On the distribution of the zeros of Dirichlet’s $L$-functions

by

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1. Let

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} \quad (s = \sigma + it)$$

be a Dirichlet’s $L$-function with a character $\chi \mod q$.

We take an arbitrary pair $(a, b)$, where $a \not\equiv b \mod q$ and $(a, q) = (b, q) = 1$, and let consider the function

$$F_{a,b}(s) = F(s) = \frac{1}{\varphi(q)} \sum_{\chi \mod q} (\overline{\chi}(b) - \overline{\chi}(a)) \frac{L'(s)}{L(s, \chi)}.$$ 

For $\sigma > 1$ this function has the expression

$$F(s) = \sum_{n \equiv a \mod q} \frac{\Lambda(n)}{n^s} - \sum_{n \equiv b \mod q} \frac{\Lambda(n)}{n^s},$$

where $\Lambda(n)$ is the von Mangoldt function.

Let $\pi(x; q, a)$ denote the number of primes $\leq x$ which are congruent to $a \mod q$. In the well-known series of papers, “Comparative Prime Number Theory”, Knapowski and Turán developed the deep theory on the difficult problem of whether $\pi(x; q, a) - \pi(x; q, b)$ changes sign infinitely often and how large the discrepancy is.

In their first paper of the series ([4], p. 306), they mentioned that the singularities of $F(s)$ play a vital part, and set out the problem of whether there is a zero of $L(s, \chi)$ in the critical strip such that the expression

$$\mu_{a,b}(\xi) = \mu(\xi) = \frac{1}{\varphi(q)} \sum_{\chi \mod q} (\overline{\chi}(b) - \overline{\chi}(a)) m_\xi(\chi)$$

does not vanish, where $m_\xi(\chi)$ denotes the multiplicity of $\chi$ as a zero of $L(s, \chi)$. Obviously $\xi$ is a singular point of $F(s)$ if and only if $\mu(\xi) \neq 0$. 
The existence of infinitely many $\rho$'s with $\mu(\rho) \neq 0$ has been proved by Kátaif (unpublished) and Grosswald [2] independently. Later Turán [7] took up this problem again and obtained the following result:

Let $f_{a,b}(T) = f(T)$ be the quantity

$$
\sum_{p \leq T \log p \leq f(T)} 1.
$$

Then,

(I) For $T > \psi(q)$ we have the inequality

$$
f(T) > C_1 \exp((\log T)^{1/4}).
$$

(II) Under the assumption of the generalized Riemann hypothesis we have

$$
f(T) > C_2 T^{1/2} \quad \text{for} \quad T > \psi(q),
$$

where $C_1, C_2$ are numerical constants and $\psi(q)$ an explicit function of $q$, and moreover the estimations are independent of $a$ and $b$.

The aim of this short paper is to improve substantially the inequality (I) by proving

**Theorem 1.** For $T > \psi(q)$ we have

$$
f(T) > T^{1/10} (\log T)^{-3},
$$

where the estimation is independent of $a$ and $b$.

It is desirable to obtain a similar result which is uniform in $q$ and holds for small $T$. In this case the problem becomes very difficult, and we have proved only

**Theorem 2.** For any sufficiently large $T$ there exists at least one $q$, with

$$\frac{1}{2} T^{1/2} (\log T)^{-\delta} < q < T^{1/2} (\log T)^{-\delta}$$

such that the inequality

$$f(T) > T^{1/12} (\log T)^{-\delta}$$

holds for any pair $(a, b)$.

2. Proof of Theorem 1. Let $N(\alpha, T; q)$ be the number of the zeros of all $L(s, \chi)\mod q$ in the rectangular region

$$0 \leq \sigma \leq T, \quad a \leq \sigma \leq 1.$$

According to the recent work of Montgomery [3] we have

$$N(\alpha, T; q) \leq C_4 (q^2 T)^{1/2 \theta} (\log q T)^{1/2}.
$$

We divide the horizontal strip $T/2 \leq \sigma \leq T$ into thinner strips

$$T/2 + jU \leq \sigma \leq T/2 + (j+1)U, \quad j = 0, 1, 2, \ldots, [T/2U]
$$

where $U$ satisfies

$$T^{23} \gg U \gg T^{11},$$

and is to be determined explicitly later. Then we have $[T/2U]$ rectangular regions

$$D_j(\alpha) : T/2 + jU \leq \sigma \leq T/2 + (j+1)U, \quad a \leq \sigma \leq 1.$$

Now it is easy to see that, if we have the inequality

$$[T/2U] > N(\alpha, T; q),$$

then at least one of $D_j(\alpha)$'s is free from the zeros of all $L(s, \chi)\mod q$. Let $D_{\mu}(\alpha)$ be one of such regions.

From now on we proceed on the line of Turán [7].

Now if we take $T \gg \psi(q)$, then by the condition (2.2) we have

$$a \left( \frac{U}{100 \log T}; q, \alpha \right) - \frac{U}{200 \log T}; q, \alpha \right) > 0,$$

and so there is a prime number $P$ such that

$$P = a \mod q, \quad \frac{U}{200 \log T} \leq P \leq \frac{U}{100 \log T}.$$ 

Let

$$\delta = \log P \quad \text{and} \quad \lambda = \frac{1}{100 P^{1/2} \log P}$$

and let consider the integral

$$J = \frac{1}{2\pi i} \int_{(\delta)} F(\alpha + iV)e^{2\pi i \alpha + \frac{i}{2} V} d\alpha,$$

where $V$ is equal to $T/2 + (j_{\alpha} + 1)U$, so that $1 + iV$ is the middle point of the right edge of the rectangle $D_{\mu}(\alpha)$.

