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A problem involving simultaneous binary compositions*

by

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1. Introduction. Throughout the following, l, h, and k will denote fixed positive integers with $h, k \geqslant 2$. Let $S(m, n) \equiv S_{l;h,k}(m, n)$ represent the number of ordered sets of positive integers x_1, x_2, y_1, y_2 satisfying

$$x_1 + y_1 = m, \quad x_2 + y_2 = n, \tag{1.1}$$

such that

$$(x_1, x_2) \text{ is } l\text{-free}, \ y_1 \text{ is } k\text{-free, and } y_2 \text{ is } h\text{-free}. \tag{1.2}$$

(In case $l = 1$, the first restriction in (1.2) is simply that x_1 and x_2 are relatively prime.) It is the purpose of this paper to find an asymptotic representation for S. In particular we show that

$$S(m, n) = c\alpha(m, n)mn + O\left(m^{\frac{k+h-2}{kh-1}+\varepsilon} n\right) \tag{1.3}$$

for any $\varepsilon > 0$, where c denotes the positive constant (dependent only on l, h, and k)

$$c = \zeta^{-1}(h)\zeta^{-1}(k)\prod_p \left(1 - \frac{p^{k+h-2l}}{(p^k-1)(p^h-1)}\right)$$

(ζ is the Riemann zeta function), and where

$$\alpha(m, n) \equiv \alpha_{l;h,k}(m, n) = \tag{1.4}$$

$$\prod_{\substack{p^u|m \\ p^v|n}} \left(1 - \frac{p^{2l-k-h}(p^k-1)(p^h-1)}{\left(p^{2l-h-u}(p^k-1)(p^h-1) - p^{k-u}+1\right)\left(p^{2l-k-v}(p^k-1)(p^h-1) - p^{h-v}+1\right)}\right)$$

$$\times \prod_{p^u|m} \left(1 + \frac{1}{p^{2l-h-u}(p^k-1)(p^h-1) - p^{k-u}}\right) \times$$

$$\times \prod_{p^v|n} \left(1 + \frac{1}{p^{2l-k-v}(p^k-1)(p^h-1) - p^{h-v}}\right)$$

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is a positive function which is both bounded and bounded away from zero. In (1.4) and in what follows we let u represent the smaller of l and k, and v the smaller of l and h.

The problem treated in this paper is analogous to one considered by Cohen [1], who proved an asymptotic formula for the number of ordered sets of positive integers x_1, x_2, y_1, y_2 satisfying (1.1) such that

$$(x_1, x_2) \text{ and } (y_1, y_2) \text{ are } l\text{-free}.$$

In our proof we adapt Cohen's modifications of a method used by Estermann [2] in giving an elementary solution to a problem of Evelyn and Linfoot: Find an asymptotic representation for the number of positive integral solutions of $x+y = n$ such that x and y are k-free. Specifically we make appropriate divisions of the intervals $[1, \sqrt[k]{m}]$ and $[1, \sqrt[h]{n}]$, and decompose the summation for S into four parts corresponding to combinations of these subintervals. It is noteworthy that our remainder term is not improved by additionally dividing the interval $[1, \sqrt[l]{m}]$, and considering the corresponding eight-term decomposition of the summation for S. Indeed, l does not contribute to the remainder term, even in the special case $l = 1$. This is unlike the formula obtained by Cohen, in which the case $l = 1$ yields a more complex O-term dependent upon the relationship between n and m^2.

2. Preliminaries. In this section we collect several known lemmas, and make some observations concerning functions peculiar to our problem. The following notation is observed: The letter p will be reserved for primes, μ is the Möbius function, J_k is the Jordan totient function of order k, τ is the divisor function, and q_k is the characteristic function of the set of k-free integers; vacuous products and sums are given the values 1 and 0, respectively. For a discussion of the O-notation the reader is referred to [3], § 1.6.

LEMMA 2.1 (cf. [3], Theorem 303).

$$\sum_{d^k|n} \mu(d) = q_k(n).$$

LEMMA 2.2 (cf. [3], Theorem 315). *For any fixed $\varepsilon > 0$, $\tau(n) = O(n^\varepsilon)$.*

From the familiar evaluation

$$J_k(n) = n^k \prod_{p|n} (1-p^{-k})$$

we have

LEMMA 2.3.

$$0 < \zeta^{-1}(k) \leqslant \frac{J_k(n)}{n^k} \leqslant 1.$$

LEMMA 2.4. *For any positive integer r,*

$$\sum_{\substack{d=1\\(d,r)=1}}^{\infty} \frac{\mu(d)}{d^k} = \frac{r^k}{\zeta(k) J_k(r)}.$$

LEMMA 2.5 (cf. [4]). *For positive integers a and b let $N_{a,b}(n)$ denote the number of positive integral solutions of $ax+by = n$. If $(a, b)|n$ then*

$$N_{a,b}(n) = \frac{n(a, b)}{ab} + O(1)$$

uniformly in a, b, and n.

