D' | D$, $D' \neq 1$. Hence by Lemma 6 (iii), if d' denotes the unitary divisor of d associated with the divisor D' of D, and if $\bigcup_{A' < A'} S(D', A') \neq \emptyset$, we have

$$l(d') = \min_{\substack{1 \leqslant i \leqslant r \\ p_i | D'}} \varkappa_i;$$

in particular, since $S_i \neq \emptyset$ for every $p_i | D$ by (10) and hypothesis, $l(p_i^{a_i}) = \varkappa_i$ for $i = 1, 2, \ldots, r$. If d^* ranges over those unitary divisors of d for which $\bigcup_{A^* < A^*} S(D^*, A^*) \neq \emptyset$, it now follows from (15) that

$$m(d) = \max_{d^*} l(d^*) = \max_{d^*} \min_{p_i | D^*} \varkappa_i = \max_{1 \leqslant i \leqslant r} \varkappa_i.$$

We may suppose without loss of generality that

$$\varkappa_1 = \max_{1 \leqslant i \leqslant r} \varkappa_i, \quad \varkappa_i = \varkappa_1 \text{ for } 1 \leqslant i \leqslant j, \quad \varkappa_i < \varkappa_1 \text{ for } j < i \leqslant r;$$

then $m(d) = \varkappa_1$. Let $D_1 = p_1 \ldots p_j$; if $D' | D$ but $D' \nmid D_1$, then by Lemma 6 (iii) $l(d') < \varkappa_1$ and hence $M(D', A') < \varkappa_1$ for all $A' < A'$ for which $M(D', A')$ is defined. It now follows from (17) that if $\varkappa_1 > 0$

$$\mathscr{M}(D, A; x) = \{B_2(D, A) + o(1)\} x (\log\log x)^{\varkappa_1 - 1} (\log x)^{-1} \tag{23}$$

where

$$B_2(D, A) = -\sum_{\substack{D' | D_1 \\ D' \neq 1}} \mu(D') \sum_{\substack{A' < A' \\ M(D', A') = \varkappa_1}} C_2(D', A');$$

if $\varkappa_1 = 0$, we have by (18)

$$\mathscr{M}(D, A; x) = O(x^{1-\varepsilon(d)}) \quad \text{where } \varepsilon(d) > 0. \tag{24}$$

To prove that $B_2(D, A) > 0$ and to find a precise value for $\varepsilon(d)$, we resort to indirect methods.

Lemma 8. *Let $k = p_1^{\tau_1+1} \ldots p_j^{\tau_j+1}$ where τ_i is defined in (7) and j has the same significance as above; assume that $\varkappa_1 > 0$ and write $\varkappa$ for $\varkappa_1$. Then there exists h coprime to k such that*

$$\mathscr{M}(D, A; x) \geqslant \pi_\varkappa(h, k; x),$$

where $\pi_\varkappa(h, k; x)$ is defined in the sentence containing (5).

Proof. It is sufficient to show that if n contributes 1 to $\pi_\varkappa(h, k; x)$, so that by (5)

$$n = q_1 q_2 \ldots q_\varkappa \quad \text{where} \quad q_i \equiv h \pmod{k} \quad (i = 1, 2, \ldots, \varkappa) \tag{25}$$

and where $q_1, \ldots, q_\varkappa$ are distinct primes, then $c(D, A; n) = 1$, provided that we define h suitably.

Since $\alpha(p_i) = 0$ for each $p_i | D_1$, it follows from (9) and (7) that τ_i is defined for $1 \leqslant i \leqslant j$, and hence k is properly defined. From (7) we obtain that there exists a least integer x_i satisfying

$$1 \leqslant x_i \leqslant p_i^{\tau_i+1}, \quad p_i \nmid x_i \quad \text{and} \quad p_i^{\tau_i} \| W(x_i).$$

We now define h to be a solution of the simultaneous congruences

$$x \equiv x_i \pmod{p_i^{\tau_i+1}} \quad (i = 1, 2, \ldots, j);$$

since the x_i's are uniquely defined and $(x_i, p_i) = 1$, it follows that h is unique $\pmod{k}$ and that $(h, k) = 1$. Furthermore if $x \equiv h \pmod{k}$, $p_i^{\tau_i} \| W(x)$ for $i = 1, 2, \ldots, j$, and hence in particular for any prime $q \equiv h \pmod{k}$, $p_i^{\tau_i} \| W(q) = f(q)$ for $1 \leqslant i \leqslant j$. Thus, since f is multiplicative, we have that for any n satisfying (25),

$$p_i^{\varkappa\tau_i} \| f(n) \quad \text{for} \quad 1 \leqslant i \leqslant j.$$

Since $\varkappa\tau_i = \varkappa_i\tau_i \leqslant a_i - 1$, it follows that $p_i^{a_i} \nmid f(n)$ if n satisfies (25) and $1 \leqslant i \leqslant j$, and hence $d \nmid f(n)$, giving $c(D, A; n) = 1$. The proof of the Lemma is now complete, for we have shown that if h is defined as above and if n satisfies (25), so that n contributes 1 to $\pi_\varkappa(h, k; x)$ for any $x \geqslant n$, then n also contributes 1 to $\mathscr{M}(D, A; x)$.

From Lemmas 3 and 8, we deduce that if $\varkappa = \varkappa_1 > 0$,

$$\mathscr{M}(D, A; x) \geqslant \big(2(\varkappa-1)!\big)^{-1} \big(\varphi(k)\big)^{-\varkappa} x (\log\log x)^{\varkappa-1} (\log x)^{-1}$$

for all sufficiently large x, and hence comparing this with (23), we obtain

$$B_2(D, A) > 0. \tag{26}$$

We now turn to the case $\varkappa_1 = 0$ and (24).