As in [7] we have easily

$$2\sqrt{\lambda} \pi J = P^{-1/2} \log P + o(1).$$

On the other hand, shifting the line of integration to $\sigma = -\frac{1}{2}$ we have

$$J = \sum_{\mu} \mu(q) e^{(a-\tfrac{1}{2} b)(\sigma - \tfrac{1}{2}) + \sigma} + o(1),$$

where $\mu$ runs over all non-trivial zeros of all $L(s, \chi)\mod q$.

Now, denoting $\nu = \beta + i\gamma$, the contribution of the zeros with

$$|V - \gamma| > U/4,$$
to the above sum does not exceed
\[ \sum_{\mid P - n \mid < U^{1/4}} |\mu(q)| e^{-\epsilon \log q} P \leq C_2(q) \sum_{n \leq U^{1/4}} \log^2 n \cdot e^{-\frac{\epsilon}{2} x}, \]
since we have
\[ N(a, T+1; q) - N(a, T; q) \leq C_3(q \log T). \]
The last sum does not exceed
\[ \int_{U^{1/4}}^\infty \log^2 x e^{-\frac{\epsilon}{2} x} \, dx \leq e^{-\frac{\epsilon}{2} \log (2)} = o(1), \]
since we have (2.4).
Hence we get
\[ f(T) = \sum_{\mid P - n \mid < U^{1/4}} \mu(q) e^{i \rho(q) + \epsilon \log q} + o(1). \]
From this and (2.5) we have
\[ \sum_{\mid P - n \mid < U^{1/4}} \mu(q) e^{i \rho(q) + \epsilon \log q} \geq P \log^{3/4} P. \]
Because of the definition of \( V, \) in the range of the above summation we have
\[ \beta \leq \alpha. \]
Hence the above inequality gives
\[ \sum_{\mid P - n \mid < U^{1/4}} |\mu(q)| \geq \frac{1}{2} P^{1-\epsilon} \log^{3/4} P. \]
This means that we have obtained
\[ f(T) \geq C_2(q) P^{1-\epsilon} \log^{3/2} P. \]
Finally we put
\[ a = \frac{4}{5}, \]
and then from (2.1), (2.3) we have to set
\[ U = C_3(q) T^{1/3} \log^{-\epsilon} T, \]
which is in the range (2.2) and gives the estimate (1.1) with (2.4) and (2.6).

3. Proof of Theorem 2. We now enter into the proof of the inequality (1.2), but we shall show only important points.
From Bombieri's theorem ([1], p. 159) we get the inequality
\[ \sum_{e < \mid \rho(q) \mid < 2} \max_{\alpha, q \neq 0} |\pi(x; q, a) - \pi(x/2; q, a) - \frac{1}{\varphi(q)} \int_{x/2}^x \frac{du}{\log u}| < x(\log x)^{-2} \]
for sufficiently large \( x. \)
4. Concluding Remark. As a natural generalization of the Lindelöf hypothesis on the Riemann $\zeta$-function we may introduce the hypothesis

\[(*) \quad L(\frac{1}{2} + it, \chi) \leq C_1(g, \varepsilon)|t|^{\sigma'},\]

where $\varepsilon$ is an arbitrarily small positive number.

From this we can deduce

\[N(a, T; q) \leq C_1(g, \varepsilon)T^{\sigma'}, \quad \sigma' = \frac{1}{2} + \varepsilon\]

by the method of Halász and Turán [3]. This strong result gives

**Theorem 3.** Under the assumption of the "generalized Lindelöf hypothesis" (*), the inequality

\[f(T) \leq C_1(g, \varepsilon)T^{1/2 - \varepsilon}\]

holds.

References


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An "exact" formula for the $m$-th Bernoulli number

by

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§ 1. Defining the Bernoulli numbers by

\[
\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}B_n x^n}{(2n)!}
\]

we prove the

**Theorem.** For $m > 1$,

\[(1) \quad 2 [2^{2m} - 1] B_m = [x_m] + 1\]

where $[x]$ denotes the greatest integer $\leq x$, and

\[(2) \quad x_m = \frac{2 [2^{2m} - 1] (2m)!}{\zeta(2m - 1) \pi^{2m}} \sum_{n=1}^{2m} \frac{1}{n^{2m}}\]

§ 2. As is well-known, writing $\zeta(s)$ for the Riemann zeta function, we have, for $m > 1$,

\[(3) \quad \zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \frac{2^{2m-1} \pi^{2m} B_m}{(2m)!}\]

In what follows we shall suppose $m \geq 2$ and use (3) and von Staudt's theorem to prove (1) and (2). Now

\[(4) \quad \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \sum_{n=1}^{3m} \frac{1}{n^{2m}} + \sum_{n=3m+1}^{\infty} \frac{1}{n^{2m}}\]

Write

\[\sigma(x) = \sum_{n=1}^{x} \frac{1}{n^{2m}}\]