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On the distribution of the zeros of Dirichlet's L-functions

by

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1. Let

$$L(s,\chi) = \sum_{n=1}^{\infty}\chi(n)n^{-s} \qquad (s = \sigma+it)$$

be a Dirichlet's L-function with a character $\chi \bmod q$.

We take an arbitrary pair (a, b), where $a \not\equiv b \bmod q$ and $(a, q) = (b, q) = 1$, and let consider the function

$$F_{a,b}(s) = F(s) = \frac{1}{\varphi(q)}\sum_{\chi \bmod q}\big(\bar{\chi}(b)-\bar{\chi}(a)\big)\frac{L'}{L}(s,\chi).$$

For $\sigma > 1$ this function has the expression

$$F(s) = \sum_{n\equiv a \bmod q}\frac{\Lambda(n)}{n^s} - \sum_{n\equiv b \bmod q}\frac{\Lambda(n)}{n^s},$$

where $\Lambda(n)$ is the von Mangoldt function.

Let $\pi(x; q, a)$ denote the number of primes $\leqslant x$ which are congruent to $a \bmod q$. In the well-known series of papers, "Comparative Prime Number Theory", Knapowski and Turán developed the deep theory on the difficult problem of whether $\pi(x; q, a)-\pi(x; q, b)$ changes sign infinitely often and how large the discrepancy is.

In their first paper of the series ([4], p. 306), they mentioned that the singularities of $F(s)$ play a vital part, and set out the problem of whether there is a zero of $L(s, \chi)$ in the critical strip such that the expression

$$\mu_{a,b}(\varrho) = \mu(\varrho) = \frac{1}{\varphi(q)}\sum_{\chi \bmod q}\big(\bar{\chi}(b)-\bar{\chi}(a)\big)m_{\varrho}(\chi)$$

does not vanish, where $m_{\varrho}(\chi)$ denotes the multiplicity of ϱ as a zero of $L(s, \chi)$. Obviously ϱ is a singular point of $F(s)$ if and only if $\mu(\varrho) \neq 0$.

The existence of infinitely many ϱ's with $\mu(\varrho) \neq 0$ has been proved by Kátai (unpublished) and Grosswald [2] independently. Later Turán [7] took up this problem again and obtained the following result:

Let $f_{a,b}(T) = f(T)$ be the quantity

$$\sum_{\substack{0 \leqslant \operatorname{Im} \varrho \leqslant T \\ \mu(\varrho) \neq 0}} 1.$$

Then,

(I) For $T > \psi(q)$ we have the inequality

$$f(T) > C_1 \exp\big((\log T)^{1/5}\big).$$

(II) Under the assumption of the generalized Riemann hypothesis we have

$$f(T) > C_2 T^{1/2} \quad \text{for} \quad T > \psi(q),$$

where C_ν are numerical constants and $\psi(q)$ an explicit function of q, and moreover the estimations are independent of a and b.

The aim of this short paper is to improve substantially the inequality (I) by proving

THEOREM 1. *For $T > \psi(q)$ we have*

$$f(T) > T^{1/10}(\log T)^{-3}. \tag{1.1}$$

Here the estimation is independent of a and b.

It is desirable to obtain a similar result which is uniform in q and holds for small T. In this case problem becomes very difficult, and we have proved only

THEOREM 2. *For any sufficiently large T there exists at least one q, with*

$$\tfrac{1}{2} T^{1/2}(\log T)^{-51} \leqslant q \leqslant T^{1/2}(\log T)^{-51},$$

such that the inequality

$$f(T) > T^{3/28}(\log T)^{-45} \tag{1.2}$$

holds for any pair (a, b).

2. Proof of Theorem 1. Let $N(\alpha, T; q)$ be the number of the zeros of all $L(s, \chi) \bmod q$ in the rectangular region

$$0 \leqslant t \leqslant T, \quad \alpha \leqslant \sigma \leqslant 1.$$

According to the recent work of Montgomery [5] we have

$$N(\alpha, T; q) \leqslant C_3 (q^2 T)^{\frac{5}{2}(1-\alpha)} (\log qT)^{13}. \tag{2.1}$$

We devide the horizontal strip

$$T/2 \leqslant t \leqslant T$$

into thinner strips

$$T/2 + jU \leqslant t < T/2 + (j+1)U, \quad j = 0, 1, 2, \ldots, [T/2U]$$

where U satisfies

$$T^{2/3} \geqslant U \geqslant T^{1/3} \tag{2.2}$$

and is to be determined explicitly later. Then we have $[T/2U]$ rectangular regions

$$\Delta_j(\alpha)\colon\ T/2 + jU \leqslant t \leqslant T/2 + (j+1)U, \quad \alpha \leqslant \sigma \leqslant 1.$$

Now it is easy to see that, if we have the inequality

$$[T/2U] > N(\alpha, T; q), \tag{2.3}$$

then at least one of $\Delta_j(\alpha)$'s is free from the zeros of all $L(s, \chi) \bmod q$. Let $\Delta_{j_0}(\alpha)$ be one of such regions.