Lemma 9. *If $\varkappa_1 = 0$,*

$$\mathscr{M}(D, A; x) = O(x^{1/2}). \tag{27}$$

Proof. As in Lemma 8, τ_i is defined for $1 \leqslant i \leqslant r$, but since now $\varkappa_i = 0$ for each i, we deduce from the definition of $\varkappa_i$ in Lemma 6 that $\tau_i \geqslant a_i$ for $i = 1, 2, \ldots, r$. Hence, since $a(p_i) = 0$ for each i, we have from (7) and (9) that $p_i^{a_i} | W(x)$ whenever $p_i \nmid x$ $(i = 1, \ldots, r)$, and therefore $d | W(q) = f(q)$ for each prime $q \nmid d$. It follows that $d | f(n)$, whence $c(D, A; n) = 0$, for each integer n for which there exists a prime q satisfying $q \| n$ and $q \nmid d$. Hence if $c(D, A; n) = 1$, n is squarefull (that is $q \| n$ for no prime q) or $q | d$ for each prime $q \| n$, and so we obtain

(28) $$\mathcal{M}(D, A; x) \leqslant \sum_{\substack{n \leqslant x \\ n \in \mathcal{S}}} 1 + \sum_{\substack{n \leqslant x \\ q \| n \Rightarrow q | d \\ n \notin \mathcal{S}}} 1 = \Sigma_1 + \Sigma_2 \text{ (say)},$$

where $\mathcal{S}$ is the set of all positive squarefull integers. By a result of Erdös and Szekeres [3],

(29) $$\Sigma_1 = \sum_{\substack{n \leqslant x \\ n \in \mathcal{S}}} 1 \sim \zeta(3/2)\zeta^{-1}(3) x^{1/2}.$$

If $n \notin \mathcal{S}$ and $q \| n \Rightarrow q | d$, then $n = p_{i_1} \ldots p_{i_j} n_1$ where $p_{i_1} \ldots p_{i_j} | D$, $(p_{i_1} \ldots \ldots p_{i_j}, n_1) = 1$ and $n_1 \in \mathcal{S}$. Hence by (29)

(30) $$\Sigma_2 \leqslant \sum_{p_{i_1} \ldots p_{i_j} | D} \sum_{\substack{p_{i_1} \ldots p_{i_j} n_1 \leqslant x \\ n_1 \in \mathcal{S}}} 1$$
$$\sim \zeta(3/2)\zeta^{-1}(3) x^{1/2} \sum_{p_{i_1} \ldots p_{i_j} | D} (p_{i_1} \ldots p_{i_j})^{-1/2}.$$

Thus (27) follows from (28), (29) and (30).

If we combine (19), (22), (23), (26) and (27), we see that we have established the result of Theorem 1. The constants β, m of that Theorem are given by

(31) $$\beta = \beta(D) = \max_{1 \leqslant i \leqslant r} a(p_i) = a(p_1),$$

(32) $$m = m(d) = \max_{\substack{1 \leqslant i \leqslant r \\ a(p_i) = a(p_1)}} \varkappa_i = \varkappa_1$$

(say), where $a(p_i)$ and $\varkappa_i$ can be obtained from Lemmas 5 and 6.

As we remarked in § 1, some special cases of this Theorem were known previously, and in particular Narkiewicz considered in Theorem II of [5] the case when d is squarefree; however he did not obtain explicitly the precise power of $\log x$ that appeared. From Theorem 1, we are able to deduce immediately

COROLLARY 1. *If d is squarefree (so that $d = D$ and $A = I$), $S(p_i, 0) \neq \emptyset$ for $i = 1, 2, \ldots, r$, and $a(p_1) = \max_{1 \leqslant i \leqslant r} a(p_i)$, then as $x \to \infty$,*

(i) *if* $0 < a(p_1) < 1$,
$$\mathcal{M}(D, I; x) \sim Bx(\log x)^{a(p_1)-1};$$
(ii) *if* $a(p_1) = 1$,
$$\mathcal{M}(D, I; x) \sim Bx;$$
(iii) *if* $a(p_1) = 0$,
$$\mathcal{M}(D, I; x) = O(x^{1/2}).$$

In (i) *and* (ii), $B = B_1(D, I) > 0$, *and in* (ii), $B \leqslant 1$.

We observe that since $a_i = 1$ for all i, $\varkappa_i = 0$ and therefore $m = m(d) = 0$.

Finally we remark that by using (11), we can eliminate the condition in Theorem 1 that $S_i \neq \emptyset$ for $i = 1, 2, \ldots, r$, but our constants β, m in Theorem 1 are then related to the sum on the right of (11).