LEMMA 2.6. *For a positive integral variable r we define*

$$\varphi_k(m, r) = \sum_{\substack{d|r\\ d^u|m}} \frac{\mu(d)}{d^{k-u}},$$

where u denotes the smaller of l and k. Then $\varphi_k(m, r) = O(r^\varepsilon)$ for any $\varepsilon > 0$.

LEMMA 2.7. *For all positive integers r*

$$\sum_{\substack{d|r\\ d^u|m}} d^{u-1} = O(r^l).$$

3. Proof of the theorem. For convenience we will often indicate the conditions of summation by referring to numbered formulas. With this convention, then, we wish to examine

$$S(m, n) = \sum_{(1.1)} q_l((x_1, x_2)) q_k(y_1) q_h(y_2),$$

where the summands x_1, x_2, y_1, y_2 are positive integers satisfying (1.1). By Lemma 2.1

$$S(m, n) = \sum_{(3.1)} \mu(r)\mu(d)\mu(e),$$

where $r, d, e, f_1, f_2, g_1, g_2$ are positive integers such that

$$r^l f_1 + d^k g_1 = m, \quad r^l f_2 + e^h g_2 = n. \tag{3.1}$$

Now let $t = t(m, n)$ and $s = s(m, n)$ denote functions of m and n to be specified later, such that $1 \leqslant t \leqslant \sqrt[k]{m}$ and $1 \leqslant s \leqslant \sqrt[h]{n}$. We decompose the summation for S as follows:

$$S(m, n) = \sum_{\substack{(3.1)\\ d\leqslant t\\ e\leqslant s}} \mu(r)\mu(d)\mu(e) + \sum_{\substack{(3.1)\\ d\leqslant t\\ e>s}} \mu(r)\mu(d)\mu(e) + \sum_{\substack{(3.1)\\ d>t\\ e\leqslant s}} \mu(r)\mu(d)\mu(e) + \sum_{\substack{(3.1)\\ d>t\\ e>s}} \mu(r)\mu(d)\mu(e) = S_1+S_2+S_3+S_4, \tag{3.2}$$

let us say. We consider S_4 under the assumptions that $t < \sqrt[k]{m}$ and $s < \sqrt[h]{n}$, and letting $\tau_l(n)$ denote the number of lth power divisors of n we have

$$|S_4| \leqslant \sum_{\substack{(3.1)\\ d>t\\ e>s}} 1 = \sum_{\substack{d>t\\ d^k g_1<m}} 1 \sum_{\substack{e>s\\ e^h g_2<n}} \tau_l\big((n-e^h g_2,\ m-d^k g_1)\big)$$

$$\leqslant \sum_{\substack{d>t\\ d^k g_1<m}} 1 \sum_{\substack{e>s\\ e^h g_2<n}} \tau(m-d^k g_1).$$

Hence by Lemma 2.2 we obtain for any $\varepsilon > 0$,

$$S_4 = O\Big(\sum_{\substack{d>t\\ d^k g_1<m}} (m-d^k g_1)^{\varepsilon} \sum_{\substack{e>s\\ e^h g_2<n}} 1\Big) = O\Big(m^{\varepsilon} \sum_{\substack{d>t\\ d^k g_1<m}} 1 \sum_{\substack{e>s\\ e^h g_2<n}} 1\Big)$$

$$= O\Big(m^{\varepsilon} \sum_{d>t} \frac{m}{d^k} \sum_{e>s} \frac{n}{e^h}\Big) = O\Big(\frac{m^{1+\varepsilon} n}{t^{k-1} s^{h-1}}\Big).$$

