4. Concluding remark. As a natural generalization of the Lindelöf hypothesis on the Riemann ζ-function we may introduce the hypothesis

$$(*)\qquad L(\tfrac{1}{2}+it,\ \chi) \leqslant C_1(q,\varepsilon)|t|^{\varepsilon},$$

where ε is an arbitrarily small positive number.

From this we can deduce

$$N(\alpha, T; q) \leqslant C_2(q,\varepsilon)T^{\sqrt{\varepsilon}}, \qquad \sigma \geqslant \tfrac{3}{4}+\sqrt{\varepsilon}$$

by the method of Halász and Turán [3]. This strong result gives

THEOREM 3. *Under the assumption of the "generalized Lindelöf hypothesis"* (*), *the inequality*

$$f(T) \geqslant C_3(q,\varepsilon)T^{1/4-10\sqrt{\varepsilon}}$$

holds.

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An "exact" formula for the m-th Bernoulli number

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§ 1. Defining the Bernoulli numbers by

$$\frac{x}{e^x-1} = 1-\frac{x}{2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_n x^{2n}}{(2n)!}$$

we prove the

THEOREM. *For* $m \geqslant 1$,

$$(1)\qquad 2(2^{2m}-1)B_m = [\varphi_m]+1$$

where $[x]$ *denotes the greatest integer* $\leqslant x$, *and*

$$(2)\qquad \varphi_m = \frac{2(2^{2m}-1)(2m)!}{2^{2m-1}\pi^{2m}}\sum_{n=1}^{3m}\frac{1}{n^{2m}}.$$

§ 2. As is well-known, writing $\zeta(s)$ for the Riemann zeta function, we have, for $m \geqslant 1$

$$(3)\qquad \zeta(2m) = \sum_{n=1}^{\infty}\frac{1}{n^{2m}} = \frac{2^{2m-1}\pi^{2m}B_m}{(2m)!}.$$

In what follows we shall suppose $m \geqslant 2$ and use (3) and von Staudt's theorem to prove (1) and (2). Now

$$(4)\qquad \sum_{n=1}^{\infty}\frac{1}{n^{2m}} = \sum_{1}^{3m}\frac{1}{n^{2m}}+\sum_{3m+1}^{\infty}\frac{1}{n^{2m}}.$$

Write

$$\sigma(x) = \sum_{n=1}^{x}\frac{1}{n^{2m}}.$$

Thus (4) becomes

$$\zeta(2m) = \sigma(3m) + \theta_1 \int_{3m}^{\infty} \frac{dx}{x^{2m}} \qquad (0 < \theta_1 < 1)$$

$$= \sigma(3m) + \frac{(3m)^{1-2m}}{2m-1}\,\theta_1$$

$$= \sigma(3m) + \frac{2\theta_2}{(3m)^{2m}} \qquad (0 < \theta_2 < 1)$$

since $\frac{3m}{2m-1} \leqslant 2$ for $m \geqslant 2$.

Thus

(5) $$B_m = \frac{(2m)!\left\{\sigma(3m) + \frac{2\theta_2}{(3m)^{2m}}\right\}}{2^{2m-1}\pi^{2m}}$$

$$= \frac{(2m)!\,\sigma(3m)}{2^{2m-1}\pi^{2m}} + \frac{2\theta_3(2m)^{2m}}{(3m)^{2m}2^{2m-1}\pi^{2m}}$$

where $0 < \theta_3 < 1$. We used here the crude inequality: $n^n \geqslant n!$.

We now establish a lemma, of interest in itself, which we derive from von Staudt's theorem — but, a direct independent proof is, most surely, easy.

LEMMA. $2(2^{2m}-1)B_m \in Z$ (one easily verifies this for $m = 1, 2, 3$).

Proof. By von Staudt's theorem we have

$$B_m = \frac{s}{q} \qquad (s, q \in Z)$$

where

$$q = \prod_{\substack{p \text{ prime} \\ (p-1)|2m}} p.$$

Now if p is an odd prime and $(p-1)|2m$ we have

$$p\,|(2^{p-1}-1)|\,(2^{2m}-1).$$

Hence $2^{2m}-1$ is divisible by all odd primes p such that $(p-1)|2m$. As for $p = 2$, we know from von Staudt's theorem that $B_m = g/2h$ where h and g are odd. Thus we have proved that

$$2(2^{2m}-1)B_m$$

is an integer (positive, of course).

Multiplying both sides of (5) by $2(2^{2m}-1)$ and using the lemma, we see that for $m \geqslant 2$

(6) $$2(2^{2m}-1)B_m = I \quad \text{(an integer)}$$

$$= \frac{2(2^{2m}-1)\,(2m)!\,\sigma(3m)}{2^{2m-1}\pi^{2m}} + \frac{4\theta_3(2m)^{2m}(2^{2m}-1)}{(3m)^{2m}2^{2m-1}\pi^{2m}}$$

$$= \varphi_m + \varrho_m$$

where $\varrho_m \to 0$ as $m \to \infty$. In fact

(7) $$\varrho_m = O\left(\left(\frac{2}{3\pi}\right)^{2m}\right).$$

Since $\varrho_m \leqslant 8/\pi^2$ for $m \geqslant 2$ we see that (6) gives

(8) $$2(2^{2m}-1)B_m = [\varphi_m]+1 \quad \text{for} \quad m \geqslant 2.$$

We easily verify that (8) holds for $m = 1$. Thus we have proved:

THEOREM. *For* $m \geqslant 1$, *we have*

(9) $$2(2^{2m}-1)B_m = [\varphi_m]+1$$

where

(10) $$\varphi_m = \frac{2(2^{2m}-1)\,(2m)!}{2^{2m-1}\pi^{2m}} \sum_{n=1}^{3m} \frac{1}{n^{2m}}.$$

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