ACTA ARITHMETICA
XXII (1973)

On Ramanujan's formula for values of Riemann zeta-function at positive odd integers

by

KOJI KATAYAMA (Tokyo)

1. Let $\zeta(s)$ be the Riemann's zeta-function and $B_{2\nu}$ the Bernoulli number in the positive sense ([1]). We consider the following formula $R(\nu, \eta)$ with (i), (ii) below, for every positive integer $\nu \geqslant 1$ and for a positive real η:

$$R(\nu, \eta):\ \frac{1}{(4\alpha)^\nu}\Big\{\tfrac{1}{2}\zeta(2\nu+1)+\sum_{m=1}^{\infty}\frac{1}{m^{2\nu+1}(e^{2\alpha m}-1)}\Big\} -$$

$$-\frac{1}{(-4\beta)^\nu}\Big\{\tfrac{1}{2}\zeta(2\nu+1)+\sum_{m=1}^{\infty}\frac{1}{m^{2\nu+1}(e^{2\beta m}-1)}\Big\}$$

$$=\frac{B_{2\nu+2}}{(2\nu+2)!}\{(-\alpha)^{\nu+1}+\beta^{\nu+1}\}-$$

$$-\sum_{k=1}^{[\frac{1}{2}(\nu+1)]}(-1)^k\pi^{2k}\frac{B_{2k}}{(2k)!}\,\frac{B_{2\nu+2-2k}}{(2\nu+2-2k)!}\{(-\alpha)^{\nu+1-2k}+\beta^{\nu+1-2k}\}.$$

(i) $\alpha = \pi/\eta$, $\beta = \pi\eta$, $\eta > 0$,

(ii) if ν is odd, the term corresponding to $k = \frac{1}{2}(\nu+1)$ in the last summation is multiplied by $\frac{1}{2}$.

([1]) By this we mean Bernoulli numbers such that $B_0 = 1$, $B_2 = 1/6$, $B_4 = 1/30$, $B_6 = 1/42$, $B_8 = 1/30, \ldots$ Therefore we have, for example,

$$\zeta(2\nu) = \frac{(2\pi)^{2\nu}}{2(2\nu)!}B_{2\nu}$$

and not

$$\zeta(2\nu) = \frac{(-1)^{\nu-1}(2\pi)^{2\nu}}{2(2\nu)!}B_{2\nu}.$$

In his note book, Ramanujan asserts that the above formula is valid with $\alpha\beta = \pi^2$ instead of (i) (2) and without (ii). The condition (ii) is due to the recent work of Grosswald [1], where he proved

(G1) $R(\nu, 1)$ is valid for positive odd integers ν,

(G2) the formula obtained by differentiating $R(\nu, \eta)$ with respect to η is valid for positive even integers ν and $\eta = 1$.

Grosswald, in his paper [1], says that: "Leider sind keinerlei Angaben darüber zu finden, wie Ramanujan diese beiden Sätze entdeckte, geschweige denn zu beweisen gedachte".

Therefore it will be worthy to prove $R(\nu, \eta)$ for special ν. The purpose of the present paper is to prove, as an application of the theory of the zeta-theta function, that $R(1, \eta)$ and $R(2, \eta)$ are valid.

2. We shall prove the following

THEOREM. *For $\alpha\beta = \pi^2$, $\alpha > 0$, $\beta > 0$, we have*

(a)
$$\frac{1}{(4\alpha)}\left\{\tfrac{1}{2}\zeta(3)+\sum_{m=1}^{\infty}\frac{1}{m^3(e^{2\alpha m}-1)}\right\}+\frac{1}{(4\beta)}\left\{\tfrac{1}{2}\zeta(3)+\sum_{m=1}^{\infty}\frac{1}{m^3(e^{2\beta m}-1)}\right\} = \frac{B_4}{4!}\{\alpha^2+\beta^2\}+\pi^2\frac{B_2}{2!}\frac{B_2}{2!},$$

i.e., $R(1, \eta)$ is valid,

(b)
$$\frac{1}{(4\alpha)^2}\left\{\tfrac{1}{2}\zeta(5)+\sum_{m=1}^{\infty}\frac{1}{m^5(e^{2\alpha m}-1)}\right\}-\frac{1}{(4\beta)^2}\left\{\tfrac{1}{2}\zeta(5)+\sum_{m=1}^{\infty}\frac{1}{m^5(e^{2\beta m}-1)}\right\} = \frac{B_6}{6!}\{-\alpha^3+\beta^3\}+\pi^2\frac{B_2}{2!}\frac{B_4}{4!}\{-\alpha+\beta\},$$

i.e., $R(2, \eta)$ is valid.