Similarly

$$S_3 = O\Big(\frac{m^{1+\varepsilon} n}{t^{k-1}}\Big) \quad \text{and} \quad S_2 = O\Big(\frac{m^{1+\varepsilon} n}{s^{h-1}}\Big)$$

for any $\varepsilon > 0$. Since $S_3 = S_4 = 0$ if $t = \sqrt[k]{m}$, and $S_2 = S_4 = 0$ if $s = \sqrt[h]{n}$, we have

$$(3.3) \qquad S_2 + S_3 + S_4 = O\Big(\frac{m^{1+\varepsilon} n}{t^{k-1}}\Big) + O\Big(\frac{m^{1+\varepsilon} n}{s^{h-1}}\Big)$$

for any $\varepsilon > 0$, with the first O-term zero if $t = \sqrt[k]{m}$, and the second O-term zero if $s = \sqrt[h]{n}$. We remark that the O-constants in (3.3) are uniform in all parameters (dependent only on l, h, and k); similar observations may be made in what follows and will be assumed.

We now return to S_1 which may be rewritten as

$$S_1 = \sum_{\substack{r \leqslant \sqrt[l]{m}\\ d \leqslant t\\ e \leqslant s}} \mu(r)\mu(d)\mu(e) N_{d^k, r^l}(m) N_{e^h, r^l}(n),$$

where by (3.1) it may additionally be assumed that $(d^k, r^l)|m$, $(e^h, r^l)|n$, and $r \leqslant \sqrt[l]{m}$. Thus in view of Lemma 2.5

$$(3.4) \qquad S_1 = S' + O\Big(m \sum_{(3.6)} \frac{(d,r)^u}{r^l d^k}\Big) + O\Big(n \sum_{(3.6)} \frac{(e,r)^v}{r^l e^h}\Big) + O\Big(\sum_{(3.6)} 1\Big),$$

where

$$(3.5) \qquad S' = mn \sum_{(3.6)} \frac{\mu(r)\mu(d)\mu(e)(d,r)^u (e,r)^v}{r^{2l} d^k e^h}$$

and where d, e, r are positive integers with

$$(3.6) \qquad d \leqslant t,\ e \leqslant s,\ r \leqslant \sqrt[l]{m},\ (d,r)^u|m, \text{ and } (e,r)^v|n.$$

The first O-term in (3.4) may be written as

$$(3.7) \qquad O\Big(m \sum_{r \leqslant \sqrt[l]{m}} \frac{1}{r^l} \sum_{\substack{d \leqslant t\\ (d,r)^u|m}} \frac{(d,r)^u}{d^k} \sum_{\substack{e \leqslant s\\ (e,r)^v|n}} 1\Big).$$

Since the last summation in (3.7) is bounded by s, and the second by t, (3.7) becomes

$$O\Big(mst \sum_{r \leqslant \sqrt[l]{m}} \frac{1}{r^l}\Big) = O(m^{1+\varepsilon} st)$$

for any $\varepsilon > 0$ (recalling that $l \geqslant 1$). Analogously the second and third O-terms in (3.4) are $O(m^{\varepsilon} nst)$ for any $\varepsilon > 0$ and $O(mst)$, respectively. That is

$$(3.8) \qquad S_1 = S' + O(m^{\varepsilon} nst)$$

for any $\varepsilon > 0$. Combining (3.2) with (3.3) and (3.8) we see that

$$(3.9) \qquad S(m,n) = S' + O(m^{\varepsilon} nst) + O\Big(\frac{m^{1+\varepsilon} n}{t^{k-1}}\Big) + O\Big(\frac{m^{1+\varepsilon} n}{s^{h-1}}\Big)$$

for any $\varepsilon > 0$, where the third and fourth O-terms are zero when $t = \sqrt[k]{m}$ and $s = \sqrt[h]{n}$, respectively.

We now consider S'. By (3.5) and (3.6)

$$S' = mn \sum_{r \leqslant \sqrt[l]{m}} \frac{\mu(r)}{r^{2l}} \sum_{\substack{d \leqslant t\\ (d,r)^u|m}} \frac{\mu(d)(d,r)^u}{d^k} \sum_{\substack{e \leqslant s\\ (e,r)^v|n}} \frac{\mu(e)(e,r)^v}{e^h};$$

and via substitutions similar to those made above,

$$(3.10) \qquad S' = mn \sum_{r \leqslant \sqrt[l]{m}} \frac{\mu(r)}{r^{2l}} \sum_{\substack{x|r\\ x^u|m}} \frac{\mu(x)}{x^{k-u}} \sum_{\substack{y|r\\ y^v|n}} \frac{\mu(y)}{y^{h-v}} \sum_{\substack{d \leqslant t/x\\ (d,r)=1}} \frac{\mu(d)}{d^k} \sum_{\substack{e \leqslant s/y\\ (e,r)=1}} \frac{\mu(e)}{e^h}.$$