6. An estimate for $N(n \leqslant x\colon d | f(n))$.

THEOREM 4. *As* $x \to \infty$,
$$N(n \leqslant x\colon d | f(n)) \sim x$$
except when $\bigcup_{i=1}^{r} S_i \neq \emptyset$, *where* S_i *is defined in* (10), *and* $\max_{\substack{1 \leqslant i \leqslant r \\ S_i \neq \emptyset}} a(p_i) = 1$. *In the exceptional case, either* $T = \{n\colon d | f(n)\} = \emptyset$, *in which case*
$$N(n \leqslant x\colon d | f(n)) = 0 \quad \textit{for all } x,$$
or
$$N(n \leqslant x\colon d | f(n)) \sim Bx \quad \textit{where} \quad 0 < B < 1.$$

Proof. Apart from the exceptional case, the result follows immediately from (1), (11) and Theorem 1, for
$$N(n \leqslant x\colon d | f(n)) = [x] - N(n \leqslant x\colon d \nmid f(n)) = [x] - o(x) \sim x.$$
Hence suppose that
$$\bigcup_{1 \leqslant i \leqslant r} S_i \neq \emptyset \quad \text{and} \quad \max_{\substack{1 \leqslant i \leqslant r \\ S_i \neq \emptyset}} a(p_i) = 1.$$
Then by (1), (11) and Theorem 1 (ii),

(33) $$N(n \leqslant x\colon d | f(n)) = \{1 - B_1(D_1, A_1) + o(1)\}x$$

where
$$D_1 = \prod_{\substack{1 \leqslant i \leqslant r \\ S_i \neq \emptyset}} p_i > 1, \quad \text{and} \quad 0 < B_1(D_1, A_1) \leqslant 1.$$
If $T = \emptyset$, it is clear that
$$N(n \leqslant x\colon d | f(n)) = 0, \quad N(n \leqslant x\colon d \nmid f(n)) = [x], \quad B_1(D_1, A_1) = 1.$$

Hence suppose that $T \neq \emptyset$; then we must show that $B_1(D_1, A_1) < 1$.

For this remaining case, we prove first that if $k = \inf T$, then

$$N(n \leqslant x:\ d \mid f(n)) \geqslant \sum_{\substack{u \leqslant x/k \\ (u,k)=1}} 1. \tag{34}$$

Since T is a non-empty set of positive integers, k exists and $k \in T$, and since $d > 1$ and $f(1) = 1$, $k > 1$. If n is any positive integer such that $k \| n$, so that $n = ku$ where $(u, k) = 1$, then since f is multiplicative, $d \mid f(n) = f(k)f(u)$. This is sufficient to establish (34). From Lemma 2 (i), we deduce that for all sufficiently large x,

$$N(n \leqslant x:\ d \mid f(n)) \geqslant \varphi(k)x/(2k^2).$$

It follows from (33) that $1 - B_1(D_1, A_1) > 0$, and the Theorem is proved in all cases.

Corollary. *If case* (ii) *of Theorem* 1 *holds, then* $B_1(D, A) < 1$ *unless* $T = \emptyset$, *in which case* $B_1(D, A) = 1$ *and* $\mathscr{M}(D, A;\ x) = [x]$.

7. Applications of Theorem 1. In this section we consider the special cases of Theorem 1 obtained by taking $f(n)$ to be the well known functions $\varphi(n)$ (Euler's function) and the divisor functions $\tau(n)$, $\sigma_\nu(n)$ (ν a positive integer). We state the results obtained as further Corollaries to Theorem 1.

Corollary 2. (i) *If* $p_1 > 2$ *is the largest prime divisor of* d, *then as* $x \to \infty$

$$N(n \leqslant x:\ d \nmid \varphi(n)) \sim B_1 x (\log\log x)^{a_1 - 1} (\log x)^{-1/(p_1 - 1)},$$

where $B_1 = B_1(D, A) > 0$;

(ii) *if* $a \geqslant 2$,

$$N(n \leqslant x:\ 2^a \nmid \varphi(n)) \sim B_2 x (\log\log x)^{a-2} (\log x)^{-1},$$

where $B_2 = B_2(D, A) > 0$.

Proof. Since $f(n) = \varphi(n)$ here, we have $W(x) = x - 1$. It is easily seen that $S(p, \lambda) = \{n:\ p^\lambda \| \varphi(n)\} \neq \emptyset$ for all primes p and all $\lambda \geqslant 0$. To find the constants β, m of Theorem 1, defined in (31) and (32), we observe that

(a) $\alpha(p) = (p-2)/(p-1)$ by (9) and hence

$$\beta = \max_{1 \leqslant i \leqslant r} \alpha(p_i) = 1 - 1/(p_1 - 1) > 1 - 1/(p_i - 1) \quad \text{for} \quad i > 1,$$

where p_1 is the largest prime divisor of d;

(b) since $W(x) = x - 1 \equiv p \pmod{p^2}$ is solvable with $p \nmid x$ for any prime p, it follows from (7) that $\tau = 1$ for each prime p and hence (see Lemma 6) $\varkappa_i = a_i - 1$ whence by (32)

$$m = \max_{\substack{1 \leqslant i \leqslant r \\ \alpha(p_i) = \alpha(p_1)}} \varkappa_i = \varkappa_1 = a_1 - 1.$$

The result of the Corollary now follows from Theorem 1 (i) and (iii).

We note that for the case omitted in the above Corollary, we have that $N(n \leqslant x:\ 2 \nmid \varphi(n)) = 2$ for all $x \geqslant 2$. We turn now to $\tau(n)$; Corollary 3 (i) below is an immediate consequence of a result of Sathe [8].

Corollary 3. (i) *If* d *has an odd prime divisor, then as* $x \to \infty$,

$$N(n \leqslant x:\ d \nmid \tau(n)) \sim B_1 x \quad \text{where} \quad 0 < B_1 < 1;$$

(ii) *if* $a \geqslant 2$,

$$N(n \leqslant x:\ 2^a \nmid \tau(n)) \sim B_2 x (\log\log x)^{a-2} (\log x)^{-1}$$

as $x \to \infty$, *where* $B_2 > 0$.