From now on we proceed on the line of Turán [7].

Now if we take $T \geqslant \psi(q)$, then by the condition (2.2) we have

$$\pi\Big(\frac{U}{100 \log T}; q, a\Big) - \pi\Big(\frac{U}{200 \log T}; q, a\Big) > 0,$$

and so there is a prime number P such that

$$P \equiv a \bmod q, \quad \frac{U}{200 \log T} \leqslant P \leqslant \frac{U}{100 \log T}. \tag{2.4}$$

Let

$$\delta = \log P \quad \text{and} \quad \lambda = \frac{1}{100 P^2 \log P}$$

and let consider the integral

$$J = \frac{1}{2\pi i} \int_{(2)} F(s + iV) e^{\lambda s^2 + \delta s} ds,$$

where V is equal to $T/2 + (j_0 + \frac{1}{2})U$, so that $1 + iV$ is the middle point of the right edge of the rectangle $\Delta_{j_0}(\alpha)$.

As in [7] we have easily

$$2\sqrt{\lambda\pi} J = P^{-iV} \log P + o(1). \tag{2.5}$$

On the other hand, shifting the line of integration to $\sigma = -\frac{3}{2}$ we have

$$J = \sum_{\varrho} \mu(\varrho) e^{\lambda(\varrho - iV)^2 + \delta(\varrho - iV)} + o(1),$$

where ϱ runs over all non-trivial zeros of all $L(s, \chi) \bmod q$.

Now, denoting $\varrho = \beta + i\gamma$, the contribution of the zeros with

$$|V - \gamma| \geqslant U/4$$

to the above sum does not exceed

$$\sum_{|V-\gamma|\geqslant U/4} |\mu(\varrho)| e^{-\lambda(V-\gamma)^2} P^\beta \leqslant C_1(q) \sum_{n\geqslant U/4} \log^2 n\cdot e^{-\lambda n^2},$$

since we have

$$N(\alpha, T+1; q) - N(\alpha, T; q) \leqslant C_2(q)\log T.$$

The last sum does not exceed

$$\int_{U/4}^{\infty} \log^2 x e^{-\lambda x^2} dx \leqslant \frac{e^{-\lambda U^2/16}}{2\lambda} = o(1),$$

since we have (2.4).

Hence we get

$$J = \sum_{|V-\gamma|\leqslant U/4} \mu(\varrho) e^{\lambda(\varrho-iV)^2+\delta(\varrho-iV)} + o(1).$$

From this and (2.5) we have

$$\Big|\sum_{|V-\gamma|\leqslant U/4} \mu(\varrho) e^{\lambda(\varrho-iV)^2+\delta(\varrho-iV)}\Big| \geqslant P\log^{3/2} P.$$

Because of the definition of V, in the range of the above summation we have

$$\beta \leqslant \alpha.$$

Hence the above inequality gives

$$\sum_{|V-\gamma|\leqslant U/4} |\mu(\varrho)| \geqslant \tfrac{1}{2} P^{1-\alpha}\log^{3/2} P.$$

This means that we have obtained

(2.6) $$f(T) \geqslant C_3(q) P^{1-\alpha}\log^{1/2} P.$$

Finally we put

$$\alpha = 4/5,$$

and then from (2.1), (2.3) we have to set

$$U = C_4(q) T^{1/2}\log^{-13} T,$$

which is in the range (2.2) and gives the estimate (1.1) with (2.4) and (2.6).

3. Proof of Theorem 2. We now enter into the proof of the inequality (1.2), but we shall show only important points.

From Bombieri's theorem ([1], p. 159) we get the inequality

$$\sum_{q\leqslant \sqrt{x}(\log x)^{-50}} \max_{(a,q)=1} \Big|\pi(x; q, a) - \pi(x/2; q, a) - \frac{1}{\varphi(q)}\int_{x/2}^{x} \frac{du}{\log u}\Big| < x(\log x)^{-2}$$

for sufficiently large x.

Here we remark that we have

$$\sum_{q\leqslant x} \frac{1}{\varphi(q)} = A\big(1+o(1)\big)\log x$$

with an absolute constant A.

From these two facts it is easy to see that there exists at least one q, with

(3.1) $$\tfrac{1}{2}\sqrt{x}(\log x)^{-50} \leqslant q \leqslant \sqrt{x}(\log x)^{-50},$$

such that

$$\max_{(a,q)=1} \Big|\pi(x; q, a) - \pi(x/2; q, a) - \frac{1}{\varphi(q)}\int_{x/2}^{x} \frac{du}{\log u}\Big| = o\left(\frac{x}{\varphi(q)\log x}\right).$$

But this means that every reduced residue-class mod q contains at least one prime in the interval $(x/2, x)$. We now fix this q and let Q be a prime number with

$$Q \equiv a \bmod q, \quad x/2 \leqslant Q \leqslant x.$$

Such Q can be found for any $(a, q) = 1$, and so our discussion in what follows is independent of a and b.)