The proof method belongs to the elementary differential and integral calculus, once we know (G1, 2) and the series representation of the inversion formula of the zeta-theta function introduced in [2].

The proof goes on the following way, for $\nu = 1$: Differentiate $R(1, \eta)$, to be proved, with respect to η. Then it coincides with the inversion formula of the zeta-theta function at $s = 3$. Therefore the integral of the latter with respect to η is equal to $R(1, \eta)$ up to some constant. To determine it, we use (G1). Then we get $R(1, \eta)$.

For $\nu = 2$, the proof will go on the same line.

(2) If both of α and β are negative, the infinite series contained in $R(\nu, \eta)$ are divergent.

3. To begin with, we shall recall some properties of the zeta-theta function introduced in [2]. For our present purpose, we only need the series representation of it, hence we adopt here its shortcut definition (for details, see [2]). Thus the zeta-theta function $\zeta_j(\omega, s)$ for $\omega = \xi + i\eta$, $\eta > 0$, $s \in \boldsymbol{C}$, is defined as follows;

$$\zeta_j(\omega, s) = \theta_j(\omega)\mathfrak{z}(\omega, s), \quad j = 0, 1,$$

$$\theta_0(\omega) = \sum e^{-2\pi i\bar{\omega}m^2}, \quad \theta_1(\omega) = \sum e^{-2\pi i\bar{\omega}(m+1/2)^2},$$

$$\mathfrak{z}(\omega, s) = \frac{\Gamma(s/2)}{\pi^{s/2}\eta^{s/2}}\zeta(s)+\frac{\Gamma((s+1)/2)}{\pi^{(s+1)/2}}\eta^{s/2}\zeta(s+1)+ + \sum_{\substack{m\neq 0\\ n\neq 0}} e^{2\pi i\xi mn}\left|\frac{n}{m}\right|^{s/2}K_{s/2}(2\pi\eta|mn|),$$

(for $\mathrm{Re}\, s > 2$),

where $K_u(z)$ is the so-called modified Bessel function.

Put $\xi = 0$. Then the last sum in the series representation of $\mathfrak{z}(\eta i, s)$ becomes

$$4\sum_{\substack{m=1\\ n=1}}^{\infty}\left(\frac{n}{m}\right)^{s/2}K_{s/2}(2\pi\eta mn).$$

We proved in Theorem 5, [2], the transformation formula satisfied by $\mathfrak{z}(\omega, s) = {}^t(\zeta_0(\omega, s), \zeta_1(\omega, s))$, from which we can deduce the following

(1)
$$\mathfrak{z}(i/\eta, s) = \eta\cdot\mathfrak{z}(\eta i, s).$$

Put $s = 2\nu+1$. $K_{\nu+1/2}(z)$ takes the following form:

(2)
$$K_{\nu+1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\sum_{r=0}^{\nu}\frac{(\nu+r)!}{r!(\nu-r)!(2z)^r},$$

(see [3]. Warning: there is a misprint in the book, p. 72. The exponent $\frac{1}{2}$ is necessary for $\left(\frac{\pi}{2z}\right)$).

Further we know that

(3)
$$\zeta(2\nu) = \frac{(2\pi)^{2\nu}}{2\cdot(2\nu)!}B_{2\nu}.$$

By the series representation of (1) and by (2), (3), we have the following formula, which is denoted by $T(\nu,\eta)$:

$$T(\nu,\eta):\ \frac{\Gamma(\nu+1/2)\,\eta^{\nu+1/2}}{\pi^{\nu+1/2}}\,\zeta(2\nu+1)+\frac{\Gamma(\nu+1)}{\pi^{\nu+1}\eta^{\nu+1/2}}\,\frac{(2\pi)^{2\nu+2}}{2(2\nu+2)!}\,B_{2\nu+2}+$$

$$+4\sum_{\substack{m=1\\ n=1}}^{\infty}\left(\frac{n}{m}\right)^{\nu+1/2}\left(\frac{\eta}{4mn}\right)^{1/2}e^{-2\pi mn/\eta}\sum_{r=0}^{\nu}\frac{(\nu+r)!\,\eta^{r}}{r!(\nu-r)!(4\pi mn)^{r}}$$

$$=\eta\left\{\frac{\Gamma(\nu+1/2)}{\pi^{\nu+1/2}\eta^{\nu+1/2}}\,\zeta(2\nu+1)+\frac{\Gamma(\nu+1)\,\eta^{\nu+1/2}}{\pi^{\nu+1}}\,\frac{(2\pi)^{2\nu+2}}{2(2\nu+2)!}\,B_{2\nu+2}+\right.$$

$$\left.+4\sum_{\substack{m=1\\ n=1}}^{\infty}\left(\frac{n}{m}\right)^{\nu+1/2}\left(\frac{1}{4\eta mn}\right)^{1/2}e^{-2\pi\eta mn}\sum_{r=0}^{\nu}\frac{(\nu+r)!}{r!(\nu-r)!(4\pi\eta mn)^{r}}\right\}.$$