Writing each of the last two inner sums in (3.10) as the difference of two infinite sums, we apply Lemmas 2.3 and 2.4 to their product to obtain

$$\sum_{\substack{d\le t/x\\(d,r)=1}} \frac{\mu(d)}{d^k} \sum_{\substack{e\le s/y\\(e,r)=1}} \frac{\mu(e)}{e^h} = \frac{r^{k+h}}{\zeta(k)\zeta(h)J_k(r)J_h(r)} + O\left(\left(\frac{x}{t}\right)^{k-1}\right) + O\left(\left(\frac{y}{s}\right)^{h-1}\right),$$

so that (3.10) becomes, on simplification,

$$S' = \frac{mn}{\zeta(k)\zeta(h)} \sum_{r\le \sqrt[l]{m}} \frac{\mu(r)}{r^{2l-k-h}J_k(r)J_h(r)}\varphi_k(m,r)\varphi_h(n,r) + O\Bigg(\frac{mn}{t^{k-1}}\sum_{r\le\sqrt[l]{m}}\frac{1}{r^{2l}}|\varphi_h(n,r)|\sum_{\substack{x\mid r\\ x^u\mid m}} x^{k-1}\Bigg) + O\Bigg(\frac{mn}{s^{h-1}}\sum_{r\le\sqrt[l]{m}}\frac{1}{r^{2l}}|\varphi_k(m,r)|\sum_{\substack{y\mid r\\ y^v\mid n}} y^{h-1}\Bigg).$$

By Lemmas 2.6 and 2.7

$$\begin{aligned} S' &= \frac{mn}{\zeta(k)\zeta(h)} \sum_{r\le \sqrt[l]{m}} \frac{\mu(r)}{r^{2l-k-h}J_k(r)J_h(r)}\varphi_k(m,r)\varphi_h(n,r) + O\Bigg(\frac{mn}{t^{k-1}}\sum_{r\le\sqrt[l]{m}} r^{\varepsilon-l}\Bigg) + O\Bigg(\frac{mn}{s^{h-1}}\sum_{r\le\sqrt[l]{m}} r^{\varepsilon-l}\Bigg)\\ &= \frac{mn}{\zeta(k)\zeta(h)} \sum_{r\le \sqrt[l]{m}} \frac{\mu(r)}{r^{2l-k-h}J_k(r)J_h(r)}\varphi_k(m,r)\varphi_h(n,r) + O\left(\frac{m^{1+\varepsilon}n}{t^{k-1}}\right) + O\left(\frac{m^{1+\varepsilon}n}{s^{h-1}}\right). \end{aligned}$$

Therefore, again using Lemmas 2.3 and 2.6, we have

$$(3.11)\qquad S' = \frac{mn}{\zeta(k)\zeta(h)} \sum_{r=1}^{\infty} \frac{\mu(r)}{r^{2l-k-h}J_k(r)J_h(r)}\varphi_k(m,r)\varphi_h(n,r) + O(m^{\varepsilon}n) + O\left(\frac{m^{1+\varepsilon}n}{t^{k-1}}\right) + O\left(\frac{m^{1+\varepsilon}n}{s^{h-1}}\right).$$

We make the definition

$$(3.12)\qquad \beta_{l;h,k}(m,n) \equiv \beta(m,n) = \zeta^{-1}(k)\zeta^{-1}(h)\sum_{r=1}^{\infty}\frac{\mu(r)}{r^{2l-k-h}J_k(r)J_h(r)}\varphi_k(m,r)\varphi_h(n,r).$$

In § 4 we will show that

$$(3.13)\qquad \beta(m,n) = ca(m,n),$$

where c and a are as described in the introduction, and we indicate a proof that β is both bounded and bounded away from zero.