Proof. From the properties of $\tau(n) = \sum_{\delta \mid n} 1$, we easily deduce that $S(p, \lambda) = \{n:\ p^\lambda \| \tau(n)\} \neq \emptyset$ for all primes p and all $\lambda \geqslant 0$ and that $W(x) = 2$. By (9), $\alpha(p) = 1$ for all odd primes p and $\alpha(2) = 0$, and hence by (31), $\beta = 1$ or 0 according as d has an odd prime divisor or d is a power of 2, respectively. Thus we only define m (see (32)) when $d = 2^a$, and since $\tau = 1$ when $p = 2$ by (7), $\varkappa = a - 1$ and so $m = a - 1$ in this case. The result now follows from Theorem 1 (ii) and (iii) provided that we show $B_1 < 1$ when d has an odd prime divisor. We use the Corollary to Theorem 4; since $d \mid \tau(p^{d-1})$ for any prime p, $T = \{n:\ d \mid \tau(n)\} \neq \emptyset$ and hence $B_1(D, A) < 1$.

The corresponding result for the case not covered by the above Corollary, namely $N(n \leqslant x:\ 2 \nmid \tau(n)) \sim x^{1/2}$, was obtained in Theorem 4 of [9].

In order to state the result of Theorem 1 in the case when

$$f(n) = \sigma_\nu(n) = \sum_{\delta \mid n} \delta^\nu \quad (\nu \text{ a positive integer}),$$

we need to introduce some notation, which was used also in [9] and [10], and we shall appeal to one of the Lemmas of [10]. If p is a prime, we define h_p (for $p \neq 2$) and γ_p by

$$h_p = (p-1)/(\nu, p-1), \qquad p^{\gamma_p} \| \nu. \tag{35}$$

Then we have

Corollary 4. (i) *If* d *has an odd prime divisor* p *such that* h_p *is odd, then as* $x \to \infty$,

$$N(n \leqslant x:\ d \nmid \sigma_\nu(n)) \sim B_1 x, \quad \text{where} \quad 0 < B_1 < 1.$$

(ii) *If* d *is not a power of* 2 *and if* h_p *is even for each odd prime divisor of* d, *then as* $x \to \infty$,

$$N(n \leqslant x:\ d \nmid \sigma_\nu(n)) \sim B_1 x (\log\log x)^m (\log x)^{-1/h},$$

where $B_1 > 0$ *and*

$$h = \max_{\substack{p \mid d \\ p \neq 2}} h_p, \qquad m = \max_{\substack{p \mid d \\ h_p = h}} \left[\frac{a-1}{\gamma_p + 1}\right] \quad (a \text{ being defined by } p^a \| d).$$

(iii) *If* $a \geqslant 2$, *then as* $x \to \infty$,

$$N(n \leqslant x:\ 2^a \nmid \sigma_\nu(n)) \sim B_2 x (\log\log x)^{a-2} (\log x)^{-1},$$

where $B_2 > 0$.

Proof. It follows from the main results of [7] and [9] that

$$S(p, \lambda) = \{n:\ p^\lambda \| \sigma_\nu(n)\} \neq \emptyset$$

for all primes p and all $\lambda \geqslant 0$. Since $W(x) = x^\nu + 1$, we can appeal to Lemma 3 of [10] to obtain the constants $\alpha(p)$ of (9) and Lemma 5, and τ of (7); we deduce that

$$\alpha(p) = \begin{cases} 1 & \text{if } p \text{ and } h_p \text{ are odd,} \\ 1 - \dfrac{(\nu, p-1)}{p-1} = 1 - \dfrac{1}{h_p} & \text{if } p \text{ is odd and } h_p \text{ is even,} \\ 0 & \text{if } p = 2, \end{cases}$$

and that

$$\tau = \begin{cases} \gamma_p + 1 & \text{if } p \text{ is odd and } h_p \text{ is even,} \\ 1 & \text{if } p = 2, \end{cases}$$

τ not being defined when p and h_p are odd. As before, let $d = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ and suppose that if d is even, $p_r = 2$; by (31) and (32) we have

(i) if d has an odd prime divisor p such that h_p is odd, then $\beta = 1$;

(ii) if p_1 is odd, if h_p is even for every odd prime divisor p of d, and if $h_{p_i} = h_{p_1}$ for $i = 1, 2, \dots, j$ and $h_{p_i} < h_{p_1}$ for $j < i \leqslant r$ and p_i odd (so that $h_{p_1} = \max_{\substack{1 \leqslant i \leqslant r \\ p_i \text{ odd}}} h_{p_i}$), then

$$\beta = 1 - (h_{p_1})^{-1}, \quad m = \max_{1 \leqslant i \leqslant j} \left[\frac{a_i - 1}{\gamma_{p_i} + 1}\right];$$

(iii) if $p_1 = 2$, so that $d = 2^{a_1}$, then $\beta = \alpha(2) = 0$ and $m = a_1 - 1$.

Most of the result of the Corollary now follows from Theorem 1. It remains to show (by using the Corollary to Theorem 4) that $B_1 < 1$ in case (i), so that we must verify that $T = \{n:\ d \mid \sigma_\nu(n)\} \neq \emptyset$. Since $\sigma_\nu(n)$ is multiplicative, it is sufficient to show that for each $p \mid d$, there are infinitely many primes q such that $p \mid \sigma_\nu(q^a)$ for some $a \geqslant 1$; for we can then construct from these prime powers q^a an integer n satisfying $d \mid \sigma_\nu(n)$, whence $T \neq \emptyset$. Now if $q \equiv 1 \pmod p$, q prime, then

$$\sigma_\nu(q^{p-1}) = 1 + q^\nu + q^{2\nu} + \dots + q^{(p-1)\nu} \equiv p \equiv 0 \pmod p;$$

by Dirichlet's Theorem, there are infinitely many such primes q. This completes the proof of Corollary 4 of Theorem 1.