4. The proof of $R(1,\eta)$. Consider $T(1,\eta)$. We know that $\Gamma(3/2)=\pi^{1/2}/2$ and $\Gamma(2)=1$. Using the well-known formulas

$$\sum_{m=1}X^m=X/(1-X),\qquad \sum_{m=1}mX^m=X/(1-X)^2,$$

for $|X|<1$, we have the following expression of $T(1,\eta)$:

$$\frac{\eta^{3/2}}{2\pi}\,\zeta(3)+\frac{1}{\pi^2\eta^{3/2}}\,\frac{(2\pi)^4}{2\cdot 4!}\,B_4+$$

$$+2\eta^{1/2}\sum_{m=1}\frac{e^{2\pi m/\eta}}{m^2(e^{2\pi m/\eta}-1)^2}+\pi^{-1}\eta^{3/2}\sum_{m=1}\frac{1}{m^3(e^{2\pi m/\eta}-1)}$$

$$=\eta\left\{\frac{1}{2\pi\eta^{3/2}}\,\zeta(3)+\frac{\eta^{3/2}}{\pi^2}\,\frac{(2\pi)^4}{2\cdot 4!}\,B_4+\right.$$

$$\left.+2\eta^{-1/2}\sum_{m=1}\frac{e^{2\pi\eta m}}{m^2(e^{2\pi\eta m}-1)^2}+\pi^{-1}\eta^{-3/2}\sum_{m=1}\frac{1}{m^3(e^{2\pi\eta m}-1)}\right\}.$$

Multiplying both sides by $2^{-2}\cdot\eta^{-3/2}$ and moving terms appropriately, we get the formula $\mathrm{Diff}(R(1,\eta))$ obtained by differentiating $R(1,\eta)$. Here we can do obviously the termwise differentiation. Therefore the formula obtained by integrating $T(1,\eta)$ (termwise) coincides with $R(1,\eta)$ up to some constant C; namely we have the following formula

$$\frac{1}{(4\alpha)}\left\{\tfrac{1}{2}\zeta(3)+\sum\frac{1}{m^3(e^{2\alpha m}-1)}\right\}+\frac{1}{(4\beta)}\left\{\tfrac{1}{2}\zeta(3)+\sum\frac{1}{m^3(e^{2\beta m}-1)}\right\}$$

$$=\frac{B_4}{4!}\{\alpha^2+\beta^2\}+C,$$

with $\alpha=\pi/\eta$, $\beta=\pi\eta$. Then put $\eta=1$. By (G1) for $\nu=1$, we have

$$C=\pi^2\frac{B_2}{2!}\frac{B_2}{2!}.$$

Thus we proved $R(1,\eta)$.

5. The proof of $R(2,\eta)$. The proof for $\nu=2$ will go on a similar way but we need further technique. We start with the formula obtained by multiplying η to both sides of $R(2,\eta)$ (to be proved) which will be denoted by $R(2,\eta)\cdot\eta$. (Hereafter, a similar notation will be used.) Differentiate $R(2,\eta)\cdot\eta$ to get the formula $\mathrm{Diff}(R(2,\eta)\cdot\eta)$ (to be proved):

$$\mathrm{Diff}(R(2,\eta)\cdot\eta):\ \frac{3\eta^2}{(4\pi)^2}\left\{\tfrac{1}{2}\zeta(5)+\sum_{m=1}\frac{1}{m^5(e^{2\pi m/\eta}-1)}\right\}+$$

$$+\frac{2\pi\eta}{(4\pi)^2}\sum_{m=1}\frac{e^{2\pi m/\eta}}{m^4(e^{2\pi m/\eta}-1)^2}+\frac{1}{(4\pi)^2\eta^2}\left\{\tfrac{1}{2}\zeta(5)+\sum_{m=1}\frac{1}{m^5(e^{2\pi\eta m}-1)}\right\}+$$

$$+\frac{2\pi}{(4\pi)^2\eta}\sum_{m=1}\frac{e^{2\pi\eta m}}{m^4(e^{2\pi\eta m}-1)^2}=\frac{B_6}{6!}\pi^3\left(\frac{2}{\eta^3}+4\eta^3\right)+2\pi^3\frac{B_2}{2!}\frac{B_4}{4!}\eta.$$