Now as in [7] we have

$$\Big|\sum_{T/2\leqslant\gamma\leqslant T} \mu(\varrho) e^{\lambda'(\varrho-iT)^2+\delta'(\varrho-iT)}\Big| > Q\log^{3/2} Q,$$

where

(3.2) $$T = 60Q\log Q, \quad \lambda' = \frac{1}{100Q^2\log Q}, \quad \delta' = \log Q.$$

Hence we have

(3.3) $$Q(\log Q)^{3/2} \leqslant \sum_{T/2\leqslant\gamma\leqslant T} |\mu(\varrho)| Q^\beta \leqslant 2\frac{Q}{\varphi(q)} N(\alpha', T; q) + C_4 \log q\cdot Q^{\alpha'} f(T)$$

for any $1/2 < \alpha' < 1$.

Here we have

$$N(\alpha', T; q) \leqslant C_5 q^{\frac{28}{3}(1-\alpha')} (\log q)^{300}$$

from a result of Tatuzawa ([6], p. 299) and (3.1), (3.2).

We now take

$$\alpha' = \frac{25}{28}\left(1+36\frac{\log\log q}{\log q}\right)$$

and with (3.3) this gives

$$Q^{3/28}(\log Q)^{-40} \leqslant f(T)$$

which proves the estimate (1.2).

4. Concluding remark. As a natural generalization of the Lindelöf hypothesis on the Riemann ζ-function we may introduce the hypothesis

$$L(\tfrac{1}{2}+it,\ \chi) \leqslant C_1(q,\ \varepsilon)|t|^{\varepsilon}, \tag{$*$}$$

where ε is an arbitrarily small positive number.

From this we can deduce

$$N(a,\ T;\ q) \leqslant C_2(q,\ \varepsilon)T^{\sqrt{\varepsilon}}, \qquad \sigma \geqslant \tfrac{3}{4}+\sqrt{\varepsilon}$$

by the method of Halász and Turán [3]. This strong result gives

THEOREM 3. *Under the assumption of the "generalized Lindelöf hypothesis" ($*$), the inequality*

$$f(T) \geqslant C_3(q,\ \varepsilon)T^{1/4-10\sqrt{\varepsilon}}$$

holds.

References

[1] H. Davenport, *Multiplicative Number Theory*, Chicago 1967.
[2] E. Grosswald, *Sur une propriété des racines complexes des fonctions $L(s,\chi)$*, C. R. Acad. Sc. Paris 260 (1965), pp. 4299–4302.
[3] G. Halász and P. Turán, *On the distribution of roots of Riemann zeta and allied functions I*, J. Number Theory 1 (1969), pp. 121–137.
[4] S. Knapowski and P. Turán, *Comparative prime-number theory I*, Acta Math. Acad. Sci. Hung. 8 (1962), pp. 299–314.
[5] H. L. Montgomery, *Zeros of L-functions*, Inventiones Math. 8 (1969), pp. 346–354.
[6] K. Pracher, *Primzahlverteilung*, Berlin 1957.
[7] P. Turán, *On a problem concerning the zeros of Dirichlet's L-functions*, Publ. Ramanujan Inst. 1 (1969), pp. 95–100.

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An "exact" formula for the m-th Bernoulli number

by

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§ 1. Defining the Bernoulli numbers by

$$\frac{x}{e^x-1} = 1-\frac{x}{2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_n x^{2n}}{(2n)!}$$

we prove the

THEOREM. *For* $m \geqslant 1$,

$$2(2^{2m}-1)B_m = [\varphi_m]+1 \tag{1}$$

where $[x]$ *denotes the greatest integer* $\leqslant x$, *and*

$$\varphi_m = \frac{2(2^{2m}-1)(2m)!}{2^{2m-1}\pi^{2m}}\sum_{n=1}^{3m}\frac{1}{n^{2m}}. \tag{2}$$

§ 2. As is well-known, writing $\zeta(s)$ for the Riemann zeta function, we have, for $m \geqslant 1$

$$\zeta(2m) = \sum_{n=1}^{\infty}\frac{1}{n^{2m}} = \frac{2^{2m-1}\pi^{2m}B_m}{(2m)!}. \tag{3}$$

In what follows we shall suppose $m \geqslant 2$ and use (3) and von Staudt's theorem to prove (1) and (2). Now

$$\sum_{n=1}^{\infty}\frac{1}{n^{2m}} = \sum_{1}^{3m}\frac{1}{n^{2m}}+\sum_{3m+1}^{\infty}\frac{1}{n^{2m}}. \tag{4}$$

Write

$$\sigma(x) = \sum_{n=1}^{x}\frac{1}{n^{2m}}.$$