Some special cases of the above result were obtained in [7] and [9]. In [7] Rankin dealt with the case when d is a prime, including the case $d = 2$ (not given in Corollary 4 above) when he obtained

$$N(n \leqslant x:\ 2 \nmid \sigma_\nu(n)) \sim (1 + 2^{-1/2}) x^{1/2};$$

an application of Theorem 1 (iv) of this paper in the case $d = 2$ would have given a less precise result. In Corollary 1 on page 280 of [9], we obtained the result above when d is a prime power, and in Corollary 2 there we established the result of Corollary 4 above in the cases when

(i) h_p is odd for exactly one odd prime divisor p of d;

(ii) h_p is even for all odd prime divisors p of d, and (see (35))

$$h_{p_i} = h_{p_1}, \quad [(a_i - 1)/(\gamma_{p_i} + 1)] = m$$

hold simultaneously for exactly one value of i.

Finally in this section we observe that Corollary 4 (ii) and (iii) above is an improvement on G. N. Watson's result stated in (2), namely

$$(36) \qquad N(n \leqslant x:\ d \nmid \sigma_\nu(n)) = O(x (\log x)^{-1/\varphi(d)}) \quad (\nu \text{ odd});$$

the condition ν odd implies that h_p is even for all odd prime divisors p of d and hence Corollary 4 (i) does not apply. In fact (36) gives the correct order of magnitude for $N(n \leqslant x:\ d \nmid \sigma_\nu(n))$ only when $d = p$ or $2p$ and $(\nu, p-1) = 1$, where p is an odd prime; for in all other cases either $d = 2^a$ for some $a \geqslant 1$ or $h < \varphi(d)$, and hence Corollaries 4 (ii) and (iii) imply

$$N(n \leqslant x:\ d \nmid \sigma_\nu(n)) = o(x (\log x)^{-1/\varphi(d)}).$$

8. A generalization of Corollary 4 of Theorem 1. In [10] we showed how a slight generalization of Narkiewicz's result in [5] could be applied to a generalized divisor function. In this section we shall obtain the result analogous to Theorem 1 for this same function by a similar process. First we need some definitions.

As in the previous sections, let $d = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$; further let Q' be a given integer > 1 and write

$$(37) \qquad Q' = Q \prod_{i=1}^{r} p_i^{e_i} = Q\delta, \quad Q'' = Q \prod_{i=1}^{r} p_i^{\max(a_i, e_i)} = Q\Delta,$$

say, where $(Q, d) = 1$ and $e_i \geqslant 0$ $(i = 1, 2, \dots, r)$, so that Q'' is the l.c.m. of Q' and d. Let χ denote a real non-principal character (mod Q'), and write $\chi = \chi_1 \chi_2$ where χ_1 is a real character (mod Q) and χ_2 is a real character (mod δ); then at least one of χ_1, χ_2 is non-principal and we shall *assume henceforth that* χ_1 *is non-principal* (so that $Q > 1$). In this section we shall be considering a generalization of the usual divisor function, namely

$$(38) \qquad \sigma_\nu(n, \chi) = \sum_{\varrho \mid n} \chi(\varrho) \varrho^\nu,$$

where ν is a positive integer, and we shall obtain an estimate for $N(n \leqslant x: d \nmid \sigma_\nu(n, \chi))$ by applying a slight generalization of Theorem 2. For any prime p, $\sigma_\nu(p, \chi) = 1 + \chi(p)p^\nu$ and hence the polynomial $W(x)$ associated with $\sigma_\nu(n, \chi)$ takes the form

$$W(x) = 1 + \chi(x)x^\nu,$$

where the value of $\chi(x)$ (which is real by assumption) depends on the value of $x \pmod{Q'}$. Thus we shall need a generalization of Theorem 2 in which the coefficients of the polynomial $W(x)$ depend on the value of $x \pmod{Q'}$. The form of the result required is very similar to that stated in Theorem 2 but we shall have to modify some of the constants involved.

Let $f(n)$ and $W(x)$ be as described in § 1 apart from the fact that the coefficients of $W(x)$ are dependent on the value of $x \pmod{Q'}$. Let $X(d, Q')$ denote the number of integers x that satisfy $1 \leqslant x \leqslant Q''$, $(x, Q'') = 1$ and $(W(x), d) = 1$, and let

$$\alpha(d, Q') = X(d, Q')/\varphi(Q''). \tag{39}$$

Similarly for any column $T_k = (t_{ik})$ of r elements, let $N(T_k, Q')$ denote the number of integers x that satisfy

$$1 \leqslant x \leqslant Q \prod_{i=1}^{r} p_i^{\max(e_i, t_{ik}+1)}, \quad (x, Qd) = 1 \quad \text{and} \quad \prod_{i=1}^{r} p_i^{t_{ik}} \,\|\, W(x);$$

define $\Theta(D, Q')$ $(D = p_1 p_2 \ldots p_r)$ to be the set of all possible columns $T_k \neq O$ such that $N(T_k, Q') > 0$. The constant $M(D, A;\ Q')$ is now defined in exactly the same way as $M(D, A)$ in § 3, but using the quantities defined above instead of $\alpha(d)$, $N(T_k)$, $\Theta(D)$. It is easily verified that the result of Theorem 2, with $\alpha(d, Q')$, $M(D, A;\ Q')$ replacing $\alpha(d)$, $M(D, A)$ respectively, remains valid in the situation described here. A particular example of this result occurs when we take $f(n) = \sigma_\nu(n, \chi)$; in this case it follows from Theorem 1 of [10] that

$$S(p, \lambda) = \{n\colon p^\lambda \,\|\, \sigma_\nu(n, \chi)\} \neq \emptyset$$

for all primes p and all $\lambda \geqslant 0$.