Now collecting the results in (3.9) and (3.11)–(3.13) we see that

$$S(m,n) = ca(m,n)mn + O\left(\frac{m^{1+\varepsilon}n}{t^{k-1}}\right) + O\left(\frac{m^{1+\varepsilon}n}{s^{h-1}}\right) + O(m^{\varepsilon}nst)$$

for any $\varepsilon > 0$. Our formula (1.3) follows on choosing

$$t = m^{\frac{h-1}{kh-1}} \quad\text{and}\quad s = m^{\frac{k-1}{kh-1}}.$$

4. Evaluation of β. Because each of the functions in the series representation for β is multiplicative in r, we may apply the Euler product formula (cf. [3], § 17.4) to the series in (3.12) to obtain

$$\begin{aligned}\beta(m,n) &= \zeta^{-1}(k)\zeta^{-1}(h)\prod_p\left(1-\frac{\varphi_k(m,p)\varphi_h(n,p)}{p^{2l-k-h}J_k(p)J_h(p)}\right)\\ &= \zeta^{-1}(k)\zeta^{-1}(h)\beta_1(m,n)\beta_2(m,n)\beta_3(m,n)\beta_4(m,n),\end{aligned}$$

where

$$\beta_1(m,n) = \prod_{\substack{p^u\| m\\ p^v\| n}}\left(1-\frac{(p^{k-u}-1)(p^{h-v}-1)}{p^{2l-u-v}(p^k-1)(p^h-1)}\right),$$

$$\beta_2(m,n) = \prod_{\substack{p^u\| m\\ p^v\nmid n}}\left(1-\frac{p^{k-u}-1}{p^{2l-h-u}(p^k-1)(p^h-1)}\right),$$

$$\beta_3(m,n) = \prod_{\substack{p^u\nmid m\\ p^v\| n}}\left(1-\frac{p^{h-v}-1}{p^{2l-k-v}(p^k-1)(p^h-1)}\right),$$

$$\beta_4(m,n) = \prod_{\substack{p^u\nmid m\\ p^v\nmid n}}\left(1-\frac{1}{p^{2l-k-h}(p^k-1)(p^h-1)}\right).$$

Rather cumbersome algebraic juggling shows that the relation (3.13) holds; we will direct our attention instead to the boundedness properties of β. For convenience we consider the four cases

$$l < \min(h,k),\qquad h\le l < k,\qquad k\le l < h,\qquad \max(h,k)\le l$$

separately, and we use the substitutions

$$x = p^{k-l}, \quad y = p^{h-l}, \quad z = (p^k-1)(p^h-1), \tag{4.1}$$

noting that $z > xy > 0$.

Case 1. $l < \min(h, k)$. Since $u = v = l$

$$\beta(m,n) = \zeta^{-1}(k)\zeta^{-1}(h)\prod_{p^l|(m,n)}\left(1-\frac{(x-1)(y-1)}{z}\right)\times$$

$$\times\prod_{\substack{p^l|m\\ p^l\nmid n}}\left(1-\frac{y(x-1)}{z}\right)\times\prod_{\substack{p^l\nmid m\\ p^l|n}}\left(1-\frac{x(y-1)}{z}\right)\times\prod_{\substack{p^l\nmid m\\ p^l\nmid n}}\left(1-\frac{xy}{z}\right)$$

$$= c\prod_{p^l|(m,n)}\left(1+\frac{x+y-1}{z-xy}\right)\times\prod_{\substack{p^l|m\\ p^l\nmid n}}\left(1+\frac{y}{z-xy}\right)\times\prod_{\substack{p^l\nmid m\\ p^l|n}}\left(1+\frac{x}{z-xy}\right).$$

With the note following (4.1) it is clear from the first equality that β is bounded above, and from the second equality that β is positive; in fact $0 < c \leqslant \beta \leqslant 1$, so that also $1 \leqslant \alpha \leqslant c^{-1}$.

The other cases are argued similarly, yielding the same bounds for α and β and we give the corresponding formulas without comment.

Case 2. $h \leqslant l < k$.

$$\beta(m,n) = \zeta^{-1}(k)\zeta^{-1}(h)\prod_{\substack{p^l|m\\ p^h\nmid n}}\left(1-\frac{y(x-1)}{z}\right)\times\prod_{\substack{p^l\nmid m\\ p^h\nmid n}}\left(1-\frac{xy}{z}\right)$$

$$= c\prod_{\substack{p^l|m\\ p^h\nmid n}}\left(1+\frac{y}{z-xy}\right)\times\prod_{p^h|n}\left(1+\frac{xy}{z-xy}\right).$$

Case 3. $k \leqslant l < h$.