Further we consider $\mathrm{Diff}\big(\mathrm{Diff}(R(2,\eta)\cdot\eta)\cdot\eta^{-1}\big)$ (to be proved). Then we can see easily that it coincides with the formula

$$T(2,\eta)\cdot\frac{1}{2\cdot 4\cdot\eta^{5/2}}.$$

Here we have to use $\Gamma(5/2)=3\cdot\pi^{1/2}/4$, $\Gamma(3)=2$ and

$$\sum_{m=1}m^2X^m=X(1+X)/(1-X)^3$$

in rewritting $T(2,\eta)$. Now we know that the latter holds for any $\eta>0$. By integrating $T(2,\eta)\cdot 2^{-1}\cdot 4^{-1}\cdot\eta^{-5/2}$, we see that it coincides with $\mathrm{Diff}(R(2,\eta)\cdot\eta)\cdot\eta^{-1}$ up to some constant C'. Namely we get the following:

$$\frac{3\eta}{(4\pi)^2}\left\{\tfrac{1}{2}\zeta(5)+\sum_{m=1}\frac{1}{m^5(e^{2\pi m/\eta}-1)}\right\}+\frac{2\pi}{(4\pi)^2}\sum_{m=1}\frac{e^{2\pi m/\eta}}{m^4(e^{2\pi m/\eta}-1)^2}+$$

$$+\frac{1}{(4\pi)^2\eta^3}\left\{\tfrac{1}{2}\zeta(5)+\sum_{m=1}\frac{1}{m^5(e^{2\pi\eta m}-1)}\right\}+\frac{2\pi}{(4\pi)^2\eta^2}\sum_{m=1}\frac{e^{2\pi\eta m}}{m^4(e^{2\pi\eta m}-1)^2}$$

$$=\frac{B_6}{6!}\cdot\pi^3\left(\frac{2}{\eta^4}+4\eta^2\right)+C'.$$

To determine C', put $\eta = 1$. Then we have

$$C' = 2\pi^3 \frac{B_2}{2!} \frac{B_4}{4!}$$

by (G2) for $\nu = 2$. Therefore we proved the validity of

$$\mathrm{Diff}\big(R(2, \eta)\cdot \eta\big)\cdot \eta^{-1}$$

and so of

$$\mathrm{Diff}\big(R(2, \eta)\cdot \eta\big).$$

Integrating the last formula, we see that $R(2, \eta)\cdot\eta$ is valid up to some constant C''. To determine C'', put $\eta = 1$. Then $C'' = -2\pi^3(B_2/2!)(B_4/4!)$. Thus $R(2, \eta)\cdot\eta$ and so $R(2, \eta)$ is valid.

6. We shall explain why we considered

$$\mathrm{Diff}\Big(\mathrm{Diff}\big(R(2, \eta)\cdot\eta\big)\cdot\eta^{-1}\Big)$$

for the proof of $R(2, \eta)$. $R(2, \eta)$ has the term

$$\pi^3 \frac{B_2}{2!} \frac{B_4}{4!} \left(\frac{1}{\eta} + \eta\right) \tag{4}$$

but $T(2, \eta)$ does not have the term corresponding to it. To get the statement such that

$$\mathrm{Diff}\Big(\mathrm{Diff}\big(R(2, \eta)\cdot\eta\big)\cdot\eta^{-1}\Big) \text{ coincides with } T(2, \eta)\cdot\frac{1}{2\cdot 4\cdot \eta^{5/2}}$$

up to some constant,

we have to remove (4) by differentiating $R(2, \eta)$ several times. In this way, we can conjecture that the proof will go on the same lines for general ν.

Added after completion of this paper: the author was informed that in [4], Professor E. Grosswald proved Ramanujan's formula in full generality. The reason why the author wishes to publish this paper is that his method is different from Grosswald's and his emphasis is in the point that the proof has been obtained as an application of the theory of the zeta-theta function. The auther expresses hearty thanks to Professor E. Grosswald for his kind letter and comments.

References

[1] E. Grosswald, *Die Werte der Riemannschen zetafunktion an ungeraden Argumentstellen*, Göttingen Nachrichten 1970, Nr. 2.

[2] K. Katayama, *On zeta-theta functions*, Journ. Math. Soc. Japan. 24(1972), pp. 307–332.

[3] Magnus, Oberhettinger and Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer.

[4] E. Grossswald, *Comments on some formulae of Ramanujan*, Acta Arith. 21 (1972), pp. 25–34.

DEPARTMENT OF MATHEMATICS
TSUDA COLLEGE
Kodaira, Tokyo, Japan

Received on 4. 11. 1971 (234)