We now use this modification of Theorem 2 and the method used to derive Theorem 1 in order to obtain an estimate for $N(n \leqslant x: d \nmid \sigma_\nu(n, \chi))$. Not all the Lemmas used to establish Theorem 1 remain valid for $W(x) = 1 + \chi(x)x^\nu$; in particular the result of Lemma 5 must be modified whilst Lemmas 1 and 4 continue to hold. We recall that h_p and γ_p were defined in (35); to replace Lemma 5, we have

LEMMA 10. *If d is odd,*

$$\alpha(d, Q') = \frac{1}{2} \prod_{\substack{p|d \\ h_p \text{ even}}} \left(1 - \frac{1}{h_p}\right) \left\{1 + \prod_{\substack{p|d \\ h_p \text{ odd}}} \left(1 - \frac{1}{h_p}\right)\right\}$$

(where an empty product has the value 1), whilst if d is even,

$$\alpha(d, Q') = 0.$$

Proof. We use the results of Lemmas 3 and 4 of [10] and ideas from the proof of Lemma 5 of that paper. Since χ is a real character $\pmod{Q'}$, when $(x, Q') = 1$ we have $W(x) = 1 \pm x^\nu$ according as $\chi(x) = \pm 1$. We see from (39) that we shall need to obtain $X(d, Q')$, the number of integers x satisfying $1 \leqslant x \leqslant Q''$, $(x, Q'') = 1$ and either $\chi(x) = 1$ and $(1 + x^\nu, d) = 1$ or $\chi(x) = -1$ and $(1 - x^\nu, d) = 1$.

By Lemma 3 of [10], the number of integers x satisfying $1 \leqslant x \leqslant p-1$ and $p \nmid (x^\nu + 1)$ is $p-1$ if p and h_p are odd, $(p-1)\left(1 - \frac{1}{h_p}\right)$ if p is odd but h_p is even, and 0 if $p = 2$; hence the number of integers x satisfying $1 \leqslant x \leqslant p^g$, $p \nmid x$ and $p \nmid (x^\nu + 1)$ is $\varphi(p^g)$, $\varphi(p^g)\left(1 - \frac{1}{h_p}\right)$, 0 respectively in the three cases. Similarly by Lemma 4 of [10], the number of integers x satisfying $1 \leqslant x \leqslant p^g$, $p \nmid x$ and $p \nmid (x^\nu - 1)$ is $\varphi(p^g)\left(1 - \frac{1}{h_p}\right)$ or 0 according as p is odd or even. Taking $p = p_i | d$ and $g = g_i = \max(a_i, e_i)$ with $i = 1, 2, \ldots, r$ and recalling that $\Delta = p_1^{g_1} p_2^{g_2} \ldots p_r^{g_r}$, we easily deduce that the number of integers x satisfying

$$\left\{\begin{array}{l} \text{(i) } 1 \leqslant x \leqslant \Delta,\ (x, \Delta) = 1 \text{ and } (x^\nu + 1, \Delta) = 1 \text{ is} \\ \displaystyle \varphi(\Delta) \prod_{\substack{i=1 \\ h_{p_i} \text{ even}}}^{r} \left(1 - \frac{1}{h_{p_i}}\right) \quad \text{if } d \text{ is odd or } 0 \text{ if } d \text{ is even;} \\ \text{(ii) } 1 \leqslant x \leqslant \Delta,\ (x, \Delta) = 1 \text{ and } (x^\nu - 1, \Delta) = 1 \text{ is} \\ \displaystyle \varphi(\Delta) \prod_{i=1}^{r} \left(1 - \frac{1}{h_{p_i}}\right) \quad \text{if } d \text{ is odd or } 0 \text{ if } d \text{ is even.} \end{array}\right. \tag{40}$$

We now argue as we did in the proof of Lemma 5 of [10]. We recall that $\chi = \chi_1 \chi_2$, where χ_1 is a non-principal real character $\pmod{Q}$ and χ_2 is a real character $\pmod{\delta}$, where Q, δ are defined in (37) and clearly $\delta | \Delta$. We determine first the number of integers x satisfying

$$1 \leqslant x \leqslant Q'', \quad (x, Q'') = 1, \quad \chi(x) = 1 \quad \text{and} \quad (x^\nu + 1, d) = 1; \tag{41}$$

we note that $(x^\nu + 1, d) = 1$ is equivalent to $(x^\nu + 1, \Delta) = 1$ since $p | d$ if and only if $p | \Delta$. Let x_1 be a fixed integer for which $1 \leqslant x_1 \leqslant \Delta$, $(x_1, \Delta) = 1$ and $(x_1^\nu + 1, \Delta) = 1$; then the value of $\chi_2(x_1)$ is already determined and is non-zero. Since $(Q, \Delta) = 1$, the integers y satisfying $1 \leqslant y \leqslant Q\Delta$, $(y, Q) = 1$ and $y \equiv x_1 \pmod{\Delta}$ form a reduced residue system $\pmod{Q}$, and hence, since χ_1 is real and non-principal, $\chi_1(y) = \chi_2(x_1)$ for exactly