$$\beta(m,n) = \zeta^{-1}(k)\zeta^{-1}(h)\prod_{\substack{p^l|n\\ p^k\nmid m}}\left(1-\frac{x(y-1)}{z}\right)\times\prod_{\substack{p^l\nmid n\\ p^k\nmid m}}\left(1-\frac{xy}{z}\right)$$

$$= c\prod_{\substack{p^l|n\\ p^k\nmid m}}\left(1+\frac{x}{z-xy}\right)\times\prod_{p^k|m}\left(1+\frac{xy}{z-xy}\right).$$

Case 4. $\max(h, k) \leqslant l$.

$$\beta(m,n) = \zeta^{-1}(k)\zeta^{-1}(h)\prod_{\substack{p^k\nmid m\\ p^h\nmid n}}\left(1-\frac{xy}{z}\right) = c\prod_{\substack{p^k|m\\ p^h\nmid n}}\left(1+\frac{xy}{z-xy}\right)\times\prod_{p^h|n}\left(1+\frac{xy}{z-xy}\right).$$

5. Some corollaries. As an immediate corollary of (1.3) and the results of the previous section we have the following

COROLLARY 5.1. *If $m \leqslant n$ then*

$$S(m,n) \sim c\alpha(m,n)mn \quad \text{as} \quad m\to\infty.$$

COROLLARY 5.2. *If $h = k = 2$ and $l = 1$ then*

$$S(m,n) = \frac{36}{\pi^4}\gamma(m,n)mn + O(m^{2/3+\varepsilon}n) \quad (m \leqslant n)$$

for any $\varepsilon > 0$, where

$$\gamma(m,n) = \prod_p\left(1-\frac{p^2}{(p^2-1)^2}\right)\times\prod_{p|(m,n)}\left(1+\frac{1}{(p^2-1)(p+1)-p}\right)\times$$

$$\times\prod_{p|mn}\left(1+\frac{p}{(p^2-1)^2-p^2}\right).$$

COROLLARY 5.3. *If $h = k = l = 2$ then*

$$S(m,n) = \frac{36}{\pi^4}\varrho(m,n)mn + O(m^{2/3+\varepsilon}n) \quad (m \leqslant n)$$

for any $\varepsilon > 0$, where

$$\varrho(m,n) = \prod_{\substack{p^2\nmid m\\ p^2\nmid n}}\left(1-\frac{1}{(p^2-1)^2}\right).$$

COROLLARY 5.4. *If $m = n$ then*

$$S(m,m) = \frac{\sigma(m)}{\zeta(k)\zeta(h)}m^2 + O\left(m^{\frac{kh+k+h-3}{kh-1}+\varepsilon}\right)$$

for any $\varepsilon > 0$, where

$$\sigma(m) = \prod_{p|m}\left(1-\frac{(p^{k-u}-1)(p^{h-v}-1)}{(p^k-1)(p^h-1)}\right)\times\prod_{p^w\nmid m}\left(1-\frac{p^{k+h-2l}}{(p^k-1)(p^h-1)}\right),$$

and where w denotes the smallest of l, h, and k.

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On Goldbach's problem

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1. Introduction. Goldbach conjectured in 1742 that every even number greater than two is the sum of two odd primes.

In 1923 Hardy and Littlewood developed a method ([4], [5]) which enabled them to show that

(i) if no Dirichlet L-function has a zero in the region $\operatorname{Re} s > 3/4$, then every sufficiently large odd natural number is the sum of three odd primes,

and

(ii) if every Dirichlet L-function has all its zeros in the region $\operatorname{Re} s \leqslant 1/2$ and if $E(N)$ is the number of even numbers less than N for which Goldbach's conjecture is false, then

$$E(N) = O_\varepsilon(N^{1/2+\varepsilon})$$

for every positive ε.

In 1937 Vinogradov obtained estimates ([12], [13]) (for an account of which, see [14]) for trigonometric sums of the form

$$\sum_{p\leqslant N} e^{2\pi i x p} \tag{1.1}$$

which, combined with Page's work [9] on the zeros of L-functions, enabled him to show unconditionally by the Hardy–Littlewood method that every sufficiently large odd number is the sum of three odd primes.

Using these ideas, Van der Corput [1], Tchudakoff [11] and Estermann [3] were able to show unconditionally that

$$E(N) = O_A(N\log^{-A} N). \tag{1.2}$$

In the mid 1940's, Linnik [7], [8] and Tchudakoff [10], by finding estimates for the number of zeros of L-functions in certain regions, were able to dispense with Vinogradov's method for sums of the type (1.1) and thus obtained essentially new proofs of the Goldbach–Vinogradov theorem and (1.2).