4. Concluding remark. As a natural generalization of the Lindelöf hypothesis on the Riemann $\zeta$ function we may introduce the hypothesis

\[ L(\frac{1}{2} + it, \chi) \leq C_1(t, \epsilon)|t|^\epsilon, \]

where $\epsilon$ is an arbitrarily small positive number.

From this we can deduce

\[ \mathcal{N}(\alpha, T; q) \leq C_2(q, \epsilon)T^{\epsilon}, \quad \sigma > \frac{1}{2} + \sqrt{\epsilon} \]

by the method of Halász and Turán [3]. This strong result gives

**Theorem 3.** Under the assumption of the "generalized Lindelöf hypothesis" (*), the inequality

\[ f(T) \geq C_3(q, \epsilon)T^{1/4 - \epsilon} \]

holds.

References


Mathematical Institute
Of the Hungarian Academy of Sciences, Budapest

Received on 15.5.1971

\[ 2^{2m} - 1 \sum_{n=1}^{\varphi(m)} \frac{(-1)^{n-1}B_n a^n}{(2n)!} \]

we prove the

**Theorem.** For $m \geq 1$,

\[ 2 \frac{[2^{2m} - 1]B_m}{[\varphi_m] + 1} \]

where $[x]$ denotes the greatest integer $\leq x$, and

\[ \varphi_m = \frac{2^{2m} - 1}{2^{2m} - 1 - 2^{2m}} \sum_{n=1}^{2m} \frac{1}{n^{2m}}. \]

§ 2. As is well-known, writing $\zeta(s)$ for the Riemann zeta function, we have, for $m \geq 1$

\[ \zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \frac{2^{2m} - 1 - \pi^{2m} P_m}{(2m)!}. \]

In what follows we shall suppose $m \geq 2$ and use (3) and von Staudt's theorem to prove (1) and (2). Now

\[ \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \sum_{n=1}^{3m} \frac{1}{n^{2m}} + \sum_{n=1}^{\infty} \frac{1}{n^{2m}}. \]

Write

\[ \sigma(x) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}}, \]
Thus (4) becomes
\[
\zeta(2m) = \sigma(3m) + 2m \left( \frac{\sigma(3m)}{(2m)!} \right) + \theta_1 \left( \frac{\sigma(3m)}{(3m)^{2m}} \right) + \delta \left( \frac{2\theta_1}{(3m)^{2m}} \right) (0 < \theta_1 < 1)
\]
\[
= \sigma(3m) + \frac{2\theta_1}{(3m)^{2m}} (0 < \theta_1 < 1)
\]
since \( \frac{3m}{2m-1} \leq 2 \) for \( m \geq 2 \).

Thus
\[
B_m = \frac{(2m)! \left\{ \sigma(3m) + \frac{2\theta_1}{(3m)^{2m}} \right\}}{2^{2m-1}(2m)^{2m}}.
\]
(5)

\[
= \frac{(2m)! \sigma(3m)}{2^{2m-1}(2m)^{2m}} + \frac{2\theta_2}{(3m)^{2m}} (0 < \theta_2 < 1)
\]

where \( 0 < \theta_2 < 1 \). We used here the crude inequality: \( n^k \geq n! \).

We now establish a lemma, of interest in itself, which we derive from von Staudt's theorem — but, a direct independent proof is, most surely, easy.

LEMMA. \( 2(2^{2m}-1)B_m \in \mathbb{Z} \) (one easily verifies this for \( m = 1, 2, 3 \)).

Proof. By von Staudt's theorem we have

\[
B_m = \frac{s}{u} \quad (s, u \in \mathbb{Z})
\]

where

\[
u = \prod_{p \text{ prime}} p.
\]

Now if \( p \) is an odd prime and \( (p-1)\mid 2m \) we have

\[
p \mid (2^{p-1}-1)(2^{2m}-1).
\]

Hence \( 2^{2m}-1 \) is divisible by all odd primes \( p \) such that \( (p-1)\mid 2m \). As for \( p = 2 \), we know from von Staudt's theorem that \( B_m = g/2h \) where \( h \) and \( g \) are odd. Thus we have proved that

\[
2(2^{2m}-1)B_m
\]
is an integer (positive, of course).

Multiplying both sides of (5) by \( 2(2^{2m}-1) \) and using the lemma, we see that for \( m \geq 2 \):

\[
2(2^{2m}-1)B_m = I \quad \text{(an integer)}
\]

\[
= \frac{2(2^{2m}-1)(2m)! \sigma(3m)}{2^{2m-1} \pi^{2m}} + \frac{4\theta_2}{(3m)^{2m}} \frac{(2^{2m}-1)}{(3m)^{2m}}
\]

\[
= \varphi_m + e_m
\]

where \( e_m \to 0 \) as \( m \to \infty \). In fact

\[
e_m = O \left( \frac{2}{3\pi} \cdot \left| \frac{1}{m} \right| \right).
\]

Since \( e_m \leq \frac{8}{\pi^2} \) for \( m \geq 2 \) we see that (6) gives

\[
2(2^{2m}-1)B_m = \varphi_m + 1 \quad \text{for} \quad m \geq 2.
\]

We easily verify that (8) holds for \( m = 1 \). Thus we have proved:

THEOREM. For \( m \geq 1 \), we have

\[
2(2^{2m}-1)B_m = [\varphi_m] + 1
\]

where

\[
\varphi_m = \frac{2(2^{2m}-1)(2m)!}{2^{2m-1} \pi^{2m}} \sum_{n=1}^{\infty} \frac{1}{n^{2m}}.
\]

INSTITUTE FOR ADVANCED STUDY, Princeton, N.J.
BLOOMSBURG STATE COLLEGE, Bloomsburg, Pa.

Received on 18.10.1971