$\frac{1}{2}\varphi(Q)$ of these integers y. It follows from this and (40 (i)) that the numbe[r] of integers x satisfying (41) is

$$\tfrac{1}{2}\varphi(Q)\varphi(\Delta)\prod_{\substack{i=1\\ h_{p_i}\text{even}}}^{r}\left(1-\frac{1}{h_{p_i}}\right)\text{ if } d \text{ is odd or } 0 \text{ if } d \text{ is even.}$$

Similarly the number of integers x satisfying

$$1\leqslant x\leqslant Q'',\quad (x, Q'')=1,\quad \chi(x)=-1\quad\text{and}\quad (x^{\nu}-1, d)=1$$

is

$$\tfrac{1}{2}\varphi(Q)\varphi(\Delta)\prod_{i=1}^{r}\left(1-\frac{1}{h_{p_i}}\right)\text{ if } d \text{ is odd or } 0 \text{ if } d \text{ is even,}$$

on using (40(ii)). Hence by (39), since $W(x) = 1+\chi(x)x^{\nu}$,

$$\alpha(d, Q') = \tfrac{1}{2}\frac{\varphi(Q)\varphi(\Delta)}{\varphi(Q\Delta)}\left\{\prod_{\substack{i=1\\ h_{p_i}\text{even}}}^{r}\left(1-\frac{1}{h_{p_i}}\right)+\prod_{i=1}^{r}\left(1-\frac{1}{h_{p_i}}\right)\right\}$$

if d is odd, and if d is even $\alpha(d, Q') = 0$; the required result now follo[ws] since $(Q, \Delta) = 1$.

We observe that $\alpha(D, Q') = \alpha(d, Q')$. We note also that the resu[lt] of Lemma 10 remains valid if we replace d in the statement by a divis[or] of d and make the corresponding adjustment in Q, δ and Δ whilst leavi[ng] Q' unaltered, for Q will be replaced by a multiple of Q and so χ_1 will [be] replaced by a character that is also non-principal. In general howeve[r] for Q' fixed, $\alpha(d, Q')$ is not a multiplicative function of d, unlike the qua[n]tity $\alpha(d)$ of Lemma 5. We can easily deduce from the result of Lemma [10] that if $d_1|d$, $d_2|d$ and $(d_1, d_2) = 1$, then

$$\alpha(d_1 d_2, Q') \leqslant \alpha(d_1, Q'),$$

the inequality being strict if d_1 is odd and $d_2 > 1$. Hence, defining $\beta(\cdot)$ in an analogous way to (13) (see below), we have

Corollary. *If d has an odd prime divisor, and $D = p_1 p_2 \cdots$ [p_r], then*

$$\beta(D) = \max_{\substack{D'|D\\ D'\neq 1}}\alpha(D', Q') = \max_{\substack{1\leqslant i\leqslant r\\ p_i\neq 2}}\alpha(p_i, Q') = \max_{\substack{1\leqslant i\leqslant r\\ p_i\neq 2}}\left(1-\frac{\psi_i}{h_{p_i}}\right),$$

where $\psi_i = \frac{1}{2}$ or 1 according as h_{p_i} is odd or even; thus $0<\beta(D)<1$. [If] d is a power of 2, then $\beta(D) = 0$.

We also need to evaluate the constant corresponding to $m(d)$ of (13), namely the constant

$$m(d) = \max_{\substack{D'|D,\,\Lambda'<A'\\ \alpha(D',Q')=\beta(D)}} M(D', \Lambda'; Q') = \max_{\substack{1\leqslant i\leqslant r\\ \alpha(p_i,Q')=\beta(D)\\ 0\leqslant\lambda_i<a_i}} M(p_i, \lambda_i; Q')$$

by the remark before the Corollary above; since, as we have already noted, $S(p_i, \lambda_i)\neq\emptyset$ for each p_i and λ_i, it follows that $M(p_i, \lambda_i; Q')$ is always defined. We have

Lemma 11. *If d has an odd prime divisor,*

$$m(d) = \max_{\substack{1\leqslant i\leqslant r\\ \alpha(p_i,Q')=\beta(D)}}\left[\frac{a_i-1}{\gamma_{p_i}+1}\right],$$

whilst if $a\geqslant 1$, $m(2^a) = a-1$.

Proof. The proof is similar to that of Lemma 6 of this paper. Define τ as in (7), so that τ is the least positive integer such that there exists a positive integer x satisfying $p\nmid x$ and $p^{\tau}\| (1+\chi(x)x^{\nu})$, and then we have from Lemma 5 of [10] that $\tau = \gamma_p+1$ if p is odd and $\tau = 1$ if $p = 2$. By considering, for p odd, the row matrix T with $[(a-1)/(\gamma_p+1)]$ equal elements with value γ_p+1, we see, by the arguments used to establish Lemma 6, that

$$\max_{0\leqslant\lambda<a} M(p, \lambda; Q') = [(a-1)/(\gamma_p+1)],$$

and the result of the Lemma follows when d has an odd prime divisor. Similarly if $d = 2^a$, we can proceed as above but with 1 replacing γ_p+1 everywhere.

By taking $f(n) = \sigma_\nu(n, \chi)$ (see (38)) in (12), by recalling that $S(p, \lambda)\neq\emptyset$ for all primes p and all $\lambda\geqslant 0$, by using the modification of Theorem 2 described above and by appealing to Lemmas 10 and 11, we can deduce the following result by the method used to prove Theorem 1 (i) (case I of § 5) when d has an odd prime divisor, and by a similar method when $d = 2^a$, $a\geqslant 2$:

Theorem 5. *Let χ_1 (defined after (37)) be a non-principal character. If d has an odd prime divisor, then as $x\to\infty$,*

$$N\big(n\leqslant x\colon\, d\nmid\sigma_\nu(n,\chi)\big)\sim B_1 x(\log\log x)^m(\log x)^{-\psi/h},$$

where

$$B_1>0,\qquad \frac{\psi}{h} = \min_{\substack{1\leqslant i\leqslant r\\ p_i\neq 2}}\frac{\psi_i}{h_{p_i}},\qquad \textit{and}\qquad m(d) = \max_{\substack{1\leqslant i\leqslant r\\ p_i\neq 2\\ \psi_i/h_{p_i}=\psi/h}}\left[\frac{a_i-1}{\gamma_{p_i}+1}\right].$$

If $a \geqslant 2$, *then as* $x \to \infty$,

$$N\big(n \leqslant x:\ 2^a \nmid \sigma_\nu(n, \chi)\big) \sim B_2 x (\log\log x)^{a-2} (\log x)^{-1},$$

where $B_2 > 0$.

From Theorem 1 of [10], it follows that in fact

$$B_2 = \frac{\pi^2}{2^{a+1}(a-2)!} \prod_{p|Q} \frac{p+1}{p},$$

where Q is defined by (37) with $d = 2^a$, and furthermore that

$$N\big(n \leqslant x:\ 2 \nmid \sigma_\nu(n, \chi)\big) \sim \prod_{p|2Q} (1+p^{-1/2}) x^{1/2}.$$

References

[1] E. Cohen, *Arithmetical functions associated with the unitary divisors of an integer*, Math. Zeitschr. 74 (1960), pp. 66–80.

[2] H. Delange, *Sur la distribution des entiers ayant certaines propriétés*, Ann. Sci. École Norm. Sup. (3) 73 (1956), pp. 15–74.

[3] P. Erdös und G. Szekeres, *Über die Anzahl der abelschen Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem*, Acta Scientiarum Mathematicarum, Szeged, 7 (1934), pp. 95–102.

[4] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 3rd ed., Oxford 1954.

[5] W. Narkiewicz, *Divisibility properties of a class of multiplicative functions*, Colloq. Math. 18 (1967), pp. 219–232.

[6] — *Divisibility properties of some multiplicative functions*, Colloquia Mathematica Societatis János Bolyai, 2. Number Theory, edited by P. Turán, Debrecen (Hungary) 1968, pp. 147–159.

[7] R. A. Rankin, *The divisibility of divisor function*, Proc. Glasgow Math. Assoc. 5 (1961), pp. 35–40.

[8] L. G. Sathe, *On a congruence property of the divisor functions*, Amer. J. Math. 67 (1945), pp. 397–406.

[9] E. J. Scourfield, *On the divisibility of $\sigma_\nu(n)$*, Acta Arith. 10 (1964), pp. 245–285.

[10] — *On the divisibility of a modified divisor function*, Proc. London Math. Soc. (3) 21 (1970), pp. 145–159.

[11] G. N. Watson, *Über Ramanujansche Kongruenzeigenschaften der Zerfällungsanzahlen (I)*, Math. Zeitschr. 39 (1935), pp. 712–731.

WESTFIELD COLLEGE
London NW3 7ST

Received on 13. 11. 1971 (238)

ACTA ARITHMETICA
XXII (1973)

On Waring's Problem in p-adic fields

by

M. M. DODSON (Heslington)

In this paper, which is a sequel to [5], we show that for a large enough exponent k, any p-adic integer can be represented non-trivially as a sum of less than $k^{7/8}$ kth powers of integers in any p-adic field Q_p with $(k, p-1) < \frac{1}{2}(p-1)$. As is well known the problem of representing any p-adic integer by a sum of s kth powers of p-adic integers is equivalent to finding a primitive solution of the congruence

$$x_1^k + \ldots + x_s^k \equiv N \ (\mathrm{mod}\ p^\gamma) \tag{1}$$

for any rational integer N, where $\gamma = \tau + 1$ and where τ is the exact power of the prime p which divides $2k$. In fact, as is also well known, a primitive solution of (1) implies that the congruence

$$x_1^k + \ldots + x_s^k \equiv N \ (\mathrm{mod}\ p^n) \tag{2}$$

has a primitive solution for every integer $n \geqslant 1$. The number $\Gamma(k, p^n)$ is defined to be the least s such that the congruence (2) has a primitive solution for any integer N so that $\Gamma(k, p^n) \leqslant \Gamma(k, p^\gamma)$ for every $n \geqslant 1$, i.e.

$$\Gamma(k, p^\gamma) = \operatorname*{Max}_n \Gamma(k, p^n),$$

where the maximum is taken over all positive integers. Also plainly if $s \geqslant \Gamma(k, p^\gamma)$ then every p-adic integer can be represented as a non-trivial sum of s kth powers of p-adic integers.

The number $\Gamma(k, p^\gamma)$ was introduced by Hardy and Littlewood in their work on Waring's Problem ([7]) though from a different point of view and with a different notation, namely γ_p, and they proved ([8], p. 533, Theorem 4) that if $d < \frac{1}{2}(p-1)$ then $\Gamma(k, p^\gamma) \leqslant k$, where as always $d = (k, p-1)$, the highest common factor of k and $p-1$. I. Chowla ([3], p. 197, Theorem 4) showed that if k is sufficiently large, then for all primes p with $d < \frac{1}{2}(p-1)$ we have for all sufficiently large k,

$$\Gamma(k, p^\gamma) < k^{1-c+\varepsilon},$$

i.e. for all integers $n \geqslant 1$,

$$\Gamma(k, p^n) < k^{1-c+\varepsilon},$$